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The Application of Dewitt-Morette Path Integrals to General Relativity

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Abstract. The formulation of path integrals in terms of pseudomeasures by Cecile DeWitt-Morette is extended to infinite-dimensional state-spaces and to the state spaces dual to nuclear spaces appropriate to second-quantisation. In both cases a "distribution" formulation is given to allow a subsequent extension to manifolds. It is shown that the resulting theory is "correct" in that it can give rise to a wave function on state space which obeys a Schrödinger equation in appropriate circumstances. The corresponding state manifolds for quantum gravity are then defined, and the conditions under which the theory extends to them are discussed. It is shown in an appendix that the Riemannian metric required by the theory exists on one of the types of state manifold for a wide class of cases.

1. Introduction and Synopsis

(a) The Idea of Path Integrals

We consider a dynamical system whose state at time τ is represented by a point $q(\tau)$ in a configuration space E. Thus as τ varies from 0 to t, $q(\tau)$ can describe a *path* $q:[0,\mathrm{t}]=T\to E$. Given an initial state $O=q(0)\in E$, we examine the set Φ of all C^∞ paths starting at O.

The basic idea of the path-integral formalism is to quantise the system by defining a wave-function $\psi(x)$ for the state at time t by the formula

$$\int_A \mathrm{d}\Psi(x) \overset{\mathrm{def}}{=} \int_A \psi(x)\mathrm{d}\mu(x) = \int_{\Phi_1} \chi_A(q(t))\mathrm{d}\nu(q) \tag{1}$$

for any $A\subset E$. Here μ is some "standard" measure on E, χ_A is the characteristic function of A. Φ has been completed in a suitable metric to Φ_1, and ν is a specially constructed measure designed to give the correct ψ. We may regard (1) as the formal version of the informal equation[1]

$$\psi(x)=\int_{\Phi_1}\delta_\mu(x-q(t))\,\mathrm{d}\nu(q)\,. \tag{1'}$$

Unfortunately, no measure ν can produce the correct ψ, so that the following alternative strategies have been proposed for defining the expression

$$\int_{\Phi_1} f(q)\,\mathrm{d}\nu(q) \tag{2}$$

of which the right hand side of (1) is a special case.

(i) Partition $[0,t]$ by a set $\{t_0=0,\ t_1,\ t_2,\dots,t_n=t\}$, integrate over the finite collection of $q(t_i)$'s, then let $n\to\infty$ (giving a "lattice integral" [18]). This was Feynman's original approach [19].

(ii) Construct a true measure $\tilde{\nu}$ related to ν by replacing t by it (or by making $\hbar$ complex), getting the required result by analytic continuation. This is the "mainstream" approach, related to Euclidean quantum field theory [13]. The measure $\tilde{\nu}$ is then the well-understood Wiener measure, and the paths are random walks.

(iii) DeWitt-Morette's approach [3, 4][2] is to write (2) as (ν, f) and regard ν as a distribution, (2) having meaning only for f in a suitable class of test functions on Φ_1 (rather than for all continuous f, as would be the case if ν were a measure).

To construct ν we must take coordinates in E, at least in some open set. Thus we consider first the case where E is actually a vector space, hoping to apply the vector space result to a coordinate patch in a general manifold if necessary. When E is *finite-dimensional* the situation has been completely analysed by DeWitt-Morette.

The key idea (see also [20]) is the use of Fourier transforms on Φ_1' : if ν were a measure, its Fourier transform would be a function $\mathscr{F}\nu$ on Φ_1' defined by

$$\mathscr{F}\nu(q')=\int_{\Phi_1}\exp[i(q',q)]\,\mathrm{d}\nu(q)$$

where (q',q) is the pairing $\Phi_1'\times\Phi\to\mathbb{C}$. Thus one can work with Fourier transforms to cover a wider class of distributions on Φ_1 than plain measures.

In § 2 I recapitulate some of DeWitt-Morette's work, giving precise definitions of all the spaces involved and extending it to the case where E is infinite-dimensional (a Banach space). Since Φ_1 is already infinite-dimensional this extension is hardly more than a matter of notation.

The implicit choice of test-functions associated with this method is a class $\mathfrak{C}$ of C^∞ cylinder functions (defined in § 3 below). But this class, while technically simple to work with, depends for its definition in an essential way on the vector space structure of E. In § 3 I shall define a second class $\mathfrak{S}$, whose definition extends trivially to the context of at least one of the state-space manifolds of interest in General Relativity, as I shall show in § 7. The spaces $\mathfrak{S}$ and $\mathfrak{C}$ can be related, so that $\mathfrak{C}$ is still available for explicit computation.

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[1] If, as usually happens, E is a finite dimensional inner product space then there is a natural δ-function and the need for μ can be overlooked. Note that one cannot rewrite the formula as $\int_{\Phi_1}\chi(q)\,\mathrm{d}\nu(q)$ with χ the characteristic function of $\{q : q(t)=x\}$ ([4], p. 69) since this set has measure zero in Φ_1

[2] Unfortunately most of the work was done before the appearance of [5]

In § 5 the consequences of the extension of E to infinite dimensions are investigated. The discussion here is not completely rigorous, though there seems every reason to believe that the existence-assumptions made will hold in most cases. With this proviso I show that a path integral can be split into an integral over paths with both endpoints fixed, followed by an integral over the final endpoint using the measure Ψ of (1); and that Ψ satisfies a Schrödinger-like equation. These properties form a generalisation of similar conclusions in [5].

(b) The Structure of State-Space

Further development needs more information about E, a space to be determined by the nature of general relativity and the experience of quantum field theory. The analogue of "a state at time τ" would be "a 3-geometry on a slice S", so that E should be a collection of 3-geometries and Φ should be a space of stacks of 3-geometries. I choose to by-pass E completely at this stage by taking Φ to be a set of 4-geometries compatible with a given initial 3-metric [§ 6(a)]. To see what is involved, in § 6(c) I investigate the analogous construction for quantising scalar fields on $\mathbb{R}^4$. Here the relationship with conventional quantum field theory still needs further clarification: the Green's function which appears naturally in the present theory, though related to the Feynman Green's function, is not identical to it.

Quantum field theory suggests that the state space used in § 6(c) (functions on a slab of $\mathbb{R}^4$ which decrease rapidly in spatial directions) is not large enough to encompass many-particle states. This is discussed in § 4, where we are led to the dual of this space. Consequently a dual formulation is given in parallel with the standard one, the appropriate test-function space being set up in § 4, and the corresponding relativistic state space being defined in § 6(b). The difficulties of continuing with the dual formulation become too great at this stage, and subsequent sections deal only with the standard version.

The important property of the manifold of geometries $\mathscr{G}$ defined here is that it has a Riemannian metric (subject to a weak restriction on the geometries). The proof of this is given in Appendix A.

(c) The Manifold Problem

There are two difficulties in passing from vector spaces to manifolds. The first is that of defining the overall context, viz. the appropriate test-function space. This is solved in § 7, Theorem 3, by showing that there is an atlas on $\mathscr{G}$ whose coordinate transformations preserve the main defining property of $\mathfrak{S}$. Thus only a small modification of $\mathfrak{S}$ is necessary to achieve a space $\mathbb{S}$ of test functions on $\mathscr{G}$. (Some technical details are deferred to Appendix B.)

The second difficulty, which is not overcome, is that of actually defining v in any coordinate patch. (If this could be done, since the atlas constructed above admits appropriate partitions of unity, a v could be defined everywhere.) Up to now, v has been constructed using only the quadratic part of the classical action S. (The linear part can always be taken care of by a translation in Φ.) In [5] DeWitt-Morette has investigated the inclusion of higher-order terms by performing an expansion. However, in the case of interest here it is nor clear that this procedure fulfils the intuitive requirement for v; namely, that near *any* point v should approximate to the pseudomeasure derived from the quadratic and linear terms in the expansion of S about that point. In § 7 I give an alternative prescription which does not use an expansion; but no proofs are available either for the consistency of the method or for establishing rigorously that it has the required property. If such proofs could be provided, then one would have a consistent scheme for the application of path integrals in General Relativity.

2. The Path Formalism

We begin by taking the state-space E to be a normed real vector space. (Thus E might be $\mathbb{R}^3$ for a free particle, or a Banach space of fields on $\mathbb{R}^3$ for field theory.) Next, following DeWitt-Morette, we shall define a Hilbert space X, with $\Phi \subset X \Phi'$, on which the constructions will take a particularly simple form. X depends only on the nature of the state-space being considered, not on the action of the system. This action enters by defining a map $P: X \to X$—the "primitive mapping" [4, 12]—which *induces the "correct" v.* Φ_1 *is then defined as the (closure of the) image of* P.

Explicitly, set $X = L_E(T)$—the space of equivalence classes of) square-integrable functions $[0, t] = T \to E$ ([6], p. 586). Note that usually P is essentially the operation of integration $\left(\text{i.e. in the simplest case } P(\phi)(\tau) = \int_0^\tau \phi(\tau')d\tau', 0 \leqq \tau \leqq t\right)$ so that Φ_1 is the space of functions of square-integrable first derivatives [14, 20].

To define P, we first suppose that the space Φ of C^∞ paths is given the nuclear topology, so that the triple $\Phi \subset X \subset \Phi'$ forms a rigged Hilbert space [9]. The classical action S is then taken to be a function on Φ. Choose a point $q_0 \in \Phi$ at which S has a turning point (q_0 being the classical path).

We assume that S is twice continously differentiable (at least in the sense of [11], which is equivalent to Gâteaux differentiability for nuclear spaces; and in practice S will be twice differentiable in the usual sense with respect to one of the norms on Φ). Thus S can be expanded about q_0 as

$$S(q) = S(q_0) + \tfrac{1}{2} S''_{q_0}(q - q_0, q - q_0) + \Sigma .$$

For the systems of physical interest, S is defined on differentiable maps $T \to E$ by the integral of a Lagrangian along the path, and $S''_{q_0}(q, r) = \int_0^t q(\tau) D r(\tau) d\tau$ for a positive second order differential operator D, provided that Φ is defined by suitable boundary conditions.

In any case, twice-differentiability implies that $S''_{q_0}: \Phi \to \Phi'$ exists: in addition to this I shall make the assumption that it is $1-1$ and onto, having a continuous inverse $G: \Phi' \to \Phi \subset X$. *(This means that I exclude the occurrence of caustics* [3].*) It is* also convenient, but not essential, to be able to represent G by a Green's function for D, $g(\upsilon, \tau)$ via

$$G(\phi')(\upsilon) = \int_0^t g(\upsilon, \tau) \phi'(\tau) d\tau . \tag{3}$$

[3] In [5] it is shown that the formalism includes caustics naturally

The positivity of D allows us to define the operator $G^{1/2}: \Phi' \to X$ (by making a self-adjoint extension of D: specific representations can be obtained from the generalised spectral resolution [8]), which, in the cases where g exists, can be written in terms of a "Green's function" $g^{(1/2)}$ for the operator $D^{1/2}$, $g^{(1/2)}$ satisfying

$$\int_0^t g^{(1/2)}(\tau, \upsilon) g^{(1/2)}(\upsilon, \sigma) \mathrm{d}\upsilon = g(\tau, \sigma).$$

Finally P is defined by setting its adjoint $\tilde{P}: \Phi' \supset \Phi'_1 \to X$ equal to $G^{1/2}$. To summarise, we now have

$$\begin{array}{ccccc} & \xrightarrow{D^{1/2}} & & \xrightarrow{D^{1/2}} & \\ \Phi \subset \Phi_1 & \underset{i}{\hookrightarrow} & X & \subset & \Phi'_1 \subset \Phi' \\ & \xleftarrow[P]{} & & \xleftarrow[\tilde{P} = G^{1/2}]{} & \end{array}$$

3. Pseudomeasures (Standard Formulation)

We now proceed to define the space $\mathfrak{S}$ of test functions on Φ_1, choosing it (Theorem 1) so that the corresponding space of distributions consists of the inverse Fourier transforms of all continuous functions on Φ'_1 (in a sense defined by the Theorem). We can then immediately define the pseudomeasure (distribution) ν via its Fourier transform. To compare this with the standard definition in terms of cylinder functions $\mathfrak{C}$ we embed $\mathfrak{S}$ in the closure of $\mathfrak{C}$; it can then be shown (Theorem 2) that our ν is identified in a natural way with the pseudo-measure of DeWitt-Morette that is derived via P from the Gaussian promeasure on X.

We now further assume that Φ_1 is Hilbert and that the topology given to Φ to form Φ_1 is such that the inclusion $i: \Phi_1 \to X$ is a Hilbert-Schmidt map. (In the case just considered, this will hold—for suitable D—when the topology on Φ_1 is that induced by P from X.) Next, a space of test functions on X is constructed. If $f: X \to \mathbb{C}$ is a C^∞ function, define $\|f\|_k = \sup_{x \in X} \|x\|^{k+1} \|D^{k+1} f(x)\|$, using the obvious Hilbert-space norms. Then $\mathfrak{S}$ is the Banach space of C^∞ functions defined by the composite norm

$$\|f\| = \max_k ((k-2)!)^{-3/2} e^{-k/2} \|f\|_k.$$

Thus $\mathfrak{S}$ consists of "rapidly decreasing functions" with some control over the rate of increase of the $\|f\|_k$; a control which is not so tight as to exclude functions of bounded support such as

$$f(x) = \begin{cases} \exp(1/(\|x\|^2 - 1)) & \|x\| < 1 \\ 0 & \|x\| > 1. \end{cases}$$

A pseudomeasure on Φ_1 is then defined to be a member of $\mathfrak{S}'$; i.e. it is a distribution for the test-function space $\mathfrak{S}$[4].

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[4] In defining $\mathfrak{S}$ the functions are to be restricted to Φ_1. The definition is given on X because this space is both more fundamental and computationally easier to handle

We have chosen $\mathfrak{S}$ so as to be able to state the following Theorem. Its statement would remain true if the definition of $\mathfrak{S}$ was slightly modified; one could equally well use the infinite-dimensional generalization of the ultradistributions of Beurling and Björck, for example [21].

Theorem 1. *If i is Hilbert-Schmidt, then for any continuous function k on Φ'_1 there is a pseudomeasure v_k on Φ_1 whose value on each $f \in \mathfrak{S}$ is given by*

$$(v_k, f)_{\Phi_1} = (\mu_f, k)_{\Phi'_1}$$

where μ_f is a measure on Φ'_1 whose Fourier transform $\mathscr{F}\mu_f$ is f.

[Note that if we *define* "the Fourier transform $\mathscr{F}v_k$" to be k then we can write

$$(v_k, f)_{\Phi_1} = (\mathscr{F}^{-1}f, \mathscr{F}v_k)$$

in analogy with the situation for finite-dimensional distributions. In other words, one can specify a pseudomeasure v_k by its Fourier transform.]

The proof follows by simple verification of the continuity of v_k once one can construct μ_f. To do this, consider a subspace V of finite dimension n in Φ_1 and set $f_V = f|V$. Then define a measure μ_V on the dual space V' by $d\mu_V(x') = \tilde{f}_V d^n x'$, $\tilde{f}_V(x') = \frac{1}{(2\pi)^n}\int f_V(x)e^{-i(x',x)}d^n x$ so that the Fourier transform of μ_V is f_V, the coordinates being orthonormal with respect to the Hilbert structure induced from Φ_1. Following [9], p. 349 we can see that the family $\{\mu_V | V \subset \Phi_1\}$ constitutes a (complex-valued) cylinder set measure on Φ'_1 (or promeasure, in the terminology of Bourbaki [2]). The measures μ_V can be estimated by the inequality

$$\int_{\|x'\| > R} |\tilde{f}_V| d^n x' \leqq (4/(1+R)(n-2)!) \, \|f\|_n^1$$

(where $\|f\|_n^1$ is the norm $\| \ \|_n$ introduced above, but evaluated with the Hilbert metric on Φ_1); while the Hilbert-Schmidt character of i gives $\|f\|_n^1 \leqq Kn^{-n/2}\|f\|_n$. This estimate therefore shows both that $\{\mu_V\}$ is a measure (using the criterion of [9], p. 318) and gives the required continuity in f.

Corollary. *Suppose the conditions of the theorem hold, and that G is defined and continuous on the whole of Φ'_1. Then there is pseudomeasure $v = v_{S,q_0}$ on Φ_1 whose Fourier transform is*

$$(\mathscr{F}v)(\phi') = \exp\left\{-\frac{i}{2}(\phi', G(\phi'))\right\}.$$

The "variance" $(\phi', G(\phi'))$ is given by $\int \phi'(\upsilon)\phi'(\tau)g(\upsilon,\tau)d\upsilon d\tau$, if we use the notation $\int \phi'(\upsilon)\chi(\upsilon)d\upsilon$ for (ϕ', χ) $(\upsilon, \tau \in T)$.

An alternative space of test-function is the set $\mathfrak{C}$ of C^∞ cylinder functions on Φ_1 with compact support on their base—that is, functions of the form $f_V \circ \pi_V$ where $\pi_V : \Phi_1 \to \Phi_1/V$ is the canonical projection onto cosets of a subspace V of finite codimension and $f_V : \phi_1/V \to \mathbb{C}$ is C^∞ with compact support. This is the implicit choice of DeWitt-Morette. The use of cylinder functions is computationally simpler, but cannot be carried over to manifolds; hence the higher priority given to $\mathfrak{S}$. On the other hand, the two approaches—distributions for $\mathfrak{S}$ and cylinder-set measures for $\mathfrak{C}$—are easily related as follows.

Regard $\mathfrak{C}$ as a subspace of $C^\infty(\Phi_1)$ with the topology of pointwise convergence in all derivatives. If $\mathfrak{S}$ is any test function space with $\mathfrak{S} \subset \bar{\mathfrak{C}} \subset C^\infty(\Phi_1)$, and if $\mu = \{\mu_V\}$ is a (not necessarily bounded, complex valued) promeasure ([2] = cylinder set measure [9]), then we shall say that μ defines (or "is") a pseudomeasure ν if $\mathfrak{S}$ lies in the domain of the closure of the map $\boldsymbol{\mu}: \mathfrak{C} \ni f_V \circ \pi_V \mapsto \int_V f_V \, d\mu_V(x) \in \mathbb{C}$ and $\bar{\boldsymbol{\mu}} | \mathfrak{S} = \nu$.

This enables us to state the following:

Theorem 2. *The pseudo-measure ν_{S,q_0} is the image under P of the Gaussian promeasure on X with variance $-(i/2)\,\|x'\|^2$.*

The proof is mainly technical, on the same lines as that of Theorem 1.

The representation in terms of the space $\mathfrak{C}$ of cylinder functions allows us to give one of the main computational formulae of the subject, which follows immediately from Theorem 2 (DeWitt-Morette [4], Equation (1), p. 68)

$$(f_V \circ P_n, \nu_{S,q_0}) = \int_{R_n} f_V(u)(2\pi i)^{-n/2}(\det \mathscr{W})^{-1/2} \cdot \exp\left(\frac{i}{2} u^i u^j (\mathscr{W}^{-1})_{ij}\right) du \ldots du^n \tag{4}$$

where

$$\mathscr{W}^{ij} = (\tilde{P}_n(e^i), G(\tilde{P}_n(e^j))$$

with $e_1 \ldots e_n$ the standard basis of $\mathbb{R}^n$.

4. Pseudomeasures (Dual Formulation)

If one wishes to apply the forgoing to quatum field theory, it would seem reasonable to take E to be the space of, say, L^2 fields on $\mathbb{R}^3$ and proceed in the way described by, for instance, Abers and Lee [1]. In this case, however, one has to resort to devices relying on results from the conventional formalism to convert the one-particle Green's function that is at first obtained into the n-particle function. This was to be expected, since this choice of E is not the full n-particle state-space. As pointed out by Isham in the context of quantum gravity [10], a possible candidate for E would be the *dual* space of rapidly-decreasing functions $\mathscr{S}$ on $\mathbb{R}^3$. (See, for instance, [13] for a full discussion of the use of this state-space.)

Thus we should recast the path integral approach in terms of path integrals over Φ', the dual space of a nuclear path space Φ of C^∞ functions $T \to \mathscr{S}$.

Fortunately, this poses no problem: indeed the resulting theory is, at this vector-space stage (but not at the manifold stage), even simpler than before.

The test function space will now be the space $\mathfrak{C}_2$ of continuous functions $f: \Phi' \to \mathbb{C}$ whose support $\operatorname{Supp} f$ is such that for each cofinite subspace V, $\pi_V(\operatorname{Supp} f)$ is compact. In fact, we then have a simpler characterisation: π_V can always be realised as a map

$$\Phi' \ni x' \mapsto ((x', \phi_1), (x', \phi_2), \ldots, (x', \phi_n)) \in \mathbb{R}^n$$

for some fixed $\{\phi_1, \ldots, \phi_n\}$, vectors in ϕ, and so the requirement is equivalent to the weak boundedness of Supp$f$. But in the dual of a nuclear space this implies compactness ([9], p. 73) and so $\mathfrak{C}_2$ is just the continuous functions of compact support. While this is neat, it must be considered a defect of the present theory since this is a very restricted class of test functions.

We can relate promeasures to pseudomeasures in $\mathfrak{C}_2'$ (as in Theorem 2) with the help of the following lemma:

Lemma. *If K is compact, then each (unbounded) promeasure μ on ϕ' determines a measure $\mu^{(K)}$ with support on K.*

Proof. For cofinite V define $\mu_V' = \chi_{\pi_V(K)}\mu_V$, a measure on Φ'/V. Then the collection $\{\mu_V'\}$ satisfies $\mu_V^W := p_{VW}^* \mu_W' \leqq \mu_V'$, for any cofinite W with $W \subset V$, p_{VW} being the natural projection $\Phi'/W \to \Phi'/V$. The set of cofinite W in V forms a directed set, and the measure

$$\mu_V^{(K)} := \lim_W \mu_V^W$$

is well-defined (possibly zero). It is now possible to verify that the collection $\{\mu_V^{(K)}\}$, is a bounded promeasure provided that $|\mu_V(\Phi'/V)| > 0$ for some V (Appendix C). But since Φ' is the dual of a nuclear space, this promeasure coincides with some measure $\mu^{(K)}$ ([9], p. 320).

Now let μ_{S,q_0} be the (unbounded) promeasure whose Fourier transform is the Gaussian of variance $\exp\{-(i/2)(G(x), x)\}$, where G is defined by restriction to $\Phi \subset X \subset \Phi'$, and the inner product is induced from X. Define

$$(v, f) = (\mu_{S,q_0}^{(K)}, f)$$

for Supp$f \subset K$, compact. Then (v, f) is continuous in f (under the topology of uniform convergence in a fixed K), and so v is a pseudomeasure for the test-function space $\mathfrak{C}_2$.

5. Two Fixed Endpoints and Schrödinger's Equation

(a) Schrödinger's Equation

Having established this formalism, we now return to (1) and examine the object Ψ thereby defined. If we take v to be a promeasure on $\mathfrak{C}$ and set $\pi^e(q) := q(t)$ (the projection of Φ_1 onto the path endpoints), then we see that Ψ is a *promeasure on E* given by

$$(\Psi, f) = (v, f \circ \pi^e) \qquad (f \text{ a cylinder function on } E). \tag{5}$$

Whereas Ψ always exists, the wave-function ψ exists only if we can form a sort of Radon-Nikodym derivative $d\Psi/d\mu = \psi$ with respect to a natural μ. This has the best chance of working in the dual theory where such a μ is introduced from a consideration of the canonical commutation relations ([13], p. 22). From now on I shall deal only with Ψ.

To define G uniquely we need a second boundary condition, and, since the formalism requires the final endpoint to be free to range over E, we impose the condition $\dot{q}(t)=0$; i.e. G is defined by imposing this restriction on Φ, and Φ_1 is then the image of the resulting $G^{1/2}$. We shall see shortly how this relates to the usual prescription of fixing the second endpoint. (Note that an alternative procedure would be to modify the action S by the inclusion of endpoint terms, a course which is natural in General Relativity where one includes terms involving the second fundamental form of the final space-like slice.)

With this choice of G, and assuming the existence of its representation (3) by a Green's function g, we can use (4) to write (5) as

$$(\Psi, f_V \circ P_n) = (v, f_V \circ (P_n \circ \pi^e))$$
$$= \int_{\mathbb{R}^n} f_V(u)(2\pi i)^{-n/2}(\det \mathscr{W}')^{-1/2} \exp\left(\frac{i}{2} u^i u^j (\mathscr{W}'^{-1})_{ij}\right) \mathrm{d}^n u \qquad (6)$$

where $\mathscr{W}'^{ij} = (\tilde{P}_n(e^i), g^t(t,t)(\tilde{P}_n(e^j)))$, writing g as g^t to indicate its dependence on the parameter value t at which the second boundary condition is imposed.

We can now derive a Fourier-transformed Schrödinger equation for the Fourier transform of Ψ, which, from (6) is

$$\mathscr{F} := \mathscr{F}\Psi = \exp\left[-\frac{i}{2}(\,.\,, g^t(t,t)\,.\,)\right].$$

If $D = A\mathrm{d}^2/\mathrm{d}t^2 + B\mathrm{d}/\mathrm{d}t + C$, *where* A, B, and C are continuous invertible linear maps $E \to E$ (Hilbert) then we can verify (putting now g for $g^t(t,t)$) that

$$\mathrm{d}g/\mathrm{d}t = (-1 + g(B + \dot{A}))A^{-1} - gCg$$

giving

$$2i\mathrm{d}\mathscr{F}/\mathrm{d}t = -(x', A^{-1}x') - iD\mathscr{F}_{x'}((B + \dot{A})A^{-1}(x'))$$
$$+ \mathrm{Tr}(D^2\mathscr{F}_{x'} \circ C) + i(\mathrm{Tr}\, g)\mathscr{F}(x')$$

providing the right hand side exists.

If E is finite dimensional then we can put $\mathrm{d}\Psi(x) = :\psi(x)\mathrm{d}^n x$ and Fourier invert to get

$$i\mathrm{d}\psi/\mathrm{d}t = \tfrac{1}{2}\{\mathrm{Tr}(D^2\psi \circ A^{-1}) - i(D\psi(B + \dot{A})A^{-1}x)$$
$$-(x, Cx)\psi + i(\mathrm{Tr}\, g)\psi\}. \qquad (7)$$

Here the first and third terms correspond to the usual terms in the Hamiltonian, for the case of a particle in a quadratic potential, and the fourth is a normalisation term that is required because Ψ is normalised by $\int \mathrm{d}\Psi(x') = 1$ (in so far as it is defined) and not $\int \bar{\psi}\psi \mathrm{d}^n x = 1$.

(b) Two Fixed Endpoints

I shall now argue that an integral over v can in a sense be split into an integral over a set of paths with both endpoints fixed, followed by an integral over the final endpoint. We continue with the same boundary condition for G as in (a) above.

Let $P^1 : \Phi_1 \to \mathbb{R}^{n_1}$ be a projection, as in §3, and let $P^2 = P^{2'} \circ \pi^e : \Phi_1 \to E \to \mathbb{R}^{n_2}$ be a projection of the sort considered in (a), depending only on the final endpoint. We consider a composite function $f(x)=f^1(x)\,f^2(x)$, where $f^1=f^1_V\circ P^1$ depends on the whole path, but $f^2=f^2_V\circ P^2$ depends only on the endpoint. Thus f is a cylinder function of $n=n_1+n_2$ dimensions, associated with the projection $P_n=P^1\times P^2:\Phi_1\to\mathbb{R}^n$.

If we now apply (4), we can partition $\mathbb{R}^n$ into $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, with $u=(u_1,u_2)$ and $\mathcal{W}$ becoming[5]

$$\mathcal{W}=\begin{pmatrix}\mathcal{W}^{11} & \mathcal{W}^{12}\\ \mathcal{W}^{21} & \mathcal{W}^{22}\end{pmatrix}$$

so that

$$\mathcal{W}^{-1}=\begin{pmatrix}\mathcal{V}^{11} & \mathcal{V}^{12}\\ \mathcal{V}^{21} & \mathcal{V}^{22}\end{pmatrix}$$

with

$$(\mathcal{V}^{11})^{-1}=\mathcal{W}^{11}-\mathcal{W}^{12}(\mathcal{W}^{22})^{-1}\mathcal{W}^{21}$$
$$(\mathcal{V}^{22})^{-1}=\mathcal{W}^{22}-\mathcal{W}^{21}(\mathcal{W}^{11})^{-1}\mathcal{W}^{12}$$
$$\mathcal{V}^{12}=-(\mathcal{W}^{11})^{-1}\mathcal{W}^{12}\mathcal{V}^{22}=-\mathcal{V}^{1}\mathcal{W}^{12}(\mathcal{W}^{22})^{-1}$$
$$\mathcal{V}^{21}=-(\mathcal{W}^{22})^{-1}\mathcal{W}^{21}\mathcal{V}^{11}=-\mathcal{V}^{22}\mathcal{W}^{21}(\mathcal{W}^{11})^{-1}$$

(assuming the invertibility of $\mathcal{W}^{11}$ and $\mathcal{W}^{22}$).

To see the significance of this decomposition introduce the Green's function $\tilde{g}$ corresponding to the boundary condition $q(t)=0$, instead of $\dot{q}(t)=0$. We can verify that $\tilde{g}$ is given by

$$\tilde{g}(v,\tau)=g(v,\tau)-g(v,t)\,(g(t,t))^{-1}\,g(t,\tau) \tag{8}$$

(assuming the necessary invertibility and differentiability). Let $\tilde{\mathcal{W}}$ be $\mathcal{W}$ defined on $\mathbb{R}^{n_2}$ but using $\tilde{g}$ instead of g, and define $\tilde{v}$, the Feynman promeasure with fixed endpoints, by (4) with $\mathcal{W}$ replaced by $\tilde{\mathcal{W}}$.

Then I claim the following

Nearly-Theorem

$$(v,f^1f^2)=(\Psi,\tilde{v}[f^1,.\,]\,f^{2\prime}) \tag{9}$$

where

$$\tilde{v}[f^1,x]:=(\tilde{v},f^1\circ T_x),$$
$$T_x(q)\,(\tau):=q(\tau)-g(\tau,t)\,(g(t,t)^{-1})\,(x)$$

and

$$f^{2\prime}:=f^2_V\circ P^{2\prime}.$$

[One could write (9) more suggestively as

$$\int f^1(q)\,f^{2\prime}(q(t))\,\mathrm{d}v(q)=\int \mathrm{d}\Psi(x)\,f^{2\prime}(x)\int f^1(q-T_x(q))\,\mathrm{d}\tilde{v}(q) \tag{10}$$

if one allows the integral notation.]

[5] These techniques have also been developed, more generally, in [5]

Not-Quite-Proof. In the following I shall assume that various limits can be taken without yielding surprising answers. In particular, we shall increase the dimension n_2 to N, so that now $f^2 : \Phi_1 \to E \to \mathbb{R}^N \underset{p}{\to} \mathbb{R}^{n_2}$, and let $N \to \infty$.

The projection $E \to \mathbb{R}^N$ is given by $x \mapsto (\phi^1(x), \ldots, \phi^N(x))$ for $\phi^1, \ldots, \phi^N \in E'$, so that we can form the map

$$\Gamma = \phi^i \mathscr{W}^{22-1}{}_{ij} \phi^j : E \to E',$$

where $\mathscr{W}^{22ij} = \phi^i g(t,t) \phi^j$.

When Γ occurs inside integral expressions I shall assume that, in the limit as $N \to \infty$, Γ can be replaced by $(g(t,t))^{-1}$.

Now, the expression for Ψ in Equation (6), together with the result $\det \mathscr{W} = \det \mathscr{W}^{22} / \det \mathscr{W}^{11}$, allows us to write (4), using an integral notation, as

$$(v, f^1 f^2) = \int_E d\Psi(x) f^{2'}(x) \int_{\mathbb{R}^N} (f^1 \circ p)(v - (\mathscr{V}^{11})^{-1} \mathscr{V}^{12} u_2(x)) \cdot \exp\left(\frac{i}{2} v^i v^j \mathscr{V}^{11}{}_{ij}\right) \cdot (\det \mathscr{V}^{11} / (2\pi^i)^{n_1})^{1/2} d^{n_1} v. \tag{11}$$

Now

$$\begin{aligned} ((\mathscr{V}^{11})^{-1} \mathscr{V}^{12} u_2(x))^i &= -(\mathscr{W}^{12} (\mathscr{W}^{22})^{-1} u_2(x))^i \\ &= -\phi^i g(.,t) \Gamma(x) \\ &\to -\phi^i g(.,t) (g(t,t))^{-1}(x) \end{aligned}$$

and a similar calculation yields, from (8),

$$(\mathscr{V}^{11})^{-1ij} \to \phi^i \tilde{g} \phi^j = \tilde{\mathscr{W}}^{ij}$$

as $N \to \infty$. Inserting these limits in (11) gives the required result (10).

6. State-Manifolds[6]

In general relativity one replaces the set of all paths $I \to M$ by the set of all 4-geometries compatible with some initial and final conditions. There is considerable freedom to choose topologies for such a manifold, and one must be guided by the procedures already tested for vector spaces. In the paper [10] already referred to, Isham has convincingly argued that one should use a nuclear topology for the underlying space, as this achieves the simplest (though not the only) realisation of the canonical commutation relations. Moreover, by working with a nuclear topology nothing is lost that cannot subsequently be regained by completion to a Banach manifold.

(a) Standard Formulation

The underlying manifold for space-time will be $M = T \times \mathbb{R}^3$, $T = [0, t]$. M being a manifold-with-boundary, we shall make the usual conventions that "C^k" on M means "extensible to a neighbourhood of M in $\mathbb{R}^4$ and C^k in that neighbourhood".

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6 See [7] for the basic ideas and notation

The set $\mathscr{M}_a$ is then defined to consist of all Lorentz metrics that are C^∞ on M and induce a fixed positive definite 3-metric a on $\{0\} \times \mathbb{R}^3 =: S_0$.

Next we subdivide $\mathscr{M}_a$ into equivalence classes of metrics with the same assymptotic behaviour. This is done automatically by placing a nuclear topology on the metrics which regards two metrics as "nearby" if their difference falls off rapidly at infinity, at all orders of differentiation: this topology causes $\mathscr{M}$ to fall into disconnected components, each component being characterised by a certain assymptotic behaviour.

To do this explicitly we can regard $\mathscr{M}_a$ as a subset of the vector space $\mathscr{N}$ of all symmetric C^∞ tensors of rank two on M, in which we can distinguish the subspace $\mathscr{N}_0$ of tensors for which the inner products

$$(g^1, g^2)_n = \int_M (1 + \|x\|)^n \sum_{k=0}^{n} \sum_{p,q,i_1,\ldots,i_k=0}^{3} g^1_{pq,i_1\ldots i_k} g^2_{pq,i_1\ldots i_k} \mathrm{d}^4 x$$

exist. We topologise $\mathscr{N}$ by taking all the open balls of $\mathscr{N}_0$ in these inner products, and their translates, as a basis, noting that this does *not* make $\mathscr{N}$ a *TVS*; $\mathscr{N}_0$ is the connected component of the identity in $\mathscr{N}$.

We are interested not in metrics, but in *geometries*, by which I mean an equivalence class of isometric metrics. Thus we must take the quotient of $\mathscr{M}_a$ by a class of diffeomorphism (noting that this will identify some metrics that are in different components of $\mathscr{M}_a$: the diffeomorphism group may not act continuously).

If $f: M \to M$ is a C^∞ diffeomorphism that is identity on S_0 such that f^* maps $\mathscr{M}_a$ into itself and is continuous on $\mathscr{N}_0$, then it is continuous on $\mathscr{M}_a$ (with the relative topology induced from $\mathscr{N}$). The set of all such diffeomorphisms is a group $\mathscr{D}$ and we write $\mathscr{G}_a = \mathscr{M}_a/\mathscr{D}$.

$\mathscr{G}_a$ is a nuclear manifold. To see this, it is sufficient to decompose $\mathscr{M}_a$ locally, in a neighbourhood of some g, into $V \times \mathscr{D}$, $V \subset \mathscr{G}_a$, by fixing gauge conditions. Explicitly, given an extension of g to a neighbourhood of M, one can show[7] that there is a neighbourhood U of g in $\mathscr{M}_a$ such that for each g' in U there is a unique transformation $f: M \to \mathbb{R}^+ \times \mathbb{R}^3$ so that $g'' = f^* g'$ satisfies $g''_{0i} = g_{0i}$ $(i = 0, 1, 2, 3)$ everywhere on $f(M)$, and $g''_{ij} = g_{ij}$, $f|S_0 =$ identity on $S_0 := \{0\} \times \mathbb{R}^3$. We can then compose this with a specified C^∞ map which "levels up" the surface $f(\{t\} \times \mathbb{R}^3)$ to coincide with $\{t\} \times \mathbb{R}^3$, thus producing a map $\mathscr{M}_a \to \mathscr{M}_a$. By its construction, the map can be shown to be in $\mathscr{D}$, thus giving the required decomposition. The proof of the assertion then becomes a routine verification.

$\mathscr{G}_a$ is *not* connected: its components represent classes of geometries with inequivalent assymptotic behaviours.

We can also restrict to the class $\mathscr{M}^b_a$ of metrics which also induce a fixed metric b on $\{t\} \times \mathbb{R}^3$, and define $\mathscr{G}^b_a$ as the set of classes of points in $\mathscr{M}^b_a$; this will be a nuclear submanifold of $\mathscr{G}_a$.

Finally, *we shall assume* that $\mathscr{G}_a$ has a naturally defined linear connection for some class including those metrics of physical interest. This is proved in the appendix for a class of spaces in which all the geodesics eventually escape to infinity.

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[7] Appendix B

(b) Dual Formulation

$\mathscr{G}_a^*$, the "dual manifold" to $\mathscr{G}_a$, will be constructed so that its tangent space at a point is isomorphic to the dual of a tangent space to $\mathscr{G}_a$, the isomorphism being natural for a given choice of gauge. This means that we *cannot* use the manifold $\mathscr{M}^*/\mathscr{D}$ (quotient by the obvious dual action of $\mathscr{D}$) which is too big: the gauge freedom is dualised in $\mathscr{M}^*$, and only part of this dualised freedom is then taken out by $\mathscr{D}$. Instead we use a direct consideration of the gauge conditions.

Define a *gauge surface* G to be a closed linear subspace of $\mathscr{M}$ that is everywhere transverse to the orbits of $\mathscr{D}$. [Such G's exist, by the arguments of (a) above.] Then let $\mathfrak{G}$ be the set of all gauge surfaces. We note that if $\phi\in\mathscr{D}$ then $G'=\phi^*G\in\mathfrak{G}$, and dualisation gives a map $\phi_*: G'^*\to G^*$. We regard this as an action of $\mathscr{D}$ on the disjoint union of all the G^*, finally defining $\mathscr{G}_a^*$ as the quotient by this action:

$$\mathscr{G}_a^* = \Big(\bigcup_{G\in\mathfrak{G}} G^*\Big)\Big/\mathscr{D}\,.$$

This space then acquires the structure of a Hausdorff manifold, modelled on a typical G^*.

(c) (3 + 1)- *and* 4-*Dimensional Formalisms*

We have seen in (a) that a neighbourhood in $\mathscr{G}_a$ can be described in terms of space-times having certain coordinate systems assigned in them, which exhibit a particular homeomorphism with $T\times\mathbb{R}^3$. Using this, we see that the tangent space to a geometry $\hat{g}\in\mathscr{G}_a$ can also be represented in terms of (perturbation) fields on $T\times\mathbb{R}^3$, subject to a gauge condition. Then the application of a gauge transformation can make the gauge condition independent of $\tau\in[0,t]=T$, so that we are left with a representation which can be regarded as a map $\tau\to$(field on $\mathbb{R}^3$), i.e. a *path* with state space the fields on $\mathbb{R}^3$ which satisfy the gauge condition.

It is appropriate to note here that the manifold approach allows one to deal with geometries which cannot themselves be represented by, for instance, normal coordinates, even though the perturbations from them can be represented in terms of the spatial metric alone: there is no one gauge condition which enables us to cover the whole of $\mathscr{G}_a$ with one "coordinate patch" and hence to regard it as a vector space. But this global consideration—which might be thought but a minor complication—is not the main motivation for stressing manifolds, as opposed to vector spaces. A manifold technique is essential for piecing together the linear-space pseudo-measures which are approximately valid only in a very small neighbourhood of their base-point (§ 7).

To give a clearer idea of the relationship between the four-dimensional geometry defined in (a) and the path-formalism of the previous section, it is useful to compare the situation with the case of scalar fields on Minkowski space. Here $E=L^2(\mathbb{R}^3)$ and S is defined on a subset of the paths from $T=[0,t]\to E$ (viz. the L_1^2 closure of the differentiable ones) by

$$\begin{aligned} S(\phi) &= \int_0^t d\tau\left[\int (m^2\phi^2+(\nabla\phi)^2-\dot{\phi}^2)\,d^3\boldsymbol{x}\right] \\ &= \int_0^t d\tau\langle\phi, D\phi\rangle \end{aligned}$$

where $\phi \equiv \phi(\tau; \boldsymbol{x})$ may be regarded as the path $\tau \to \phi(\tau; .)$ [or, more conventionally, as a field $(\tau, \boldsymbol{x}) \to \phi(\tau; \boldsymbol{x})$], $\langle , \rangle$ denotes the inner product $\int(\ )d^3\boldsymbol{x}$ in E and

$$D = m^2 - \nabla^2 + \partial^2/\partial t^2 .$$

Consequently the operator G can be represented by a Green's function $g(\tau, \upsilon)$, this being, for each τ and υ, an unbounded operator defined via the Fourier transform $\hat{}$ in R^3 by

$$g(\tau,\upsilon)(\psi)\hat{}(\boldsymbol{k}) = \begin{cases} -\dfrac{\cos k^0(\upsilon - t)\sin k^0\tau}{(2\pi)^3 k^0 \cos k^0 t}\hat{\psi}(\boldsymbol{k}) & (\tau < \upsilon) \\ -\dfrac{\sin k^0 \upsilon \cos k^0(\tau - t)}{(2\pi)^3 k^0 \cos k^0 t}\hat{\psi}(\boldsymbol{k}) & (\tau > \upsilon) \end{cases}$$

$(k^{02} = \boldsymbol{k}^2 + m^2)$.

Here the expressions are to be interpreted as defining g as the unbounded operator obtained by closing the operator defined above when ψ is such that the right-hand side is Fourier-invertible [e.g. with ψ restricted to rapidly decreasing functions for which $\hat{\psi}(\boldsymbol{k})$ vanishes at $k^0 = (n + \frac{1}{2})\pi/t$].

The Green's function g should not be confused with the related quantum-mechanical "Green's function". Thus the fixed-point function $\tilde{g}$ does tend to a limit as τ, υ and $t \to \infty$, with $(\tau - \upsilon)$ finite, but the result is a *real* time-symmetric kernel for $m^2 + \Box$ and *not* the Feynman Green's function.

7. The Manifold Problem

The translation of the preceding formalism to the setting of the manifold $\mathscr{G}$ (or $\mathscr{G}^*$) involves two steps. In the first, which is fairly clear in outline, the "test function" space $\mathfrak{S}$ (or $\mathfrak{C}_2$) must be extended to a space of functions on the manifold. In the second step, where there remain several uncertainties, the pseudo-measure must be freed from its dependence on the coordinates and on the base-point q_0.

(a) Test-Functions on Manifolds

In the case of $\mathscr{G}$ the situation is particularly tractable because we can form partitions of unity out of the functions of $\mathfrak{S}$. The construction can be modelled closely on the work of Bonic and Frampton [15], except that $\mathfrak{S}$ does not, of course, correspond to any $\mathscr{S}$-category. The definition of $\mathfrak{S}$ is, however, preserved by composition because we have the following.

Lemma. *Let $f: E \to F$, $g: G \to E$ be C^∞ functions between Banach spaces defined on domains in which $\|D^n g\| \leqq (C_n)^n$, $\|D^n f\| \leqq (C_n')^n$. Then $\|D^n(f \circ g)\| \leqq (C_n'')^n$, where $C_n'' = C_n C_n'(n/2\pi^{1/2})^{1/n}$. (We assume that $C_1 \leqq C_2 \leqq \ldots$ and so on.)*

Proof. Repeated differentiation gives

$$\begin{aligned} &D^n(f \circ g)(x)(h_1, \ldots, h_n) \\ &= \sum_{k=1}^{n} (D^k f)(g(x)) \sum_{P_k^n} (D^{n_1} g(x)(h_{i_1}, \ldots, h_{i_{n_1}}), \ldots, D^{n_k} g(x) \\ &\quad \cdot (h_{i_{n_{k-1}+1}}, \ldots, h_{i_n})) , \end{aligned} \tag{12}$$

the second summation being over the set P_k^n of every arrangement of the integers $1, 2, \ldots, n$ into an unordered set of k disjoint exhaustive nonordered nonempty subsets, of the form $\{\{i_1, i_2, \ldots, i_{n_1}\}, \{i_{n_1+1}, \ldots, i_{n_2}\}, \ldots, \{i_{n_{k-1}+1}, \ldots, i_n\}\}$. Consequently

$$\|D^n(f \circ g)\| \leqq \sum_{k=1}^{n} C_n'^k k! \Big(\sum_{P_k^n} C_n^n n_1!(n_2-n_1)! \ldots (n-n_{k-1})! \Big).$$

But the number $\sum_{P_k^n} n_1! \ldots (n-n_{k-1})!$ is just the number of unordered arrangements into k nonempty disjoint exhaustive *ordered* subsets, $= n!(n-1)!/(n-k)!(k-1)!k!$. Hence

$$\|\mathrm{D}^n(f \circ g)\| \leqq (C_n C_n')^n \sum_{k=1}^{n} n!(n-1)!/(n-k)!(k-1)!$$

$$\leqq \frac{n}{2\pi^{1/2}} (2C_n C_n')^n n! \quad \text{q.e.d.}$$

We can now show (Appendix B) that there is an atlas on $\mathscr{G}_a$ whose transition functions have n'th derivatives that are bounded (pointwise) by $K^n n!$, if we measure derivatives by reference to the L^2, rather than the nuclear, topology on M — as is needed for investigating the class $\mathfrak{S}$ which is defined on X rather than Φ.

If we restrict to functions of bounded support then the dependence on $\|x\|$ in the definition of $\mathfrak{S}$ can be ignored, and we obtain a class which is invariant under certain coordinate changes on $\mathscr{G}_a$, using the lemma to compose functions with coordinate changes. This enables us to apply the method of Bonic and Frampton [15] to deduce:

Theorem. *There is an atlas $\mathscr{A}$ on $\mathscr{G}_a$ such that for every open covering $\mathscr{U}$ of $\mathscr{G}_a$ there is a refinement $\mathscr{U}'$ and a partition of unity subordinate to $\mathscr{U}'$, each of whose functions are in a class $\mathfrak{S}$ for some suitably chosen chart in $\mathscr{A}$.*

This result makes reasonable the following extension of $\mathfrak{S}$ to the manifold $\mathscr{G}_a$: we define $\mathbb{S}$ to be the class of functions $f: \mathscr{G}_a \to \mathbb{C}$ such that $f = f_1 + f_2 + \ldots + f_n$ (i.e. *finite*), with each f_i being in an $\mathfrak{S}$-class for some chart of $\mathscr{A}$. Then $\mathbb{S}$ inherits a topology from $\mathfrak{S}$ in a natural way, and pseudomeasures on $\mathscr{G}_a$ are defined as members of $\mathbb{S}'$.

(b) The Construction of a Pseudomeasure

The essential criterion for the choice of pseudomeasure v on $\mathscr{G}_a$ is that it should, near each point q_0 of $\mathscr{G}_a$, approximate to the vector-space pseudomeasure v_{S,q_0} *constructed in normal coordinates* (with respect to the Riemannian structure on $\mathscr{G}_a$) based on q_0. Where q_0 is not a turning point of S the first derivative is incorporated by translating the pseudomeasure, writing (rather informally)

$$S(q) = \tfrac{1}{2} \mathrm{D}^2 S(q_0)(q-q_1, q-q_1) + \Sigma$$

with

$$\Sigma = \text{const} + o(q-q_0),$$

where

$$q_1 = q_0 - (D^2S(q_0))^{-1}(DS(q_0)),$$

regarding the second derivative as $D^2S(q_0): \Phi_1 \to \Phi_1'$ and $DS(q_0) \in \Phi_1'$. Then one can proceed to construct a pseudomeasure as before, only using the translated coordinate $q - q_1$.

The important feature is that the second derivative is to be evaluated in normal coordinates at the current base point, and not at some fixed turning point, if we are to achieve a true global formulation. This is essential to the philosophy of the path-integral approach, in which the classical path emerges from the formalism as one where the action is stationary: it would be otiose to fix a classical path at the outset as a base-point from which the formalism was to be developed.

Thus the problem, which has yet to be solved, is to construct a pseudomeasure v which is in some sense "tangent" to v_{S,q_0} at *each* q_0—and not just the turning points. The following outline of how to do this is mainly conjectural.

The aim is to write

$$(v, f) = (v_{S,q_0}, f \cdot k_{q_0})$$

for f having support in some neighbourhood of q_0, k_{q_0} being a "compensating function", equal to unity at q_0. In the finite dimensional case we would have

$$k_{q_0}(q) = e^{-i\Sigma/2} \frac{\det D^2S(q_0)}{\det D^2S(q)}.$$

The determinants are not defined in our case [8]; but we could consider the equivalent formulation

$$k_{q_0}(q) = e^{-i\Sigma/2} \exp\left[- \int_0^1 \mathrm{Tr}\{(D^2S(tq + [1-t]q_0))^{-1} \cdot \frac{d}{dt}(D^2S(tq + [1-t]q_0))\} dt \right].$$

There is considerable evidence that this is defined in the case of general relativity. The directional derivative $\frac{d}{dt}(D^2S(tq + [1-t]q_0))$ does exist if we strengthen Φ_1 to incorporate higher derivatives of the metric, and the existence of the trace depends essentially on the behaviour of the high-frequency perturbations to the metric, which are comparatively insensitive to the lower derivatives of the perturbed Einstein operator. While simple cases suggest that the trace exists, no way has yet been found of evaluating D^2S *in normal coordinates* at each point of $\mathscr{G}_a$ (the coordinates being chosen continuously along the path from q_0 to q).

Of course, even if the above expression is well defined, this is no guarantee that the resulting v will be independent of q_0, as it is in the finite-dimensional case.

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8 See [16] for a direct approach to this problem

(c) The Dual Case

It is clear that the class $\mathfrak{C}_2$ is too narrow to allow a similar construction on $\mathscr{G}_a^*$. Here one must look, if possible, for an indirect demonstration of the existence of a v with the required properties of "tangency" to the v_{S,q_0}.

8. Conclusions

I have tried to develop the path-integral approach as a method in its own right, not dependent on results from quantum theory. It seems clear that this development can be carried a long way—but possibly at the cost of diverging from orthodox quantum theory. For instance, it is far from obvious how one could construct a Hilbert space translation of the dual path integral theory. While this presents possibly an insuperable handicap to the application of the theory to fields that are already well studied by traditional methods, it may provide new opportunities in quantum gravity, where the path integral approach seems better matched to the geometrical techniques that relativists are already accustomed to.

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Appendix A: Riemannian Structures on Manifolds of Geometries

It will be helpful to generalise the context slightly to a set $\mathscr{M}$ of metrics defined on an n-dimensional manifold-with-boundary M with the following properties:

(i) either the metrics in $\mathscr{M}$ have positive definite signature, or (for $n=4$ only) they have Lorentz signature;

(ii) for each metric in $\mathscr{M}$ and each $x \in M$ there is a compact $K \ni x$ and a diffeomorphism $M \setminus K \to \mathbb{R}^4$ which is onto either $\mathbb{R}^4 \setminus B$ or $T \times \mathbb{R}^3 \setminus B$ (B a ball in $\mathbb{R}^4$) and such that all geodesics from x leave int K and subsequently have $\mathrm{d}r/\mathrm{d}s$ and their divergence uniformly bounded positively away from zero (r = radial distance in $\mathbb{R}^4$, s = affine parameter);

(iii) the topology on $\mathscr{M}$ is such that the members of $T(\mathscr{M})$ (perturbations) die off at ∞ in an L^1 manner [with respect to the diffeomorphism in (ii) above].

The only condition requiring special comment is (ii), which is a very weak "assymptotic flatness" condition.

Let $\mathscr{D}$ be a group of surjective diffeomorphisms $f: M \to M$ for which $f^*: \mathscr{M} \to \mathscr{M}$ and suppose that the induced orbit through $g \in \mathscr{M}$, for any g, is a closed submanifold $\mathcal{O}_g$ and that the $\mathcal{O}_g$ foliate $\mathscr{M}$. (This assumption thus excludes symmetric and "nearly periodic" metrics from $\mathscr{M}$.) The tangent space $T_g(\mathcal{O}_g)$ to an orbit will, as usual, be a space of tensor fields of the form $\xi_{(k;j)}$ (covariant differentiation with respect to g), and the tangent space at $[g]_{\mathscr{D}}$ to $\mathscr{G} = \mathscr{M}/\mathscr{D}$ will be $T_g(\mathscr{M})/T_g(\mathcal{O}_g)$.

We shall construct a Riemannian metric $\langle \ , \ \rangle$ on $T(\mathscr{M})$ such that $\langle \xi, \eta \rangle_g = 0$ for any $\xi \in T_g(\mathscr{M})$, $\eta \in T_g(\mathcal{O}_g)$: in this case $\langle \ , \ \rangle$ will reduce to a metric on $T(\mathscr{G})$.

If we restrict attention to metrics of the form

$$\langle \xi, \eta \rangle = \int K^{i'j'kl}(x', x)\xi_{i'j'}(x')\eta_{kl}(x) \sqrt{(g(x)g(x'))}\, d^n x d^n x'$$

where $K^{i'j'kl} = K^{(i'j')kl} = K^{i'j'(kl)}$, then we require K to be symmetric ($K^{i'j'kl}(x', x) = K^{kli'j'}(x, x')$) and, for reduction to $T(\mathscr{G})$, to satisfy

$$K^{i'j'kl}{}_{;l} = 0 . \tag{A1}$$

There is obviously a multitude of solutions to this on a noncompact manifold, and any smooth choice would satisfy our requirements. But, fortunately, it is possible to give an explicit construction of a natural solution for the class of metrics being considered.

Take the positive definite case and work at first locally, with x in a normal neighbourhood of x'.

We would achieve a conventional L^2 metric if we set $K^{i'j'kl} = g^{(i'(k}g^{j')l)}\delta^{(g)}_x(x')$ (where $g^{i'k}$ is the two-point parallel propagator); but then, of course, (A1) would not be satisfied. Thus the δ-function must be "spread out" somewhat, and we are led to consider the form $K^{i'j'kl}(x, x') = X^{i'}X^{j'}X^k X^l f(x, x')$ where $\exp_x(X) = x'$, $\exp_{x'}(X') = x$. This will satisfy (A1) for $x \neq x'$ provided that f on the geodesic between x and x' satisfies

$$s\, df/ds + (s\theta + 4)f = 0$$

(where θ is the divergence of the geodesics from x and s is their distance parameter). In order to obtain a symmetric K we must replace (A1) by $(K^{i'j'kl}(x', x) + K^{kli'j'}(x, x'))_{;l} = 0$ and solve the resulting equation for f. The constant of integration can be fixed by the requirement $f \sim s^{-(n+1)}$ as $x \to x'$.

We actually require (A1) to be satisfied not only for $x \neq x'$, but in a distributional sense for all x. This is easily verified to be the case.

The global extension of this requires the use of our restrictions on $\mathscr{M}$. The integral expression for the inner product remains defined (f being continued through critical points on the geodesics by demanding asymptotic symmetry about the critical point), providing we write it in the form

$$\langle \xi, \eta \rangle = \int d^n x' \xi_{i'j'}(x') \int d^{n-1}\hat{X}' ds (s^2 \hat{X}'^{i'} \hat{X}'^{j'}) f(x', \hat{X}', s) \cdot X^k X^l \eta_{kl}(x) \sqrt{g(x')g(x)}\, J(x, \hat{X}', s)$$

where $\hat{X}'$ ranges over the unit tangent sphere at x', $x = \exp_{x'}(s\hat{X}')$ and X is defined as $-s$ times the parallel propagate of $\hat{X}'$ to x. J is the Jacobian of the coordinate transformation (obtained from the Hessian of the Jacobi fields on the geodesic) and vanishes at critical points. Convergence follows from the isolation of the critical points, and the assumption on the divergence of the geodesics and their asymptotic behaviour.

In the indefinite case the function f can no longer be normalised by refering to a distance parameter. However, for $n = 4$ we can look for a K of the form $X'^{i'}X'^{j'}X^k X^l f(x, x')\, \mathrm{Im}(\Gamma(x, x') + i0)^{-7/2}$, where Γ is the square of the geodesic distance and $(\Gamma + i0)^\lambda$ is the generalised function, defined analogously to $(P + i0)^\lambda$ in

[17], p. 274. We find that f must satisfy the symmetrized form of

$$df/ds+(\theta-3/s)f=0$$

and so we can achieve a normalisation by requiring $f\sim 1$ as $x\to x'$. The rest of the analysis is essentially unchanged.

Appendix B: Gauge Transformations

We carry out a coordinate transformation $M\to\mathbb{R}^4: x^{\mu'}\to x^\mu$ defined by $x^{\mu'}=x^\mu+\xi^\mu(x)$, ξ^μ being chosen so that the transformed metric satisfies $g_{0\nu}(x^\mu)=g^*_{0\nu}(x^\mu)$ $(\nu=0,1,2,3)$, with g^* being the "reference metric" which serves as a base point for imposing this gauge condition on metrics in a neighbourhood of g^*. Thus we demand

$$\begin{aligned}F(\xi,g)(x)_\nu\equiv\, & g_{0\nu}(x-\xi(x))+2g_{\varrho(0}(x-\xi(x))\xi^\varrho{}_{,\nu)}\\ & +g_{\varrho\sigma}(x-\xi(x))\xi^\sigma{}_{,0}\xi\ {}_{,\nu}-g^*_{0\nu}(x-\xi(x))=0\,. \end{aligned}\tag{B1}$$

This is a nonlinear differential equation for ξ, to be solved under the condition $\xi\equiv 0$ on S_0. Solution is clearly possible provided $g_{0\nu}-g^*_{0\nu}$, and their derivatives, are suitably small (globally).

Let us regard $F(\xi,g)$ as a function from the nuclear space of pairs of fields (ξ,g) into a nuclear space of vector fields. Differentiating we obtain from (12)

$$\begin{aligned}&(\mathrm{D}^nF)(\xi,g)(\delta_1\xi,\ldots,\delta_n\xi;\delta_1 g,\ldots,\delta_n g)\\ &=(g^{(n)}+2g^{(n)}\xi'+g^{(n)}\xi'^2+g^{*(n)})(\delta_1\xi\delta_2\xi\ldots\delta_n\xi)+(1+2\xi'+\xi'^2)\\ &\quad\cdot\left(\sum^n\delta_1 g^{(n-1)}\delta_2\xi\ldots\delta_n\xi\right)+(2g^{(n-1)}+2g^{(n-1)}\xi')\left(\sum^n\delta_1\xi'\delta_2\xi\ldots\delta_n\xi\right)\\ &\quad+(2+2\xi')\left(\sum^{n(n-1)}\delta_1 g^{(n-2)}\delta_2\xi'\delta_3\xi\ldots\delta_n\xi\right)\\ &\quad+2g^{(n-2)}\left(\sum^{n(n-1)/2}\delta_1\xi'\delta_2\xi'\delta_3\xi\ldots\delta_n\xi\right)\\ &\quad+2\left(\sum^{n(n-1)(n-2)/2}\delta_1\xi'\delta_2\xi'\delta_3^{(n-3)}g\delta_4\xi\ldots\delta_n\xi\right)\end{aligned}$$

where all the coordinate indices have been omitted for brevity, ξ', $g^{(n)}$ etc. denote partial derivatives with respect to the coordinates, and summations (over the number of terms indicated above the $\sum$) are over permutations of indices on the terms following so as to symmetrize them.

We see that, because F is polynomial, the number of terms rises modestly with increasing n. More significantly, successively higher coordinate derivatives of the metrics become involved, and there is no limit placed on the rate of increase of these with n. Thus if we are to bound D^nF by some known behaviour as $n\to\infty$ we must modify the transformation somewhat. Instead of determining ξ by (B1) we set

$$F(\xi,\tilde{g})(x)_\nu=0 \tag{B2}$$

where $\tilde{g}_{\mu\nu}(x)$ are functions obtained from $g_{\mu\nu}$ by smoothing with a fixed analytic kernel. Also g^* is assumed to be smoothed similarly. It is clearly possible to find a sub-atlas for $\mathscr{G}_a$ all of whose coordinate transformations are derived from gauge transformations of this form. (This is somewhat analogous to choosing an analytic structure on a finite-dimensional manifold.)

The derivatives of the function $\tilde{F}(\xi, g) \equiv F(\xi, \tilde{g}(g))$ are now freed from the spatial derivatives of g, so that if we define $\xi = \Xi(g)$ by $\tilde{F}(\xi, g) = 0$, we can bound $D^n\Xi$ by $A \cdot K^n n!$ with the norm on g depending only on the magnitude of $g_{\mu\nu}(x)$ (e.g. a simple supremum norm). The Lemma of § 7 is used to estimate the derivatives of compositions.

To complete the gauge transformation we "level up" the top of the slice by $x^\mu \to x''^\mu$, where

$$x''^\mu = x^\mu - \delta^\mu_0 a(x^a)x^0$$

$(a = 1, 2, 3)$. To determine a, note that the previous transformation carried a point $(t, x^{b'})$ at the top of the slice into some point $x^\mu(t, x^{b'})$ so that we now need

$$t = x^0(t, x^{b'})(1 - a(x^a(t, x^{b'}))).$$

The derivatives of this transformation can again be estimated, noting that the coordinate derivatives of ξ are now smoothed because of the smoothing imposed on g. Consequently we obtain bounds of the same form: $A'K'^n n!$

If we write the composite transformation as $x^{\mu'} = x^{\mu''} + \xi^\mu(x'')$, then we obtain for the transformed metric an expression like the first three terms of (B1). Thus the derivatives of the transformation of metrics $g' \to g''$ will again involve the spatial derivatives of the original metric. We conclude, therefore, that if we put coordinate charts on $\mathscr{G}_a$ simply by imposing gauge conditions on the corresponding metrics, then the coordinate-transformations on $\mathscr{G}_a$ will not have well-controlled higher derivatives. Consequently, the coordinates on $\mathscr{G}_a$ are to be fixed by *smoothing the metric* and applying a gauge transformation to the smoothed metric. This will produce derivatives satisfying the required bounds.

Appendix C: Submeasures

Let X be a topological vector space and $\mathscr{F}$ the class of its cofinite subspaces. A *submeasure* is a family $\{\mu'_V | V \in \mathscr{F}\}$, μ'_V being a positive bounded measure on X/V, satisfying

$$\mu^W_V := p^*_{VW}\mu'_W \leqq \mu'_V$$

for $W \subset V$: $V, W \in \mathscr{F}$; $p_{VW}: X/W \to X/V$. Set $\mu_V = \lim_W \mu^W_V$ (as in § 4).

Proposition. *$\{\mu_V | V \in \mathscr{F}\}$ is a promeasure.*

Proof. (a) μ_V is additive, since

$$\mu_V(Z_1 + Z_2) = \lim \mu^W_V(Z_1) + \lim \mu^W_V(Z_2).$$

(b) We now show that $\mu_V^W(Z) \to \mu_V(Z)$ uniformly in Z. For

$$\mu_V^W(Z) - \mu_V(Z) = \mu_V^W(X/V) - \mu_V^W(\complement Z) - \mu_V(X/V) + \mu_V(\complement Z) < \mu_V^W(X/V) - \mu_V(X/V)$$

since $\mu_V < \mu_V^W$.

Thus for all ε, *we can find* $W(\varepsilon)$ *such that*

$$W \subset W(\varepsilon) \Rightarrow 0 \leqq \mu_V^W(Z) - \mu_V(Z) < \varepsilon \quad (\forall Z),$$

where $W(\varepsilon)$ decreases with ε.

(c) We can now deduce countable additivity from (b). Put $U = W(\varepsilon/2N)$. Then

$$\varepsilon/2 > -\sum_{i=1}^{N} \mu_V(Z_i) + \sum_{i=1}^{N} \mu_V^U(Z_i) = -\sum^{N} \mu_V(Z_i) + \mu_V^U \sum^{N} Z_i .$$

Thus

$$\begin{aligned} \left|\mu_V\left(\sum^{\infty} Z_i\right) - \sum^{N} \mu_V(Z_i)\right| &\leqq \left|\sum^{\infty} \mu_V^U(Z_i) - \sum^{N} \mu_V^U(Z_i)\right| + \varepsilon \\ &= \left|\mu_V^U\left(\sum_{N+1}^{\infty} Z_i\right)\right| + \varepsilon \\ &< \left|\mu_V'\left(\sum_{N+1}^{\infty} Z_i\right)\right| + \varepsilon \end{aligned}$$

i.e. $\left|\mu_V\left(\sum^{\infty} Z_i\right) - \sum^{N} \mu_V(Z_i)\right| \to 0 \ (N \to \infty)$ as required.

(d) The projective properties can now be routinely verified, thus proving the proposition.

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