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Pressure and Variational Principle for Random Ising Model

F. Ledrappier

Université Pierre et Marie Curie, Laboratoire de Calcul Probabilités, F-75230 Paris, France

Abstract. An Ising model traditionally is a model for a repartition of spins on a lattice. Griffiths and Lebowitz ([3. 5]) have considered distributions of spins which can occur only on some randomly prescribed sites—Edwards and Anderson have introduced models where the interaction was random ([6, 7]). In both cases, the formalism of statistical mechanics reduces mainly to a relativised variational principle, which has been proved recently by Walters and the author [1]. In this note, we show how that reduction works and formulate the corresponding results on an example of either model.

1. Notations and Results

Let $Y=\{0,1\}^{\mathbf{Z}^d}$, $X=\{0,+1,-1\}^{\mathbf{Z}^d}$ be the sets of configurations of particles (respectively of particles with a spin) on a lattice $\mathbf{Z}^d$, Let $\pi: X\to Y$ denote the natural map such that $(\pi(x))_s=|x_s|$ for s in $\mathbf{Z}^d$, τ_s the shift transformations on X and Y, Λ_n the positive cube of side n containing the point $(0,0,\dots,0)$ of $\mathbf{Z}^d$. A point y is said generic for an invariant measure ν on Y if the measures $\frac{1}{n^d}\sum_{s\in\Lambda_n}\delta_{\tau_s y}$ converge towards the measure ν (δ_z denotes the Dirac measure at the point z).

Let J,h be real numbers. For x in X with $x_s=0$ except for a finite number of s, define:

$$U(x)=\sum_{s\in\mathbf{Z}^d}hx_s+\sum_{\substack{s,t\in\mathbf{Z}^d\\|s-t|=1}}Jx_sx_t\,,$$

where $|s|=\sum_i|s_i|$ if $s=(s_i,\,i=1,\dots,d)$.

For any finite subset Λ of $\mathbf{Z}^d$ and any y in Y let us consider the partition function of the box Λ above $yZ_\Lambda(y)$:

$$Z_\Lambda(y)=\sum\exp(-U(x))\,,$$

where the summation is made over the set of x such that $|x_s|=y_s$ for s in Λ, $x_s=0$ elsewhere. Let $M(X,\tau)$ denote the set of invariant probability measures on X.

For μ in $M(X,\tau)$ and A a finite measurable partition of X, we consider $H(\mu, A)$ the mean entropy of A, and define the entropy $h(\mu)$ by: $h(\mu)=\sup_A H(\mu, A)$. Let us define also the conditional entropy $h(\mu/Y)$ by:

$h(\mu/Y)=\sup_A \inf_B H(\mu, A)-H(\mu, \pi^{-1}(B))$, where A (resp. B) is a partition of X [resp. a partition of Y with $\pi^{-1}(B)$ coarser than A]. If $h(\mu\circ\bar{\pi}^{-1})$ is finite, we have the following formula:

$$h(\mu/Y)=h(\mu)-h(\mu\cdot\pi^{-1}) \qquad \text{(see [2])}\,.$$

Theorem 1. *If y is generic for some measure v then the sequence $\frac{1}{n^d}\operatorname{Log} Z_{A_n}(y)$ converges as n goes to infinity towards a number P_v called the pressure above v; the pressure above v satisfies the following variational principle:*

$$P_v=\max_{\substack{\mu\in M(X,\tau)\\ \mu\circ\pi^{-1}=v}} h(\mu)-h(v)+\int a(x)d\mu\,,$$

where $a(x)=-hx_0-\frac{J}{2}\sum_{|s|=1} x_0x_s$.

Note that if v is ergodic almost every point y is generic.

Let S be the set of pairs of neighbours in $\boldsymbol{Z}^d$; the translations of $\boldsymbol{Z}^d$ act naturally on S.

Let $\mathbb{R}$ denote the real line and fix y' in $Y'=\mathbb{R}^S$. We can define by the usual formulas the partition functions $P_\Lambda(y')$ of a finite box Λ corresponding to the interaction $J_{i,j}$

$$J_{i,j}=y_{\{i,j\}} \text{ if } i \text{ and } j \text{ are neighbours}\,,$$

on the space $X'=\{-1,+1\}^{\boldsymbol{Z}^d}$ of spins on the lattice $\boldsymbol{Z}^d$.

Theorem 2. *Let v be a $\boldsymbol{Z}^d$-invariant, ergodic probability measure on Y', such that $\sup_t \int |y_t| dv<\infty$.*

The limit $\lim_{n\to\infty}\frac{1}{n^d}\operatorname{Log} P_{\Lambda_n}(y')$ exists for almost every y' and satisfies a variational principle. (See Vuillermot [8] for a close result when the $y_{i,j}$ are independent.)

2. Proof of Theorem 1

We recall first the notation and results from [1], in a suitable form.

Let X, Y compact metric spaces, $\pi: X\to Y$ a surjection and a $\boldsymbol{Z}^d$ action on X and Y which commutes with π. Let $\varepsilon>0$, n integer be given, d denote a distance on X.

A set E in X is said (n,ε) separated if for any $x_1\neq x_2$ in E, $\sup_{i\in\Lambda_n} d(\tau_i x_1, \tau_i x_2)$ is greater than ε. For f continuous function on X, y in Y, we define:

$$P_n(\tau, f, y, \varepsilon)=\sup_E \sum_{x\in E}\exp\left(\sum_{i\in\Lambda_n} f(\tau_i x)\right),$$

where the sup is taken over the (n,ε) separated sets E with $\pi(x)=y$ for every x in E

$$p(\tau, f, y)=\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n^d}\operatorname{Log} p_n(\tau, f, y, \varepsilon)\,.$$

Theorem 3 ([1], *Proposition 3.5*). *For any invariant measure μ on X, we have:*

$$h(\mu/Y)+\mu(f)\leqq \sup_{\varepsilon} \limsup_{n\to\infty} \int \frac{1}{n^d} \operatorname{Log} p_n(\tau,f,y,\varepsilon)d\mu\circ\pi^{-1}(y)\,.$$

Remark. Actually Proposition 3.5 in [1] is stated with $\int p(\tau,f,y)d\mu\circ\pi^{-1}(y)$ instead of sup lim sup.... But this stronger result is also true by *not* applying Fatou's lemma at the end the proof of the Proposition 3.5.

Theorem 4 ([1], *Proposition 3.6*). *If y is generic for some measure v and ε positive, there exists an invariant measure on X such that $\mu\circ\pi^{-1}=v$ and*

$$h(\mu/Y)+\mu(f)\geqq \limsup_{n\to\infty} \frac{1}{n^d} \operatorname{Log} p_n(\tau,f,y,\varepsilon)\,.$$

As the entropy $h(\mu/Y)$ is upper semi-continuous on the space $M(X,\tau)$ and as the set of measures which projects onto v is a closed subset of $M(X,\tau)$ there exists a measure μ_0 such that $\mu_0\circ\pi^{-1}=v$ and:

$$h(\mu_0/Y)+\mu_0(f)=\sup_{\substack{\mu\in M(X,\tau)\\ \mu\circ\pi^{-1}=v}} h(\mu/Y)+\mu(f)\,.$$

Therefore Theorem 1 will be proved when we shall have shown the following inequalities:

$$(*)\quad \limsup_n \frac{1}{n^d}\operatorname{Log} Z_{\Lambda_n}(y)\leqq \sup_{\substack{\mu\in M(X,\tau)\\ \mu\circ\pi^{-1}=v}} h(\mu/Y)+\mu(a)$$
$$\leqq \liminf_n \frac{1}{n^d}\operatorname{Log} Z_{\Lambda_n}(y)$$

as soon as y is generic for v.

We prove these relations with two lemmas:

Let us take on X the distance δ defined by $\delta(x^1,x^2)=\alpha^k$, where $0<\alpha<1$ and k is the smallest positive integer such that there exists $s=(s_1,\dots,s_d)$ in $\boldsymbol{Z}^d$ with $\sup_J s_j=k$ and $x^1_s\neq x^2_s$.

Lemma 5. *For any y in Y, $\varepsilon>0$,* $\frac{1}{n^d}\operatorname{Log} Z_{\Lambda_n}(y)\leqq \frac{1}{n^d}\operatorname{Log} p_n(\tau,a,y,\varepsilon)+\frac{2^dJ}{n}$, *for any y in Y, $\varepsilon>0$, there exists m such that:*

$$\frac{1}{n^d}\operatorname{Log} p_n(\tau,a,y,\varepsilon)\leqq \frac{1}{n^d}\operatorname{Log} Z_{\Lambda_n}(y)+\frac{2^dJ}{n}+\frac{(2m)^d\log 2}{n}\,.$$

Proof. Take $\varepsilon>\alpha$. A set E is (n,ε) separated if and only if any two different points in E have some different coordinate in Λ_n. So the set of x such that $|x_s|=y_s$ for s in Λ_n, $x_s=0$ elsewhere is (n,ε) separated and we may write, by estimation of the boundary effect

$$Z_{\Lambda_n}(y)\leqq p_n(\tau,a,y,\varepsilon)\cdot\exp(2^dn^{d-1}J)\,.$$

On the other hand for any ε there exists m such that $\varepsilon>\alpha^m$ and so if a set E is (n,ε) separated any two different points in E have some different coordinate s with $-m\leqq s_j<n+m$. If z is some point with $|z_s|=y_s$ for s in Λ_n, $z_s=0$ elsewhere, there are

at most $2^{(2m)^d n^{d-1}}$ different points in E with $x_s = z_s$ for all s in Λ_n. For any (n, ε) separated set E in $\pi^{-1}(y)$ we have:

$$\sum_{x \in E} \exp\left(\sum_{i \in \Lambda_n} a(\tau_i x) \right) \leqq 2^{(2m)^d n^{d-1}} \cdot \exp(2^d n^{d-1} J) \cdot Z_{\Lambda_n}(y), \quad \text{q.e.d.}$$

Corollary 6. *For any y in Y, any measure ν on Y:*

$$\limsup_n \frac{1}{n^d} \operatorname{Log} Z_{\Lambda_n}(y) = p(\tau, a, y),$$

$$\limsup_n \frac{1}{n^d} \int \operatorname{Log} Z_{\Lambda_n}(y) d\nu(y) = \lim_{\varepsilon \to 0} \limsup_n \frac{1}{n^d} \int \operatorname{Log} p_n(\tau, a, y, \varepsilon) d\nu(y).$$

Lemma 7. *If y is generic for some measure ν, we have:*

$$\limsup_n \frac{1}{n^d} \int \operatorname{Log} Z_{\Lambda_n}(y) d\nu(y) \leqq \liminf_n \frac{1}{n^d} \operatorname{Log} Z_{\Lambda_n}(y).$$

Proof. Let us take $m > n$, j in Λ_n. The box Λ_m is made of disjoint boxes $\Lambda_n + j + ns$, where $ns = (ns_1, \ldots, ns_d)$, s_i is a positive integer smaller than $\frac{m}{n} - 1$, and of less that $(2n)^d m^{d-1}$ other points.

There are less than $\left(\frac{m}{n}\right)^d 2^d n^{d-1}$ points in the boundaries of the small $\Lambda_n + j + ns$ boxes. Therefore we may write:

$$\operatorname{Log} Z_{\Lambda_m}(y) \geqq \sum_{s, 0 \leqq s_i < \frac{m}{n} - 1} \operatorname{Log} Z_{\Lambda_n}(\tau_{ns+j} y) - J 2^d n^{d-1} \left(\frac{m}{n}\right)^d - (h + 2J)(2n)^d m^{d-1}.$$

Averaging over all j in Λ_n, dividing by m^d and taking $\liminf_m$, we get by the genericity of y:

$$\liminf_m \frac{1}{m^d} \operatorname{Log} Z_{\Lambda_m}(y) \geqq \int \frac{1}{n^d} \operatorname{Log} Z_{\Lambda_n}(y) d\nu(y) - J \frac{2d}{n}.$$

The lemma follows by taking $\limsup_n$.

The inequalities $(*)$ are proved by comparison of Theorems 3 and 4, Corollary 6, and Lemma 7.

3. Proof of Theorem 2

Let us choose a sequence of continuous real functions g_k on $\boldsymbol{R}$ with compact support such that

$$\delta_k = \sup_t \int |g_k(y_t) - y_t| d\nu \text{ goes to } 0 \text{ as } k \text{ goes}$$

to infinity.

Let $P_\Lambda^k(y')$ be the partition function on X' corresponding to the interaction $J_{i,j}^k$:

$$J_{i,j}^k = g_k(y_{\{i,j\}}) \text{ if } i \text{ and } j \text{ are neighbours}.$$

Let a_k and a be real continuous functions on the product space $Y' \times X'$ defined by

$$a_k(y', x') = -\frac{1}{2d} \sum_{|s|=1} g_k(y_{\{0,s\}}) x_0 x_s$$

$$a(y', x') = -\frac{1}{2d} \sum_{|s|=1} y_{\{0,s\}} x_0 x_s .$$

For any k, the following lemma is got by considering Y' as a factor of $Y' \times X'$.

Lemma 8. *For ν almost every y', we have*

$$\lim_{n\to\infty} \frac{1}{n^d} \operatorname{Log} P^k_{xo}(y') = \max_{\substack{\mu \in M(Y' \times X', \tau) \\ \mu \circ \pi^{-1} = \nu}} h(\mu | Y') + \int a_k d\mu .$$

Let us consider the compact spaces $\bar{\boldsymbol{R}} = \boldsymbol{R} \cup \{\infty\}$ and $\bar{Y}' = \bar{\boldsymbol{R}}^S$. The space Y' is naturally continuously imbedded in $\bar{Y}'$, the function a_k is the restriction to $Y' \times X'$ of a continuous function $\bar{a}_k$ on $\bar{Y}' \times X'$, the measure ν is the measure induced on the invariant set $Y' \times X'$ by an invariant ergodic measure $\bar{\nu}$ on $\bar{Y}' \times X'$.

We get then by the same estimations as in § 2: If y is generic for $\bar{\nu}$, we have:

$$\lim_{n\to\infty} \frac{1}{n^d} \operatorname{Log} P^k_{\Lambda_n}(y) = \max_{\substack{\bar{\mu} \in M(\bar{Y}' \times X', \tau) \\ \bar{\mu} \circ \pi^{-1} = \bar{\nu}}} h(\bar{\mu} | \bar{Y}) + \int \bar{a}_k d\bar{\mu} .$$

Lemma 8 follows by observing that almost every y' in Y' is generic for $\bar{\nu}$ and that measures on $\bar{Y}' \times X'$ which projects onto $\bar{\nu}$ are actually carried by $Y' \times X'$.

We also have the following uniform approximations:

Lemma 9. *For any measure μ such that $\mu \circ \pi^{-1} = \nu$,*

$$\left| \int a_k d\mu - \int a d\mu \right| \leqq \delta_k$$

obvious.

Lemma 10. *The sequence of functions on Y', $s_k(y)$*

$$s_k(y) = \sup_n \frac{1}{n^d} |\operatorname{Log} P_{\Lambda_n}(y) - \operatorname{Log} P^k_{\Lambda_n}(y)|$$

converges to zero in probability (i.e. for any $\alpha \nu(s_k \geqq \alpha) \to 0$).

Proof of Lemma 10. We have for any y and any n

$$|\operatorname{Log} P_{\Lambda_n}(y) - \operatorname{Log} P^k_{\Lambda_n}(y)| \leqq \sum_t |y_t - g_k(y_t)| ,$$

where the sum extends over all pairs of neighbours in Λ_n. Let τ denote the action of $\boldsymbol{Z}^d$ on Y, $G_k(y) = \sum_{t, t \ni (0,0,0)} |y_t - g_k(y_t)|$.

We have then:

$$|\operatorname{Log} P_{\Lambda_n}(y) - \operatorname{Log} P^k_{\Lambda_n}(y)| \leqq \sum_{\Lambda_n} G_k(\tau^i y)$$

and

$$s_k(y) \leqq \sup_n \frac{1}{n^d} \sum_{\Lambda_n} G_k(\tau^i y) .$$

By a maximal ergodic lemma for a $\boldsymbol{Z}^d$ action ([9], Theorem IV′), there exists a number λ such that

$$\nu\left\{\sup_n \frac{1}{n^d} \sum_{\Lambda_n} G_k \circ \tau^i \geqq \alpha\right\} \leqq \frac{\lambda}{\alpha} \int |G_k| d\nu \leqq \frac{\lambda}{\alpha} 2d\delta_k$$

and the lemma follows.

We can now proof Theorem 2. Let us choose a sequence k_i such that $s_{k_i}(y)$ converges to zero almost everywhere. For almost every y, the conclusion of Lemma 8 holds for every k_i, we have $s_{k_i}(y) \to 0$ and

$$\sup_{\substack{\mu \in \boldsymbol{M} \\ \mu \circ \pi^{-1} = \nu}} \int a_{k_i} d\mu - \int a d\mu \to 0 \quad \text{by Lemma 9} .$$

The conclusion of Theorem 2 follows.

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