Commun. math. Phys. 55, 299—315 (1977)

Communications in Mathematical Physics

Phase Transitions for a Continuous System of Classical Particles in a Box*

Guy A. Battle

Duke University, Durham, North Carolina 27706, USA

Abstract. A continuous classical system involving an infinite number of distinguishable particles is analyzed along the same lines as its quantum analogue, considered in [1]. A commutative C^*-algebra is set up on the phase space of the system, and a representation-dependent definition of equilibrium involving the static KMS condition is given. For a special class of interactions the set of equilibrium states is realized as a convex Borel set whose extremal states are characterized by solutions to a system of integral equations. By analyzing these integral equations, we prove the absence of phase transitions for high temperature and construct a phase transition for low temperature. The construction also provides an example of a translation-invariant state whose decomposition at infinity yields states that are not translation-invariant. Thus we have an example in the classical situation of continuous symmetry breaking.

Section 0. Introduction

We wish to study the equilibrium states of an infinite collection of distinguishable, classical-mechanical particles in a one-dimensional box, where the interaction is infinitely weak as in the quantum analogue considered in [1].

The first step is to single out a certain class of states p for which there exists a limiting derivation δ_p on the underlying algebra $\mathscr{A}$ of observables with range in $\pi_p(\mathscr{A})''$, where $(\mathscr{H}_p, \pi_p, \Phi_p)$ is the canonical cyclic representation of $\mathscr{A}$ with respect to p. The second step is to introduce a notion of equilibrium, and we do so by using an extension of the static KMS condition (which is studied in [2] for another classical situation) to the type of derivation considered here. We do not consider the dynamical KMS condition [3] because it is not clear whether the derivation δ_p generates a dynamics in the representation associated with p. The third step is to prove that the decomposition at infinity of an equilibrium state yields equilibrium states that are trivial at infinity, and this is done for a special class of interactions. The fourth step is to characterize the equilibrium states that are trivial at infinity as finite sets of functions satisfying the system (0.2) of equations, and, again, this is done for a special class of interactions. Using this explicit characterization we prove

* This article is a part of the author's doctoral thesis, which was submitted to the mathematics

that phase transitions are absent for high temperature and construct an example of a phase transition for low temperature.

We begin by indexing the particles with Z^+ and letting $[-\pi, \pi]$ be the interval of space to which the system is confined. Let X_j be the jth copy of $[-\pi, \pi]$ and Y_j the jth copy of $\mathbb{R}$. Let x_j and p_j be the independent variables for X_j and Y_j, respectively. The interaction is defined in the following operational way:

Introduce the particles into the system one after another. When the first N particles are present, the interaction is a pair interaction, and the potential energy for the pair (j,k) of particles is $\mu_{jk}(N)f(x_j - x_k)$, where $\mu_{jk}(N)$ is the coupling coefficient and f is the functional form of the potential. In this case the energy of the system is given by

$$H_N = \frac{1}{2}\sum_{k=1}^{N} p_k^2 + \frac{1}{2}\sum_{j,k=1}^{N} \mu_{jk}(N) f(x_j - x_k) - \frac{1}{2} f(0) \sum_{k=1}^{N} \mu_{kk}(N). \tag{0.1}$$

Our assumptions on the $\mu_{jk}(N)$ will include the assumption that $\lim\limits_{N\to\infty} \mu_{jk}(N) = 0$, so the limiting situation as $N \to \infty$ may be thought of as an infinite system governed by an "infinitely weak" interaction.

We take $\mathscr{A}_j = C_\infty(X_j \times Y_j)$, the algebra of continuous functions on $X_j \times Y_j$ that either vanish at infinity with respect to p_j and are periodic with respect to x_j or are constant, to be the algebra of observables for the jth particle. For a finite $\Omega \subset Z^+$, let $\mathscr{A}_\Omega = C_\infty\left(\prod_{j\in\Omega} (X_j \times Y_j)\right)$, which is isomorphic to $\bigotimes_{j\in\Omega} C_\infty(X_j \times Y_j) = \bigotimes_{j\in\Omega} \mathscr{A}_j$. Since the particles are distinguishable, $\mathscr{A}_\Omega$ is the algebra of observables for the set Ω of particles. The family $\{\mathscr{A}_\Omega\}$ is a directed system of C^*-algebras in the usual sense. We take the C^*-closure of $\bigcup_\Omega \mathscr{A}_\Omega$ to be the C^*-algebra of the infinite system. We will refer to $\mathscr{A}$ as a quasi-local algebra and an element of $\bigcup_\Omega \mathscr{A}_\Omega$ as a local observable. Note that "local" refers to the number of particles and has no spatial meaning. Notice also that $\mathscr{A}$ is a separable C^*-algebra.

In Section 1 we single out a certain class of states and show that the set of all such states is a convex, Borel set. We give a result concerning integral decompositions of such states. Further, we give a definition of β-equilibrium state that involves the derivation δ_p, whose existence is proven for p in the class of states. We show that the set of β-equilibrium states is a convex, Borel set for each inverse temperature β. We prove in Section 2 a preliminary result concerning the support of the measure at infinity of a β-equilibrium state.

We develop a Fourier analysis of the interaction in Section 3, and in Section 4 we apply this analysis in our examination of β-equilibrium states with trivial algebra at infinity.

In Section 5 we restrict ourselves to a special class of interactions (interactions of the Pth kind) and prove that:

a) The measure at infinity of a β-equilibrium state is concentrated on a Borel set of β-equilibrium states with trivial algebra at infinity.

b) The β-equilibrium states with trivial algebra at infinity are exactly the extremal β-equilibrium states.

c) There is a one-to-one correspondence between β-equilibrium states with trivial algebra at infinity and P-tuples $(g_1, \ldots, g_p)$ of continuous functions satisfying the system of equations

$$g_j(x) = \sum_{k=1}^{P} \lambda_{jk} \frac{1}{\int\limits_{-\pi}^{\pi} e^{-\beta g_k(y)} dy} \int\limits_{-\pi}^{\pi} f(y-x) e^{-\beta g_k(y)} dy, \tag{0.2}$$

where the λ_{jk} are given numbers related to the interaction.

This explicit characterization of extremal β-equilibrium states allows us to show in Section 7 that for an interaction of the Pth kind there is a temperature above which there is only one equilibrium state. Also, it is possible to give an example of a phase transition at low temperature and show that it is also an example where the decomposition at infinity breaks symmetry.

The quantum analogue of this situation was previously considered by the author [1]. We state without proof results which are parallel to those in [1]. In fact, some of the proofs that we do present are analogous to the corresponding proofs in [1], and some of the discussion carries over almost verbatim.

The author would like to thank Michael Reed for reading a first draft of the manuscript and making many valuable suggestions.

Section 1. Definition of Equilibrium

Throughout we assume that f is an even, real-valued, C^1 function on $\mathbb{R}$ which is periodic with period 2π. Let $\{a_n | n \in Z\}$ be the sequence of Fourier coefficients of f, and note that the a_n are real, $a_{-n} = a_n$, and $\sum_{n=-\infty}^{\infty} |a_n| < \infty$. We assume that $\sum_{n=-\infty}^{\infty} |na_n| < \infty$. Let μ_{jk} denote the function $N \to \mu_{jk}(N)$ on Z^+. As in [1], the double sequence $\{\mu_{jk}\}$ is assumed to have the following properties:

$$\mu_{jk} = \mu_{kj}, \tag{1.1}$$

$$\lim_{N \to \infty} \mu_{jk}(N) = 0, \tag{1.2}$$

$$\sup_{N,j} \sum_{k=1}^{N} |\mu_{jk}(N)| < \infty. \tag{1.3}$$

We also introduce the following notation:

$$W_{j,N} = \sum_{k=1}^{N} \mu_{jk}(N) f(x_j - x_k), \tag{1.4}$$

$$V_{j,N}(x) = \sum_{k=1}^{N} \mu_{jk}(N) f(x - x_k), \tag{1.5}$$

$$F_{j,N} = \sum_{\substack{k=1 \\ k \neq j}}^{N} \mu_{jk}(N) f'(x_j - x_k), \tag{1.6}$$

$$E_{j,N}(x) = \sum_{k=1}^{N} \mu_{jk}(N) f'(x - x_k) \tag{1.7}$$

for $+\infty < x < \infty$. These quantities are obviously uniformly bounded in j, N, and x, by (1.3).

For every finite $\Omega \subset Z^+$, we denote by $\mathscr{B}_\Omega$ the set of all C^∞ functions in $\mathscr{A}_\Omega$ that are either of rapid decrease in the momentum variables or constant. Let $\mathscr{B} = \bigcup_\Omega \mathscr{B}_\Omega$ and define δ_N as the unbounded derivation on $\mathscr{A}$ with domain $\mathscr{B}$ such that

$$\delta_N(A) = \sum_{j \in \Omega} p_j \frac{\partial A}{\partial x_j} - \sum_{j \in \Omega} F_{j,N} \frac{\partial A}{\partial p_j}$$

for $A \in \mathscr{B}_\Omega$. Notice that $\mathscr{B}$ is dense in $\mathscr{A}$.

The states of the physical system are states on the C^*-algebra $\mathscr{A}$. Let $\mathscr{S}$ be the set of states on $\mathscr{A}$. It will be understood that $\mathscr{S}$ has the weak* topology. Let I denote the set of all states p such that the strong limits

$$\operatorname*{s-lim}_{N \to \infty} \pi_p(W_{j,N}) \equiv W_{j,p}$$

$$\operatorname*{s-lim}_{N \to \infty} \pi_p(V_{j,N}(x)) \equiv V_{j,p}(x)$$

$$\operatorname*{s-lim}_{N \to \infty} \pi_p(F_{j,N}) \equiv F_{j,p}$$

$$\operatorname*{s-lim}_{N \to \infty} \pi_p(E_{j,N}(x)) \equiv E_{j,p}(x)$$

exist for $-\infty < x < \infty$, where $(\mathscr{H}_p, \pi_p, \Phi_p)$ will always denote the canonical cyclic representation of $\mathscr{A}$ with respect to p. We will call such states *asymptotic states*. For every increasing sequence S in Z^+, we denote by J_S the set of all states p such that the strong limits

$$\operatorname*{s-lim}_{N \to \infty} \pi_p(W_{j,S_N}) \equiv W_{S,j,p}$$

$$\operatorname*{s-lim}_{N \to \infty} \pi_p(V_{j,S_N}(x)) \equiv V_{S,j,p}(x)$$

$$\operatorname*{s-lim}_{N \to \infty} \pi_p(F_{j,S_N}) \equiv F_{S,j,p}$$

$$\operatorname*{s-lim}_{N \to \infty} \pi_p(E_{j,S_N}(x)) \equiv E_{S,j,p}(x)$$

exist for $-\infty < x < \infty$, and we will call such states *S-asymptotic states*.

1.1. Theorem. *For every increasing sequence S in Z^+, J_S is a convex, Borel set in $\mathscr{S}$. In particular, I is a convex, Borel set.*

Proof. The convexity of J_S is obvious.

Let $\{A_m\}$ be a dense sequence in $\mathscr{A}$. The existence of such a sequence follows from the separability of $\mathscr{A}$. It follows that $\{\pi_p(A_m)\Phi_p\}$ is dense in $\mathscr{H}_p$ for every state p. The remainder of the proof is similar to the proof of the parallel result in [1].

1.2. Theorem. *If μ is a probability measure on $\mathscr{S}$ whose resultant lies in I, then $\mu(J_S) = 1$ for some increasing sequence S in Z^+.*

Aside from using the sequence $\{A_m\}$ introduced above, one proves this theorem in the same way that the parallel theorem in [1] is proven.

Now, consider an asymptotic state p and an observable $A \in \mathscr{B}_\Omega$ for some finite $\Omega \subset Z^+$. The strong limit

$$\underset{N\to\infty}{s\text{-lim}}\, \pi_p(\delta_N(A)) \equiv \delta_p(A)$$

exists and is equal to

$$\sum_{j\in\Omega} \pi_p\left(p_j \frac{\partial A}{\partial x_j}\right) - \sum_{j\in\Omega} F_{j,p}\pi_p\left(\frac{\partial A}{\partial p_j}\right).$$

Thus δ_p is a well-defined linear mapping from $\mathscr{B}$ into $\pi_p(\mathscr{A})''$. Moreover, δ_p is an unbounded derivation on $\mathscr{A}$ in the sense that $\mathscr{B}$ is dense in $\mathscr{A}$ and

$$\delta_p(AB) = \pi_p(A)\delta_p(B) + \delta_p(A)\pi_p(B)$$

for $A, B \in \mathscr{B}$.

1.3. Definition. Let $p \in I$ and $\beta > 0$. p is a *β-equilibrium state* if and only if

$$(\delta_p(A)\pi_p(B)\Phi_p, \Phi_p) = p(\{A, B\}) \tag{1.8}$$

for $A, B \in \mathscr{B}$. We denote the set of such p by $\mathscr{E}_\beta$.

(1.8) is an obvious extension of the static β-KMS condition forced upon us by the fact that the range of the derivation does not lie in $\mathscr{A}$. Later in our discussion we will refer to the ordinary static β-KMS condition simply as the β-KMS condition.

1.4. Theorem. *$\mathscr{E}_\beta$ is a convex, Borel set in $\mathscr{S}$.*

Proof. Let $\{A_m\}$ be a sequence in $\mathscr{B}$ such that for every finite $\Omega \subset Z^+$, $\{A_m\} \cap \mathscr{B}_\Omega$ is dense in $\mathscr{B}_\Omega$ in the C^1 norm. There is such a sequence because each $\mathscr{B}_\Omega$ is separable in the C^1 norm. It is clear that $p \in I$ is a β-equilibrium state if and only if (1.8) is satisfied for $A = A_m$ and $B = A_n$ for all $m, n \in Z^+$. Thus $\mathscr{E}_\beta$ is defined by a countable number of equations. It is therefore sufficient to show that the functions $p \to p(\{A, B\})$, $p \to (\delta_p(A)\pi_p(B)\Phi_p, \Phi_p)$ are Borel functions on I for fixed $A, B \in \mathscr{B}$.

The first function is continuous. The second function is the pointwise limit of the sequence of functions $p \to p(\delta_N(A)B)$ on I, and these functions are continuous. This completes the proof.

Let p be an S-asymptotic state for some increasing sequence S in Z^+. and consider an observable $A \in \mathscr{B}_\Omega$ for some finite $\Omega \subset Z^+$. The strong limit

$$\underset{N\to\infty}{s\text{-lim}}\, \pi_p(\delta_{S_N}(A)) \equiv \delta_p^S(A)$$

exists and is equal to

$$\sum_{j\in\Omega} \pi_p\left(p_j \frac{\partial A}{\partial x_j}\right) - \sum_{j\in\Omega} F_{S,j,p}\pi_p\left(\frac{\partial A}{\partial p_j}\right).$$

1.5. Definition. Let S be an increasing sequence in Z^+, $p \in J_S$, and $\beta > 0$. p is an *$S-\beta$-equilibrium state* if and only if

$$(\delta_p^S(A)\pi_p(B)\Phi_p, \Phi_p)\beta = p(\{A, B\}) \tag{1.9}$$

for $A, B \in \mathscr{B}$. We denote the set of such p by $\mathscr{E}_\beta^S$.

1.6. Theorem. *For every increasing sequence S in Z^+, $\mathscr{E}_\beta^S$ is a convex, Borel set.*

The proof of this theorem is identical to the proof of Theorem 1.4.

Section 2. Decomposition at Infinity of an Equilibrium State

Our next step is to examine the measure at infinity of a β-equilibrium state. By [4], the decomposition at infinity of a state on a separable quasi-local algebra is concentrated on a Borel set of states with trivial algebra at infinity. We wish to answer the following question: is the measure at infinity of a β-equilibrium state concentrated on a Borel set of *β-equilibrium* states with trivial algebra at infinity?

2.1. Lemma. *Let $p \in \mathscr{E}_\beta$ and μ the measure at infinity of p. Let $\mathscr{F}$ be a Borel set of states such that $\mu(\mathscr{F})>0$. Then the state $\lambda(A)=\frac{1}{\mu(\mathscr{F})}\int_{\mathscr{F}}\sigma(A)d\mu(\sigma)$ is also a β-equilibrium state.*

Proof. By the nature of the decomposition at infinity [5], there is an orthogonal projection $E\in\bigcap_{\Omega}\pi_p(\mathscr{A}_\Omega^\perp)''$ such that

$$(E\pi_p(A)\Phi_p,\Phi_p)=\int_{\mathscr{F}}\sigma(A)d\mu(\sigma)$$

for all $A\in\mathscr{A}$, so

$$\lambda(A)=\frac{1}{\|E\Phi_p\|^2}(E\pi_p(A)\Phi_p,\Phi_p).$$

Hence $(E\mathscr{H}_p, A\to\pi_p(A)|_{E\mathscr{H}_p}, E\Phi_p/\|E\Phi_p\|)$ is the canonical cyclic representation of λ. It follows immediately that $\lambda\in I$. Also, $\delta_\lambda(a)=\delta_p((A))|_{E\mathscr{H}_p}$ for all $A\in\mathscr{B}$, so in order to show that $\lambda\in\mathscr{E}_\beta$, it suffices to show that

$$\beta(\delta_p((A))\pi_p(B)E\Phi_p,\Phi_p)=(\pi_p(\{A,B\})E\Phi_p,\Phi_p)$$

for all A, $B\in\mathscr{B}$.

Fix A, $B\in\mathscr{B}_\Omega$ for some finite $\Omega\subset Z^+$. Since $E\in\pi_p(\mathscr{A}_\Omega)''$, there exists a net $\{A_\alpha\}$ in $\mathscr{A}_\Omega$ such that $\pi_p(A_\alpha)\to E$ strongly. Since every element of $\mathscr{A}_\Omega^\perp$ can be uniformly approximated by elements of $\mathscr{B}_\Omega^\perp$, we may choose $\{A_\alpha\}$ to lie in $\mathscr{B}_\Omega^\perp$. For each α we have

$$\beta(\delta_p((A))\pi_p(BA_\alpha)\Phi_p,\Phi_p)=p(\{A,BA_\alpha\}).$$

But $\{A,BA_\alpha\}=\{A,B\}A_\alpha$ because $A_\alpha\in\mathscr{B}_\Omega^\perp$ and $A\in\mathscr{B}_\Omega$. Hence

$$\beta(\delta_p((A))\pi_p(B)\pi_p(A_\alpha)\Phi_p,\Phi_p)=p(\{A,B\}A_\alpha)$$

from which it follows that

$$\beta(\delta_p((A))\pi_p(B)E\Phi_p,\Phi_p)=(\pi_p(\{A,B\})E\Phi_p,\Phi_p).$$

This completes the proof.

2.2. Theorem. *Let $p\in\mathscr{E}_\beta$ and μ the measure at infinity of p. Then $\mu(\mathscr{E}_\beta^S)=1$ for some increasing sequence S in Z^+.*

Proof. By Theorem 1.2, $\mu(J_S)=1$ for some increasing sequence S in Z^+. Now suppose $\mu(J_S\backslash\mathscr{E}_{S,\beta})>0$. Let $\{A_m\}$ be as in the proof of Theorem 1.4 and set $I_{mn}=\{\sigma\in J_S|$ (1.9) does *not* hold for $A=A_m$, $B=A_n\}$. Then $J_S\backslash\mathscr{E}_{S,\beta}=\bigcup_{m,n=1}^{\infty} I_{mn}$ and the I_{mn} are Borel sets. Hence there exist $m_1, n_1\in Z^+$ such that $\mu(I_{m_1n_1})>0$.

Let K_{rN} be the set of all $\sigma\in I_{m_1n_1}$ such that the complex number

$$c(\sigma)\equiv\beta(\delta_\sigma^S((A_{m_1}))\pi_\sigma(A_{n_1})\Phi_\sigma,\Phi_\sigma)-\sigma(\{A_{m_1},A_{n_1}\}),$$

satisfies $|c(\sigma)|>\frac{1}{N}$ and $2\pi\frac{r-1}{17}\leqq\arg c(\sigma)<2\pi\frac{r}{17}$. Then the K_{rN} are Borel sets and $I_{m_1n_1}=\bigcup_{N=1}^{\infty}\bigcup_{r=1}^{17}K_{rN}$. Hence $\mu(K_{r_1N_1})>0$ for some r_1, N_1. Thus

$$\int_{K_{r_1N_1}} c\,d\mu\neq 0.$$

Let $\lambda(A)=\frac{1}{\mu(K_{r_1N_1})}\int_{K_{r_1N_1}}\sigma(A)d\mu(\sigma)$ for all A in $\mathscr{A}$. Then $\lambda\in\mathscr{E}_\beta$ by the preceding lemma. We also have that

$$\beta\frac{1}{\mu(K_{r_1N_1})}\int_{K_{r_1N_1}}(\delta_\sigma^S((A_{m_1}))\pi_\sigma(A_{n_1})\Phi_\sigma,\Phi_\sigma)d\mu(\sigma)$$
$$\neq\frac{1}{\mu(K_{r_1N_1})}\int_{K_{r_1N_1}}\sigma(\{A_{m_1},A_{n_1}\})d\mu(\sigma).$$

But $(\delta_\sigma^S((A_{m_1}))\pi_\sigma(A_{n_1})\Phi_\sigma,\Phi_\sigma)=\lim_{N\to\infty}\sigma(\delta_{S_N}(A_{m_1})A_{n_1})$ and the uniform boundedness of the sequence of functions $\sigma\to\sigma(\delta_{S_N}(A_{m_1})A_{n_1})$ follows from the uniform boundedness of the $F_{j,N}$, so by dominated convergence,

$$\beta\lim_{N\to\infty}\lambda(\delta_{S_N}(A_{m_1})A_{n_1})\neq\lambda(\{A_{m_1},A_{n_1}\}).$$

Since $\lambda\in I$, we certainly have that $(\delta_\lambda((A_{m_1}))\pi_\lambda(A_{n_1})\Phi_\lambda,\Phi_\lambda)=\lim_{N\to\infty}\lambda(\delta_{S_N}(A_{m_1})A_{n_1})$, so λ does *not* satisfy (1.8) for $A=A_{m_1}$, $B=A_{n_1}$. This is the desired contradiction.

Combining this result with the remark at the beginning of the section, it follows that the measure at infinity of a β-equilibrium state is concentrated on a Borel set of $S-\beta$-equilibrium states with trivial algebra at infinity for some increasing sequence S in Z^+. Therefore it is important to analyze such states.

Section 3. Fourier Analysis of the Interaction

On the space of all bounded, complex sequences select an arbitrary Banach limit λ, as was done in [1]. Ultimately there will be no dependence on our choice of λ. In order to avoid confusion with regard to indices, we will denote a sequence $\{b_N\}$ by $N\to b_N$.

3.1. Definition. Let p be a state on $\mathscr{A}$. Then $B_{j,n,p}$ is the operator on $\mathscr{H}_p$ defined by

$$(B_{j,n,p}\Psi_1, \Psi_2) = \lambda\left(N \to \sum_{k=1}^{N} \mu_{jk}(N)\,(\pi_p(e^{inx_k})\Psi_1, \Psi_2)\right) \tag{3.1}$$

for all $\Psi_1, \Psi_2 \in \mathscr{H}_p$.

It follows from (1.3) that the sequence in the argument of λ is indeed bounded, so (3.1) makes sense. Note that the $B_{j,n,p}$ are uniformly bounded, and recall that $\{a_n | n \in Z\}$ is the sequence of Fourier coefficients of f.

3.2. Proposition. *$B_{j,n,p} \in \bigcap_{\Omega} \pi_p(\mathscr{A}_\Omega^\perp)''$, and $\sum_{n=-\infty}^{\infty} na_n B_{j,n,p}\pi_p(e^{-inx_j})$ converges absolutely in norm.*

Proof. The second part of the statement follows from the fact that $\sum_{n=-\infty}^{\infty} |na_n| < \infty$.

Let Ω be a finite subset of Z^+ and $A \in \pi_p(\mathscr{A}_\Omega^\perp)'$. It suffices to show that $B_{j,n,p}$ commutes with A. Letting $\Psi_1, \Psi_2 \in \mathscr{H}_p$ and $A\Psi_1, \Psi_2$ be the vectors in (3.1), we obtain

$$(B_{j,n,p}A\Psi_1, \Psi_2) = \lambda\left(N \to \sum_{k=1}^{N} \mu_{jk}(N)\,(\pi_p(e^{inx_k})A\Psi_1, \Psi_2)\right).$$

Letting Ψ_1, $A^*\Psi$ be the vectors in (3.1), we have

$$(AB_{j,n,p}\Psi_1, \Psi_2) = \lambda\left(N \to \sum_{k=1}^{N} \mu_{jk}(N)\,(A\pi_p(e^{inx_k})\Psi_1, \Psi_2)\right).$$

Since A commutes with $\pi_p(e^{inx_k})$ when $k \in \Omega$, it follows that

$$\begin{aligned}([B_{j,n,p}, A]\Psi_1, \Psi_2) &= \lambda\left(N \to \sum_{k\in\Omega} \mu_{jk}(N)\,([\pi_p(e^{inx_k}), A]\Psi_1, \Psi_2)\right)\\ &= \lim_{N\to\infty} \sum_{k\in\Omega} \mu_{jk}(N)\,([\pi_p(e^{inx_k}), A]\Psi_1, \Psi_2)\\ &= 0.\end{aligned}$$

This completes the proof.

3.3. Proposition. *If $p \in I$, then*

$$W_{j,p} = \sum_{n=-\infty}^{\infty} a_n B_{j,n,p}\pi_p(e^{-inx_j})$$

$$V_{j,p}(x) = \sum_{n=-\infty}^{\infty} a_n B_{j,n,p} e^{-inx}$$

$$F_{j,p} = \sum_{n=-\infty}^{\infty} na_n B_{j,n,p}\pi_p(e^{-inx_j})$$

$$E_{j,p}(x) = \sum_{n=-\infty}^{\infty} na_n B_{j,n,p} e^{-inx}.$$

The proof is entirely similar to the proof of the parallel result in [1].

3.4. Definition. Let p be a state on $\mathscr{A}$ and S an increasing sequence in Z^+. $B_{S,j,n,p}$ is the operator on $\mathscr{H}_p$ such that

$$(B_{S,j,n,p}\Psi_1,\Psi_2)=\lambda\left(N\to\sum_{k=1}^{S_N}\mu_{jk}(S_N)((\pi_p(e^{inx_k})\Psi_1,\Psi_2)\right)\tag{3.2}$$

for all $\Psi_1, \Psi_2\in\mathscr{H}_p$.

It is clear that $B_{S,j,n,p}$ has the properties stated in Proposition 3.2 for $B_{j,n,p}$. Furthermore, if $p\in J_S$, then

$$W_{S,j,p}=\sum_{n=-\infty}^{\infty}a_nB_{S,j,n,p}\pi_p(e^{-inx_j})$$

$$V_{S,j,p}(x)=\sum_{n=-\infty}^{\infty}a_nB_{S,j,n,p}e^{-inx}$$

$$F_{S,j,p}=\sum_{n=-\infty}^{\infty}na_nB_{S,j,n,p}\pi_p(e^{-inx_j})$$

$$E_{S,j,p}(x)=\sum_{n=-\infty}^{\infty}na_nB_{S,j,n,p}e^{-inx}.$$

Section 4. Equilibrium States with Trivial Algebra at Infinity

Let p be a state in $\mathscr{E}_\beta$ with trivial algebra at infinity, i.e., such that $\bigcap_{\Omega}\pi_p(\mathscr{A}_\Omega^\perp)''=\mathbb{C}1_{\mathscr{H}^p}$. Then the operators $B_{j,n,p}$ of the preceding section are scalar multiples of the identity operator. By the equations in Proposition 3.3, it follows that $V_{j,p}$ and $E_{j,p}$ are real-valued continuous functions times the identity operator, and, identifying $V_{j,p}$ and $E_{j,p}$ with the corresponding functions, it also follows that

$$E_{j,p}=V'_{j,p},$$
$$W_{j,p}=\pi_p(V_{j,p}(x_j)),$$
$$F_{j,p}=\pi_p(E_{j,p}(x_j)).$$

In physical terms, $V_{j,p}$ is the potential field in which the jth particle is moving when the system is in state p, while $E_{j,p}$ is the force field.

Now recall that

$$\delta_p(A)=\sum_{j\in\Omega}\pi_p\left(p_j\frac{\partial A}{\partial x_j}\right)-\sum_{j\in\Omega}F_{j,p}\pi_p\left(\frac{\partial A}{\partial p_j}\right)$$

for all $A\in\mathscr{B}_\Omega$. Thus

$$\delta_p(A)=\pi_p\left(\sum_{j\in\Omega}p_j\frac{\partial A}{\partial x_j}-\sum_{j\in\Omega}V'_{j,p}(x_j)\frac{\partial A}{\partial p_j}\right).$$

Since $p\in\mathscr{E}_\beta$, it follows that

$$\beta p\left(\sum_{j\in\Omega}p_j\frac{\partial A}{\partial x_j}B-\sum_{j\in\Omega}V'_{j,p}(x_j)\frac{\partial A}{\partial p_j}B\right)=p(\{A,B\})$$

for all $A, B \in \mathscr{B}_\Omega$. This equation may be rewritten as

$$\beta\left(\sum_{j\in\Omega} p_j \frac{\partial A}{\partial x_j} - \sum_{j\in\Omega} V'_{j,p}(x_j)\frac{\partial A}{\partial p_j}\right) p_\Omega = \{p_\Omega, A\},$$

where the measure p_Ω is the restriction of p to $\mathscr{A}_\Omega$, and where the derivatives of p_Ω are taken in the distributional sense with respect to the space $\mathscr{B}_\Omega$. Setting $A = e^{ix_k}$, we get

$$\beta p_k p_\Omega = -\frac{\partial p_\Omega}{\partial p_k}.$$

On the other hand, if $A = \exp(-\frac{1}{2}p_k^2)$, we obtain

$$\beta p_k \exp(-\tfrac{1}{2}p_k^2) V'_{k,p}(x_k) p_\Omega = -p_k \exp(-\tfrac{1}{2}p_k^2)\frac{\partial p_\Omega}{\partial x_k},$$

or

$$\beta V'_{k,p}(x_k) p_\Omega = -\frac{\partial p_\Omega}{\partial x_k}.$$

From these equations it follows that p_Ω is an absolutely continuous measure whose density is

$$\frac{\exp\left(-\beta \sum_{j\in\Omega} (\frac{1}{2}p_j^2 + V_{j,p}(x_j))\right)}{\iint \exp\left(-\beta \sum_{j\in\Omega} (\frac{1}{2}p_j^2 - V_{j,p}(x_j))\right) dx_\Omega dp_\Omega}.$$

Thus we have the following result:

4.1. Theorem. *If $p \in \mathscr{E}_\beta$ and has trivial algebra at infinity, then the $V_{j,p}$ are real-valued functions and*

$$p(A) = \frac{\iint A \exp\left(-\beta \sum_{j\in\Omega} (\frac{1}{2}p_j^2 + V_{j,p}(x_j))\right) dx_\Omega dp_\Omega}{\iint \exp\left(-\beta \sum_{j\in\Omega} (\frac{1}{2}p_j^2 + V_{j,p}(x_j))\right) dx_\Omega dp_\Omega} \tag{4.1}$$

for $A \in \mathscr{A}_\Omega$.

4.2. Corollary. *If $p \in \mathscr{E}_\beta$ and has trivial algebra at infinity, then p is a product state.*

Now recall that $V_{j,p}(x) = \underset{N\to\infty}{s\text{-lim}}\, \pi_p\left(\sum_{k=1}^{N} \mu_{jk}(N)\, f(x_k - x)\right)$. Since $V_{j,p}(x)$ is a scalar, we have in particular that

$$\begin{aligned} V_{j,p}(x) &= (V_{j,p}(x)\Phi_p, \Phi_p) \\ &= \lim_{N\to\infty}\left(\pi_p\left(\sum_{k=1}^{N} \mu_{jk}(N) f(x_k - x)\right)\Phi_p, \Phi_p\right) \\ &= \lim_{N\to\infty} \sum_{k=1}^{N} \mu_{jk}(N) p(f(x_k - x)). \end{aligned} \tag{4.2}$$

Hence, we have the following compatibility conditions:

$$V_{j,p}(x)=\lim_{N\to\infty}\sum_{k=1}^{N}\mu_{jk}(N)\cdot\frac{\iint f(x_k-x)\exp(-\beta(\frac{1}{2}p_k^2+V_{k,p}(x_k)))dx_k dp_k}{\iint\exp(-\beta(\frac{1}{2}p_k^2+V_{k,p}(x_k)))dx_k dp_k}$$
$$=\lim_{N\to\infty}\sum_{k=1}^{N}\mu_{jk}(N)\cdot\frac{1}{\int_{-\pi}^{\pi}e^{-\beta V_{k,p}(y)}dy}\int_{-\pi}^{\pi}f(y-x)e^{-\beta V_{k,p}(y)}dy. \tag{4.3}$$

The equations for the functions $V_{j,p}$ are *necessary* conditions for the state p to be a β-equilibrium state with trivial algebra at infinity. This fact alone, of course, is not enough to make the equations themselves interesting from the standpoint of computing β-equilibrium states. There will be more to say about this problem when we consider a special class of interactions in the next section.

Suppose that S is an increasing sequence in Z^+ and $p\in\mathscr{E}_\beta^S$ with trivial algebra at infinity. Then by reasoning identical to that given above, we obtain that p is a product state,

$$p(A)=\frac{\iint A\exp(-\beta(\frac{1}{2}p_j^2+V_{S,j,p}(x_j))dx_j dp_j}{\iint\exp(-\beta(\frac{1}{2}p_j^2+V_{S,j,p}(x_j)))dx_j dp_j} \tag{4.4}$$

for $A\in\mathscr{A}_j$, and

$$V_{S,j,p}(x)=\lim_{N\to\infty}\sum_{k=1}^{S_N}\mu_{jk}(S_N)\cdot\frac{1}{\int_{-\pi}^{\pi}e^{-\beta V_{S,k,p}(y)}dy}\int_{-\pi}^{\pi}f(y-x)e^{-\beta V_{S,k,p}(y)}dy. \tag{4.5}$$

Section 5. Interactions of the *P*th Kind

We are now ready to consider special classes of interactions for which more complete results can be obtained.

5.1. Definition. $\{\mu_{jk}\}$ is said to be *of the* Pth *kind* if and only if $\{\mu_{jk}\}$ has period P with respect to j (and therefore with respect to k) and

$$\lim_{N\to\infty}\mu_{jk}(N)\,|\{n\in Z^+\,|\,nP+k\leqq N\}|\equiv\lambda_{jk} \tag{5.1}$$

exists for $1\leqq k\leqq P$, where $||$ denotes the cardinality of the set.

Suppose $\{\mu_{jk}\}$ is of the Pth kind, and let $p\in\mathscr{E}_\beta^S$ have trivial algebra at infinity, where S is an increasing sequence in Z^+. Then by reasoning similar to that given in

[1] for the parallel situation, it follows that $p\in\mathscr{E}_\beta$, $\{V_{j,p}\}$ is periodic in j with period P, and

$$V_{j,p}(x)=\sum_{k=1}^{P}\lambda_{jk}\frac{\int_{-\pi}^{\pi} f(y-x)e^{-\beta V_{j,p}(y)}dy}{\int_{-\pi}^{\pi} e^{-\beta V_{j,p}(y)}dy}.$$

Thus we have the following result.

5.2. Theorem. *If the interaction is of the* P*th kind, the measure at infinity of a β-equilibrium state is concentrated on a Borel set of β-equilibrium states with trivial algebra at infinity.*

We also have the lemma corresponding to Lemma 8.2 in [1].

5.3. Lemma. *Suppose $\{\mu_{jk}\}$ is of the* P*th kind, and let p be a product state such that $\{p(e^{inx_j})\}$ is periodic in j with period p for all n. Then $p\in I$.*

Our next step is to characterize the states in $\mathscr{E}_\beta$ that have trivial algebra at infinity for interactions of the Pth kind, and we begin by summarizing what we have accomplished so far. If p is a state in $\mathscr{E}_\beta$ with trivial algebra at infinity, then p is a product state,

$$p(A)=\frac{\int_{-\infty}^{\infty}\int_{-\pi}^{\pi} A\exp(-\beta(\tfrac{1}{2}p_j^2+V_{j,p}(x_j))dx_jdp_j}{\int_{-\infty}^{\infty}\int_{-\pi}^{\pi} \exp(-\beta(\tfrac{1}{2}p_j^2+V_{j,p}(x_j)))dx_jdp_j} \tag{5.2}$$

for all A in $\mathscr{A}_j$, and

$$V_{j,p}(x)=\sum_{k=1}^{P}\lambda_{jk}\frac{1}{\int_{-\pi}^{\pi} e^{-\beta V_{k,p}(y)}dy}\int_{-\pi}^{\pi} f(y-x)e^{-\beta V_{k,p}(y)}dy \tag{5.3}$$

for all $j\in Z^+$.

These conditions are *necessary* for p to be a β-equilibrium state with trivial algebra at infinity. The following result establishes that the consistency equations (5.3) are *sufficient* conditions for the state corresponding to the functions via (5.2) to be a state in $\mathscr{E}_\beta$ with trivial algebra at infinity.

5.4. Theorem. *Suppose $\{\mu_{jk}\}$ is of the* P*th kind, and let $\{g_j\}$ be a sequence of bounded, real-valued, measurable functions such that*

$$g_j(x)=\sum_{k=1}^{P}\lambda_{jk}\frac{1}{\int_{-\pi}^{\pi} e^{-\beta g_k(y)}dy}\int_{-\pi}^{\pi} f(y-x)e^{-\beta g_k(y)}dy. \tag{5.4}$$

Let p be the product state such that

$$p(A)=\frac{\int_{-\infty}^{\infty}\int_{-\pi}^{\pi} A\exp(-\beta(\tfrac{1}{2}p_j^2+g_j(x_j)))dx_jdp_j}{\int_{-\infty}^{\infty}\int_{-\pi}^{\pi} \exp(-\beta(\tfrac{1}{2}p_j^2+g_j(x_j)))dx_jdp_j}$$

for all $A \in \mathscr{A}_j$. Then $p \in \mathscr{E}_\beta$ and has trivial algebra at infinity. Furthermore, $V_{j,p}$ exists and is equal to g_j.

Proof. p is a product state. Moreover,

$$p(e^{inx_j}) = \frac{1}{\int\limits_{-\pi}^{\pi} e^{-\beta g_j(y)} dy} \int\limits_{-\pi}^{\pi} e^{iny} e^{-\beta g_j(y)} dy .$$

Since $\{g_j\}$ is periodic in j with period P, so is $\{p(e^{inx_j})\}$. By Lemma 5.3 it follows that $p \in I$. In particular, $V_{j,p}$ exists.

The next point to notice is that p is a β-KMS state on $\mathscr{A}_j$ with respect to the unbounded derivation

$$A \to p_j \frac{\partial A}{\partial x_j} - g_j'(x_j) \frac{\partial A}{\partial p_j}, \qquad A \in \mathscr{B}_j .$$

(g_j is clearly in $C^1(-\pi, \pi)$ because f is.) Further, p is β-KMS with respect to the unbounded derivation δ on $\mathscr{A}$ that is locally defined in this way. We wish to show that $p \in \mathscr{E}_\beta$.

Since p is a product state, it follows from [5] that p has trivial algebra at infinity. But recall that for a state in I which has trivial algebra at infinity, $V_{j,p}$ is an ordinary function and

$$\delta_p(A) = \pi_p \left(p_j \frac{\partial A}{\partial x_j} - V_{j,p}'(x_j) \frac{\partial A}{\partial p_j} \right)$$

for $A \in \mathscr{B}_j$. Since (4.2) holds for all asymptotic states with trivial algebra at infinity, we have

$$V_{j,p}(x) = \sum_{k=1}^{P} \lambda_{jk} p(f(x_k - x)) .$$

By our definition of p, we have

$$p(f(x_k - x)) = \frac{1}{\int\limits_{-\pi}^{\pi} e^{-\beta g_k(y)} dy} \int\limits_{-\pi}^{\pi} f(y - x) e^{-\beta g_k(y)} dy .$$

Since the g_j are assumed to satisfy the consistency relations, it follows that

$$V_{j,p}(x) = g_j(x) .$$

Hence $\delta_p(A) = \pi_p(\delta(A))$ for all $A \in \mathscr{B}$, so the statement that p is β-KMS with respect to δ is exactly the statement that $p \in \mathscr{E}_\beta$. This completes the proof.

Combining this theorem with the remark preceding it, we obtain

5.5 Theorem. *Suppose the interaction is of the* P*th kind. Then there is a one-to-one correspondence between states in $\mathscr{E}_\beta$ with trivial algebra at infinity and P-tuples*

$(g_1, \ldots, g_p)$ of bounded, real-valued, measurable functions satisfying the system (6.4) *of equations.*

Thus, the computation of β-equilibrium states with trivial algebra at infinity is reduced to a very concrete problem. Notice that there is one and only one solution $(g_1, \ldots, g_p)$ whose entries are constant, namely $g_j = a_0 \sum_{k=1}^{P} \lambda_{jk}$. $\left(\text{Recall that } a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)dy.\right)$ This state is a "free" state, since the effective force fields in which the particles are moving are zero. It should not be surprising that there is such a state, since the interaction we are considering is "infinitely weak". We note that the constant solution is independent of β, while the corresponding product state

$$p(A) = \frac{1}{\int_{-\infty}^{\infty} e^{-\frac{1}{2}\beta p_j^2} dp_j} \int_{-\infty}^{\infty} A e^{-\frac{1}{2}\beta p_j^2} dp_j, \qquad A \in \mathscr{A}_j,$$

is dependent on β. In particular, $\mathscr{E}_\beta \neq \emptyset$ for all $\beta > 0$.

In Section 6 we will give an example where the free solution is not the only solution. We note at this point that if $(g_1, \ldots, g_p)$ is a solution with nonconstant entries, then all of its spatial translates modulo 2π are also solutions.

Our characterization of β-equilibrium states trivial at infinity for interactions of the Pth kind enables us to prove the following theorem:

5.6. Theorem. *Suppose that the interaction is of the* Pth *kind and* $p \in \mathscr{E}_\beta$. *Then* p *is an extreme point of* $\mathscr{E}_\beta$ *if and only if* p *has trivial algebra at infinity.*

The proof of this result is identical to the proof of Theorem 7.7 in [1], so we omit it. Thus the decomposition at infinity of a β-equilibrium state is also an extremal decomposition. Whether the extremal decomposition is unique or not is an open question.

We close this section with the statement of a result which is stronger than the converse of Theorem 5.2.

5.7. Proposition. *Let* μ *be a probability measure concentrated on* $\mathscr{E}_\beta$. *Then the resultant of* μ *lies in* $\mathscr{E}_\beta$.

Proof. Let p be the resultant of μ. Since μ is concentrated on I, it follows by dominated convergence that the limits in question exist on the dense subspace $\pi_p(\mathscr{A})\Phi_p$ of $\mathscr{H}_p$, therefore on $\mathscr{H}_p$ by uniform boundedness. Thus $p \in I$.

Let $A, B \in \mathscr{B}$. For every $\sigma \in \mathscr{E}_\beta$ we have

$$(\delta_\sigma(A)\pi_\sigma(B)\Phi_\sigma, \Phi_\sigma)\beta = \sigma(\{A, B\})$$

and

$$\delta_\sigma(A) = \operatorname*{s-lim}_{N \to \infty} \pi_\sigma(\delta_N(A)),$$

so

$$\lim_{N\to\infty} \sigma(\delta_N(A)B)\beta = \sigma(\{A, B\}).$$

By dominated convergence it follows that

$$\lim_{N\to\infty} \int_{\mathscr{E}_\beta} \sigma(\delta_N(A)B)d\mu(\delta)\beta = \int_{\mathscr{E}_\beta} \sigma(\{A, B\})d\mu(\sigma).$$

Therefore,

$$\begin{aligned}(\delta_p(A)\pi_p(B)\Phi_p, \Phi_p) &= \lim_{N\to\infty} p(\delta_N(A)B) \\ &= \lim_{N\to\infty} \int_{\mathscr{E}_\beta} \sigma(\delta_N(A)B)d\mu(\sigma)\beta \\ &= \int_{\mathscr{E}_\beta} \sigma(\{A, B\})d\mu(\sigma) \\ &= p(\{A, B\}).\end{aligned}$$

Thus $p \in \mathscr{E}_\beta$.

Section 6. Phase Transitions

Our next step is to examine the bifurcation theory with respect to the inverse temperature β. We have the following result:

6.1. Theorem. *Suppose that the interaction is of the* P*th kind. Then there is only one β-equilibrium state for β sufficiently small.*

Proof. By the decomposition theorem, it is sufficient to show that there is only one β-equilibrium state with trivial algebra at infinity for β small enough. By our characterization of such states and the remarks of the preceding section, it is enough to prove that the free solution is the only solution to our system of equations for β sufficiently small.

Notice that if $(g_1, \ldots, g_p)$ solves the equations for f, then for a real constant C, $\left(g_1 + C\sum_{k=1}^{P} \lambda_{1k}, \ldots, g_p + C\sum_{k=1}^{P} \lambda_{pk}\right)$ solves the equations for $f+C$, so the abundance of solutions is unaffected by the assumption that

$$\int_{-\pi}^{\pi} f(x)dx = 0.$$

In this case $(0, \ldots, 0)$ is the free solution. Let $(g_1, \ldots, g_p)$ be an arbitrary β-solution. We need to find a $\beta_0 > 0$, independent of $(g_1, \ldots, g_p)$, such that if $\beta \leqq \beta_0$, then $g_j = 0$ for $1 \leqq j \leqq P$.

We have

$$|g_j(x)| \leqq \sum_{k=1}^{P} |\lambda_{jk}| \frac{1}{\int_{-\pi}^{\pi} e^{-\beta g_k(y)} dy} \left| \int_{-\pi}^{\pi} f(y-x) e^{-\beta g_k(y)} dy \right|$$

$$= \sum_{k=1}^{P} |\lambda_{jk}| \frac{1}{\int_{-\pi}^{\pi} e^{-\beta g_k(y)} dy} \left| \int_{-\pi}^{\pi} f(y-x)(e^{-\beta g_k(y)} - 1) dy \right|$$

$$= \sum_{k=1}^{P} |\lambda_{jk}| \frac{1}{\int_{-\pi}^{\pi} e^{-\beta g_k(y)} dy} \left| \int_{-\pi}^{\pi} f(y-x)(-g_k(y)) \int_{0}^{\beta} e^{-s g_k(y)} ds dy \right|$$

$$\leqq \sum_{k=1}^{P} |\lambda_{jk}| \, \|f\|_\infty \, \|g_k\|_\infty \frac{1}{\int_{-\pi}^{\pi} e^{-\beta g_k(y)} dy} \int_{-\pi}^{\pi} \int_{0}^{\beta} e^{-s g_k(y)} ds dy$$

$$\leqq \sum_{k=1}^{P} |\lambda_{jk}| \, \|f\|_\infty \, \|g_k\|_\infty \frac{2\pi}{2\pi e^{-\beta \|g_k\|_\infty}} \int_{0}^{\beta} e^{s\|g_k\|_\infty} ds$$

$$\leqq \beta \|f\|_\infty \sum_{k=1}^{P} |\lambda_{jk}| \, \|g_k\|_\infty \, e^{2\beta \|g_k\|_\infty}.$$

Hence

$$\|g_j\|_\infty \leqq \beta \|f\|_\infty \sum_{k=1}^{P} |\lambda_{jk}| \, \|g_k\|_\infty \, e^{2\beta \|g_k\|_\infty}.$$

But we also have the crude estimate

$$\|g_k\|_\infty \leqq \|f\|_\infty \sum_{r=1}^{P} |\lambda_{kr}|,$$

so

$$\|g_j\|_\infty \leqq \beta \|f\|_\infty \sum_{k=1}^{P} |\lambda_{jk}| \, \|g_k\|_\infty \, e^{2\beta \|f\|_\infty \sum_{r=1}^{P} |\lambda_{kr}|}$$

It is clear from this estimate that the $g_k = 0$ for β small enough.

In considering interactions of the 1st kind, the β-equilibrium states with trivial algebra at infinity correspond to functions g satisfying the equation

$$g(x) = \frac{\lambda}{\int_{-\pi}^{\pi} e^{-\beta g(y)} dy} \int_{-\pi}^{\pi} f(y-x) e^{-\beta g(y)} dy, \tag{6.1}$$

where λ is a constant depending upon the strength of the "infinitely weak" interaction. We wish to find an interaction and a temperature for which a phase transition occurs—that is, a function f with the required properties and a number $\beta > 0$ for which (6.1) has two solutions. For each $\beta > 0$, (6.1) has a constant solution $g = \lambda a_0$, so it is necessary only to find an f and β such that (6.1) has a non-constant solution. Taking the Fourier transform of (6.1), we get

$$b_n = \frac{\lambda a_{-n}}{\int\limits_{-\pi}^{\pi} e^{-\beta g(y)} dy} \int\limits_{-\pi}^{\pi} e^{-iny} e^{-\beta g(y)} dy, \tag{6.2}$$

where $\{b_n | n \in Z\}$ is the sequence of Fourier coefficients for g. Thus we need to *find* a sequence $\{a_n\}$ with the appropriate properties such that (6.2) holds and $b_k \neq 0$ for some $k \neq 0$, so we simply *pick* a convenient $\{b_n\}$ and solve for $\{a_n\}$:

$$a_n = \frac{b_{-n}}{\lambda \int\limits_{-\pi}^{\pi} e^{iny} e^{-\beta g(y)} dy} \int\limits_{-\pi}^{\pi} e^{-\beta g(y)} dy. \tag{6.3}$$

The only remaining obstacle is the possibility that the denominator may vanish for some n where $b_{-n} \neq 0$. We must find a $\beta > 0$ such that this does not happen for any such n. In this way we will have found an f and a β which yield as a solution the non-constant function g that was picked to begin with.

To this end, let $b_1 = b_{-1} = -\frac{1}{2}$ and $b_n = 0$ for all other n. Then $g(x) = -\cos x$. We need to show that there is a $\beta > 0$ such that neither $\int\limits_{-\pi}^{\pi} e^{iy} e^{-\beta g(y)} dy$ nor $\int\limits_{-\pi}^{\pi} e^{-iy} e^{-\beta g(y)} dy$ vanish. Since one is the complex conjugate of the other, it is enough to show that the sum does not vanish, so the problem is reduced to showing that

$$\int\limits_{-\pi}^{\pi} \cos y e^{\beta \cos y} dy \neq 0.$$

But the integral is also the integral of $\cos y$ with respect to the measure $e^{\beta \cos y} dy$, whose density $e^{\beta \cos y}$ is increasing for $-\pi \leqq y \leqq 0$ and decreasing for $0 \leqq y \leqq \pi$. By the nature of $\cos y$, the integral cannot vanish for *any* value of β, so we have what we want. Notice that this does not contradict Theorem 6.1 because the estimate involves $\|f\|_\infty$, and for different values of β we have different functions f yielding the same function g.

By reasoning parallel to the reasoning at the conclusion of [1], this example of a phase transition also provides an example where the decomposition at infinity breaks symmetry with respect to spatial translations.

References

1. Battle, G.: Dynamics and phase transitions for a continuous system of quantum particles in a Box (preprint)
2. Gallavotti, G., Verboven, E.: Il Nuovo Cimento **28**, 274 (1975)
3. Aizenman, M., Gallavotti, G., Goldstein, S., Lebowitz, J.: Commun. math. Phys. **48**, 1 (1976)
4. Ruelle, D.: Symmetry breakdown in statistical mechanics. In: Kastler, D. (Ed.): Cargese lectures in physics, Vol. 4, pp. 169—194. New York: Gordon and Breach 1970
5. Sakai, S.: C^*-algebras and W^*-algebras. Berlin-Heidelberg-New York: Springer 1971

Communicated by J. L. Lebowitz

Received December 20, 1976