Non-Equilibrium Dynamics of One-dimensional Infinite Particle Systems with a Hard-Core Interaction

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Abstract. An infinite system of Newton's equation of motion is considered for one-dimensional particles interacting by a finite-range hard-core potential of singularity like an inverse power of distance between the hard cores. Existence of limiting solutions is proved for initial configurations of finite specific energy and the semigroup of motion is constructed if energy fluctuations near infinity increase only as a small power of distance from the origin. In this case uniqueness of solutions is also proved and the solution is a weakly continuous function of initial data. The allowed set of initial configurations carries a wide class of probability measures including Gibbsian fields with different potentials. In the absence of hard cores limiting solutions are constructed for initial configurations with a logarithmic order of energy and density fluctuations.

1. Introduction

The aim of this paper is to extend the results by Harris [1] and Lanford [2] on the existence of non-equilibrium dynamics of infinitely many particles on the line. The interaction is given by a translation invariant hard-core pair potential U of finite range, thus the interparticle force is just the negative derivative of U and the equations of motion are those of classical mechanics with this conservative force. As it is well known (see [1, 2, 5]) a solution to such an infinite system of differential equations exists only for a relatively small set of initial configurations, and the solution, if any, is not unique in the usual sense.

For the purposes of non-equilibrium statistical mechanics the semigroup of motion should be constructed in a set of configurations large enough to carry a class of Gibbsian fields. Let us remark that equilibrium dynamics that is a semigroup acting in the support of a Gibbsian field associated with the same potential as the dynamics itself has been constructed by several authors ([3–7]) while Lang [8] considers a first order system with additional white-noise terms. As regards non-equilibrium dynamics, in Harris' collision model (zero-range interaction) U is formally the δ-function, Lanford [2] treats the case of not necessarily symmetric,

smooth interactions and proves existence of solutions for initial configurations of a logarithmic order of velocity and density fluctuations.

In this paper the main condition on the interaction potential is that its singularity near the hard core is not stronger than that of an inverse power at 0. Solutions associated with initial configurations of finite specific energy are constructed as weak limits of solutions to finite subsystems. Assuming that U is repulsive enough we show that the equations of motion satisfy a quasi-Lipschitz condition; hence we deduce that solutions form a reversible semigroup e. g. in the set $\bar{\Omega}_0$ of configurations with a logarithmic order of energy fluctuation, and the motion is a uniquely determined weakly continuous function of its initial data. $\bar{\Omega}_0$ is of full measure with respect to any Gibbsian field with potential $\tilde{U}$ such that the singularity of $\tilde{U}$ is not weaker than that of U. In the absence of hard cores solutions are constructed in the subset Ω_0 of $\bar{\Omega}_0$ characterized by a logarithmic order of density fluctuation. In the proofs the conservative nature of interparticle forces is essentially exploited, the a priori bound implying compactness of the sequence of solutions to finite subsystems is deduced from the law of energy conservation. Such ideas will be applied to prove two-dimensional existence theorems in a forthcoming paper [9] by the authors.

2. Preliminaries

First we specify notation and terminology used throughout this paper. In general, x_i and v_i, $i \in I$ denote the position and the velocity of the i-th particle; I is the set of integers, x_i and v_i are real numbers. Let Ω denote the set of all locally finite labelled configurations satisfying the hard-core condition $|x_i - x_j| > \delta$ if $i \neq j$; $\delta \geq 0$ will be the hard-core diameter of our potential U. We may and do assume that particles are numbered in an increasing order of positions so that a configuration $\omega \in \Omega$ is a doubly infinite sequence $\omega = \{(x_i, v_i); i \in I\}$ of pairs of real numbers such that $x_{i+1} > x_i + \delta$. In the special case of $\delta = 0$ we need the condition of local finiteness as well, i.e. only such configurations are considered where the sequence of positions has no limit points. Configurations differing only in the way of enumeration of particles are usually identified, but the equations of motion and some quantities will be formulated in terms of labelled configurations. If necessary, the position and the velocity of the i-th particle in ω will be denoted as $x_i = x_i(\omega)$ and $v_i = v_i(\omega)$, respectively. The configuration space Ω is equipped with the weak topology, i.e. $\lim \omega_n = \omega$ means that $\lim_n x_i(\omega_n) = x_i(\omega)$ and $\lim_n v_i(\omega_n) = v_i(\omega)$ for each i.

Trajectories in Ω are always parametrized by the time t from 0 to $+\infty$, the set of weakly continuous trajectories $\omega_t = \varphi(t)$, $\omega_t \in \Omega$ for $t \geq 0$ will be denoted by $\Omega[0, \infty)$. Due to the continuity of individual trajectories $x_i(\omega_t)$, $v_i(\omega_t)$, particles along a trajectory $\omega_t \in \Omega[0, \infty)$ preserve their initial numbering. Convergence $\lim \omega_t^n = \omega_t$ in $\Omega[0, \infty)$ is defined by

$$\lim_n \sup_{0 \leq t \leq T} [|x_i(\omega_t^n) - x_i(\omega_t)| + |v_i(\omega_t^n) - v_i(\omega_t)|] = 0$$

for each $T > 0$ and $i \in I$ without any uniformity assumption. A family $\omega_t = \varphi(t, \omega)$, $\omega \in \Omega'$, $t \geq 0$ of weakly continuous trajectories is a reversible semigroup in $\Omega' \subset \Omega$ if

$\omega_t \in \Omega'$ for each t, $\varphi(0, \omega) = \omega$ and $\varphi(t + s, \omega) = \varphi(t, \varphi(s, \omega))$, $\varphi(t, [\varphi(t, \omega)]^+) = \omega^+$, $t, s \geq 0$ are identities; in the second one $^+$ indicates that the velocities have been reversed, i.e. $x_i(\omega^+) = x_i(\omega)$, $v_i(\omega^+) = -v_i(\omega)$ for each i.

Suppose now that we are given a translation invariant symmetric pair potential U of hard-core diameter $\delta \geq 0$ and range $R > \delta$ that is an even function $U = U(x)$, $-\infty < x < +\infty$, such that $U(x) = +\infty$ if $|x| \leq \delta$, $|U(x)| < +\infty$ if $|x| > \delta$, $U(x) = 0$ if $|x| > R$, further U is continuously differentiable for $|x| > \delta$ and $\lim U(x) = +\infty$ if x approaches δ. For convenience we assume that the derivative U' of U satisfies a local Lipschitz condition for $|x| > \delta$ that means finiteness of

$$L(u) = \sup_{x, y} \left\{ \left| \frac{U'(x) - U'(y)}{x - y} \right| ; \ U(x) \leq u, U(y) \leq u, x \neq y \right\}$$

for $u < +\infty$. A weakly continuous trajectory $\omega_t \in \Omega[0, \infty)$ is called a (global) solution to the equations of motion with initial condition $\omega \in \Omega$ if $\omega_0 = \omega$, the individual trajectories $x_i = x_i(\omega_t)$, $v_i = v_i(\omega_t)$, $t \geq 0$ are differentiable and satisfy the equations of motion

$$\frac{dv_i}{dt} = -\sum_{j \neq i} U'(x_i - x_j), \quad \frac{dx_i}{dt} = v_i; \quad i \in I \tag{1}$$

for $t \geq 0$. We have assumed here that particles are of unit mass. Let us remark that the sum in (1) is always finite since only locally finite configurations are considered.

Owing to the law of energy conservation the initial value problem has a unique solution to any finite subsystem of (1), thus we have a possibility to define a sequence $\varphi_n(t, \omega) \in \Omega[0, \infty)$, $n = 1, 2, \ldots$ of approximate solutions for each initial configuration $\omega \in \Omega$, e.g. as follows. Consider the $2n + 1$ pairs of equations from (1) associated with particles numbered by i, $|i| \leq n$ and let $\{x_i(t), v_i(t); |i| \leq n\}$ denote the solution of this system with initial data $x_i(0) = x_i(\omega)$, $v_i(0) = v_i(\omega)$, $\omega \in \Omega$; then φ_n is defined by

$$\begin{aligned}
&x_i(\varphi_n) = x_i(t), \quad v_i(\varphi_n) = v_i(t) \quad \text{if} \quad |i| \leq n, \\
&x_i(\varphi_n) = x_n(t) + (i - n) R \quad \text{if} \quad i > n, \\
&x_i(\varphi_n) = x_{-n}(t) + (i + n) R \quad \text{if} \quad i < -n, \\
&v_i(\varphi_n) = 0 \quad \text{if} \quad |i| > n.
\end{aligned} \tag{2}$$

It is easy to check that $x_i(\varphi_n)$ and $\dfrac{d}{dt} x_i(\varphi_n)$ satisfy the equations of motion for each i, further $\varphi_n(t + s, \omega) = \varphi_n(t, \varphi_n(s, \omega))$, $\varphi_n(t, [\varphi_n(t, \omega)]^+) = \varphi_n(0, \omega^+)$ if $t, s \geq 0$.

A solution ω_t is called a limiting solution if we can select a subsequence n_k such that $\lim_k \varphi_{n_k}(t, \omega_0) = \omega_t$ in the topology of $\Omega[0, \infty)$. To prove existence of a limiting solution with initial configuration ω, compactness of the sequence of approximate solutions is needed that means a bound $|v_i(\varphi_n(t, \omega))| \leq q_i(t, \omega)$ such that q_i does not depend on n and it is bounded in finite intervals of time. The basic problem is to find a large enough set in Ω where such an a priori bound holds.

3. Main Result

Integrals of motion as total energy and particle number play a decisive role in the study of system (1). Let $f_0 = f_0(x, \sigma)$ denote the indicator of $[-\sigma, \sigma]$, $\sigma > 0$, i.e. $f_0 = 1$ if $|x| \leqq \sigma$ and $f_0 = 0$ otherwise, further $x_i = x_i(\omega)$, $v_i = v_i(\omega)$, $\omega \in \Omega$; then

$$N(\omega, \mu, \sigma) = \sum_{i \in I} f_0(x_i - \mu, \sigma) \tag{3}$$

and

$$H(\omega, \mu, \sigma) = \sum_{i \in I} f_0(x_i - \mu, \sigma) \tfrac{1}{2} \left[v_i^2 + \sum_{j \neq i} f_0(x_j - \mu, \sigma) U(x_i - x_j) \right] \tag{4}$$

are the particle number and the total energy of ω in the interval $[\mu - \sigma, \mu + \sigma]$.

An essential difference between the special cases $\delta > 0$ and $\delta = 0$ is that the number of nonzero summands in $U_i = \sum_{j \neq i} U(x_i - x_j)$ can't exceed $2\dfrac{R}{\delta}$ if $\delta > 0$ while it may be arbitrarily large if $\delta = 0$. From now on we assume that $\delta > 0$, the case $\delta = 0$ will be discussed in Section 8.

Since $\min\limits_x U(x) \geqq -\dfrac{a}{b}$ with $a > 0$, $b > 0$ as specified in condition (E) below, any partial sum V_i of U_i satisfies $V_i \geqq -2\dfrac{aR}{b\delta}$, thus putting $B = \dfrac{aR}{b\delta}$ we obtain a nonnegative version

$$Q(\omega, \mu, \sigma) = B N(\omega, \mu, \sigma) + H(\omega, \mu, \sigma) \tag{5}$$

of the total energy. The reason why we prefer Q rather than H is that Q is a nondecreasing interval function;

$$|v_i| \leqq [2Q(\omega, x_i, \sigma)]^{1/2}, \tag{V}$$

$$U(x_i - x_j) \leqq Q(\omega, x_i, \sigma) \quad \text{if} \quad |x_i - x_j| \leqq \sigma, \tag{D}$$

therefore any bound on Q implies bounds on velocities and for interparticle distances as well. Let us remark that Q is just the potential energy in the presence of a negative chemical potential $-B$.

The order of energy fluctuation at infinity (boundary condition in terms of Lanford [5]) is characterized by a moderately increasing (MI) function g.

Definition 1. A continuous even function $g = g(x)$ is called an MI-function if g is increasing and concave for $x \geqq 0$ with right derivative $g'(0 + 0) = 1$; for convenience we assume that $g(0) = 3R + 16(1 + b)^2$, b is given in (E) below. We say that the energy fluctuation of $\omega \in \Omega$ is only of order g if

$$\bar{Q}_g(\omega) = \sup_{m \in I} \sup_{\sigma \geqq g(x_m)} \frac{1}{2 \cdot \sigma} Q(\omega, x_m, \sigma) \tag{6}$$

is finite, the set of such configurations will be denoted by $\bar{\Omega}_g$.

The particular cases $\bar{g}(x) = g(0) + |x|$ and $g_0(x) = g(0) + \log(1 + |x|)$ are of special interest, the corresponding objects will be denoted by $\bar{Q}, \bar{\Omega}$ and $\bar{Q}_0, \bar{\Omega}_0$, respectively; $\log u$ denotes logarithm to base e. It is plain that $\bar{\Omega}_g \subset \bar{\Omega}_h$ if $g \leqq h$, thus $\bar{\Omega}_g \subset \bar{\Omega}$ for each MI-function g. Limiting solutions will be constructed for initial configurations

$\omega \in \bar{\Omega}$ and we shall see that the order of energy fluctuation, i.e. the sets $\bar{\Omega}_g$, $\bar{\Omega} - \bar{\Omega}_g$, is an invariant of motion.

Let us remark that

$$\bar{Q}_g(\omega) \leq \sup_{\mu \in I} \sup_{\sigma \geq g(|\mu| - 1)} \frac{1}{2 \cdot \sigma} Q(\omega, \mu, 1 + \sigma) \tag{7}$$

for each MI-function g. Further, $Q(\omega, \mu, \sigma) \leq Q(\omega, 0, |\mu| + \sigma)$ and $|\mu| \leq \sigma$ in (7) if $g = \bar{g}$, thus the Ergodic theorem [13] implies that $\bar{\Omega}$ is of full measure with respect to any translation invariant field such that $H(\omega, 0, R)$ has a finite expectation. On the other hand, even $\bar{\Omega}_0$ carries a wide class of Gibbsian fields:

Proposition 1. *If $\tilde{P}$ is the probability measure of a Gibbsian field associated with a finite-range pair potential $\tilde{U}$ such that*

$$U(x) \leq \tilde{a} + \tilde{b} \tilde{U}(x) \tag{8}$$

with some constants $\tilde{a}$ and $\tilde{b}$, then $\tilde{P}(\bar{\Omega}_0) = 1$.

The proof of this statement is very simple because using (8) and taking into account that $U(x) = +\infty$ if $|x| \leq \delta$, a direct calculation shows that

$$\int_{\Omega} \exp(\lambda Q(\omega, \mu, \sigma)) P(d\omega) \leq e^{2c\sigma}$$

with some $\lambda > 0$, $c < +\infty$ that do not depend on μ and σ. Hence $P\left[Q(\omega, \mu, \sigma) > \dfrac{2\sigma K}{\lambda}\right]$
$\leq \exp[(c - K)2\sigma]$, thus Proposition 1 follows from (7) by the Borel-Cantelli lemma [13]. It seems that this proposition can be generalized e.g. for multibody potentials of infinite range of interaction, but condition (8) cannot be completely removed.

We can't disprove that even in the presence of limiting solutions there may exist further solutions oscillating rather violently at remote places of the line. Anyway, only such solutions are tractable where the order of energy fluctuations is limited not only at the initial moment of time, but later as well.

Definition 2. We say that a solution ω_t is g-tempered if $\bar{Q}_g(\omega_t)$ is bounded in finite intervals of time, in the particular case $g(x) = g(0) + |x|$ the prefix g will be omitted.

In the proof of existence we need an additional regularity condition on the interaction: There exist such constants a and b that

$$|(x - \delta) U'(x)| \leq a + b U(x) \quad \text{if} \quad x > \delta. \tag{E}$$

Observe that (E) implies $U(x) \leq a_0 + b_0(x - \delta)^{-b}$ for $x > \delta$ and this relation is sufficient for (E) if U is convex in a small neighbourhood $(\delta, \delta + \varepsilon)$ of δ. Although it is not everywhere necessary, the validity of (E) will be assumed throughout this paper. The main result is

Theorem 1. *For each $\omega \in \bar{\Omega}$ there exists a limiting solution ω_t with initial condition $\omega_0 = \omega$, and any limiting solution $\bar{\omega}_t$ is tempered if $\bar{\omega}_0 \in \bar{\Omega}$.*

On the structure of tempered solutions we prove the following.

Theorem 2. *Let g denote an MI-function, then a tempered solution ω_t is g-tempered if and only if $\omega_0 \in \bar{\Omega}_g$, further if $\omega_t \in \bar{\Omega}_g$ at least for one value of $t > 0$, then $\omega_t \in \bar{\Omega}_g$ for each $t \geq 0$.*

Theorems 1 and 2 will be proved in Sections 5 and 6. Uniqueness of g-tempered solutions will be deduced from a quasi-Lipschitz condition of the following kind. We say that the right hand side of (1) satisfies a g-Lipschitz condition with an MI-function g if

$$\lim_n n^{-2} L(vg(n)) = 0 \quad \text{for each} \quad v > 0, \tag{U}$$

where $L = L(u)$ is the Lipschitz constant of U' in the domain $\{x; U(x) \leq u\}$, L is defined before (1). The principal content of (U) is that the singularity of U is not very weak, i.e. the interparticle force is repulsive enough. For example, if U' is concave in a neighbourhood $(\delta, \delta + \varepsilon)$ of δ and $U(x) \geq b_1(x - \delta)^{-c}$ in this interval, further $b_1 > 0$, $0 < c \leq 2$ and $0 < \lambda \leq \dfrac{2c}{c+2}$, then (U) holds with $g_\lambda(x) = g(0) + \dfrac{1}{\lambda}[(1+|x|)^\lambda - 1]$.

Theorem 3. *Suppose that the right hand side of (1) satisfies a g-Lipschitz condition with an* MI-*function g. Then for each $\omega \in \bar\Omega_g$ there exists the limit $U_t\omega = \lim_n \varphi_n(t, \omega)$*

of approximate solutions φ_n in the topology of $\Omega[0, \infty)$. U_t is a reversible semigroup of g-tempered solutions, and the restriction of $U_t\bar\Omega_q \mapsto \Omega[0, \infty)$ to $\bar\Omega_g^q = \{\omega; \bar Q_g(\omega) \leq q\}$ is a continuous function of $\omega \in \bar\Omega_g^q$. Further, $\omega_t = U_t\omega_0$ is the only tempered solution with initial condition $\omega_0 \in \bar\Omega_g$.

The proof of this result will be given in Section 7.

4. On the Idea of the Proof

The crucial step in the proof of Theorem 1 is to verify compactness of the sequence φ_n of approximate solutions. For this we deduce an a priori bound $\bar Q(\varphi_n(t, \omega)) \leq q(t, \omega)$ such that q is independent of n and continuous in $t \geq 0$ for each $\omega \in \bar\Omega$. To explain on a heuristic level that how such a q can be obtained, we replace $\bar Q$ by $\bar Q_{00}$ that corresponds to the constant function $g_{00}(x) = g(0)$. Although this situation is not very interesting because $\bar Q_{00}(\omega) = +\infty$ a.s. in nondegenerate Gibbsian fields, the general case is more difficult only in its technical aspects.

Consider $Q_t = Q[\varphi_n(t, \omega), x_m(\varphi_n(t, \omega)), r(t)]$ that is the quantity of Q in the $r(t)$-neighbourhood V_t of the m-th particle of $\varphi_n(t, \omega)$. In view of the law of energy conservation, the differential gain of Q_t consists of two summands: (i) transported energy Q_{tp} that amounts to the sum of chemical, kinetic and potential energy carried by particles crossing the boundary of V_t and (ii) transferred energy Q_{tf} that is the work of external forces on internal particles, this is a boundary effect, too. Let $0 \leq t \leq T$, $q_n(t, \omega) = \bar Q_{00}(\varphi_n(t, \omega))$ and

$$r(t) = \sigma + c \int_t^T \sqrt{q_n(s, \omega)}\, ds.$$

Since all velocities in ω are bounded by $2\sqrt{g(0)\bar Q_{00}(\omega)}$, for $c > 2\sqrt{g(0)}$ we have $-r'(t) \geq |v_i(\varphi_n(t, \omega))|$, thus new particles cannot enter into V_t so that $Q_{tp} \leq 0$. One

would expect that even $Q_{tp}+Q_{tf}\leq 0$ holds if c is large enough, i.e. $Q_T\leq Q_0$, or more exactly,

$$Q[\varphi_n(T,\omega),x_m(\varphi_n(t,\omega)),\sigma]\leq Q_0\leq Q\left[\omega,x_m(\omega),\sigma+c\int_0^T \sqrt{q_n(s,\omega)}\,ds\right] \tag{9}$$

for each $T\geq 0$. This heuristic argument remains in force in the d-dimensional space, too, where V_t is the sphere of radius $r(t)$ and centre $x_m(\varphi_n(t,\omega))$, further $\bar{Q}_{00}$ is the supremum of the quantities: Q per volume; the volume is bounded away from zero in the definition of $\bar{Q}_{00}$. Dividing by the volume of V_t and taking the supremum of both sides, (9) turns into

$$q_n(T,\omega)\leq\bar{Q}_{00}(\omega)\left[1+\bar{c}\int_0^T \sqrt{q_n(s,\omega)}\,ds\right]^d, \tag{10}$$

where $\bar{c}$ depends only on c and d. It is easy to check that the maximal solution $q(t,\omega)$ of this integral inequality is bounded in finite intervals of time if $d\leq 2$ and $\bar{Q}_{00}(\omega)<+\infty$, while $q(t,\omega)=+\infty$ for large values of t if $d>2$, see Bihari's Inequality in [10,11]. Therefore we have a possibility to prove existence of solutions if $d=1$ or $d=2$, but such an argument does not work if $d\geq 3$. Moreover, some examples indicate that (9) cannot be improved essentially, so that there is no hope for extending Theorem 1 in its deterministic form to three-dimensional particle systems.

Let us now turn to our main hypothesis $Q_{tp}+Q_{tf}\leq 0$. Unfortunately this is not true in general because Q_{tf} may be arbitrarily large if many particles are close to each other near the boundary of V_t. It is possible, however, to replace Q by such an additive, energy-like quantity W that the transport of W overbalances its transfer even if particles interact very strongly near the boundary of V_t. For this our system will be transformed into a new one by removing the intervals occupied by the hard cores, the definition of W for this new system will be essentially the same as that of Q for the old one. Due to this contraction of the system, the formal transport of energy has been considerably enlarged in case of strongly interacting configurations so that we shall have $W_{tp}+W_{tf}\leq 0$. In the absence of hard cores and in the two-dimensional space such a method does not work, some new ideas are needed then, see Section 8 and [9].

5. The a priori Bound

In this section we translate into mathematics the ideas outlined above. Let $f=f(x,\sigma)$ denote such a continuously differentiable version of the indicator f_0 of $[-\sigma,\sigma]$ that the partial derivatives $f_1=\dfrac{\partial f}{\partial x}$ and $f_2=\dfrac{\partial f}{\partial \sigma}$ are continuous in the domain $-\infty<x<+\infty$, $\sigma>0$, further

    (i) $0\leq f(x,\sigma)\leq 1$ for each x and $\sigma>0$,
    (ii) $f(x,\sigma)=1$ if $|x|\leq\sigma$, $f(x,\sigma)=0$ if $|x|\geq\sigma+2R$,
    (iii) $|f_1(x,\sigma)|\leq f_2(x,\sigma)$ for each x and $\sigma>0$,
    (iv) $|f_1(u,\sigma)|\leq f_2(x,\sigma)+f_2(y,\sigma)$ if $x<u<y\leq x+R$.

For example if $e(x)$ is a continuously differentiable nonincreasing function such that $e(x)=1$ if $x \leq 0$, $e(x)=0$ if $x \geq 2R$, $e(x)$ is concave for $x \leq 3\frac{R}{2}$ and convex for $x \geq \frac{R}{2}$, i.e. it is linear if $|x-R| \leq \frac{R}{2}$, then $f(x,\sigma)=e(|x|-\sigma)$ satisfies (i)–(iv). Now we introduce a smooth and additive version W of Q satisfying our hypothesis on the balance of transport and transfer. For notational convenience the reduced positions $y_i = y_i(\omega) = x_i(\omega) - i\delta$ will be used simultaneously with the original co-ordinates $x_i = x_i(\omega)$. W is defined as a function of $\omega \in \Omega$, $\sigma > 0$ and $m \in I$ by

$$W_m(\omega,\sigma) = \sum_{i \in I} f(y_i - y_m, \sigma) \left[2B + v_i^2 + \sum_{j \neq i} U(x_i - x_j) \right]. \tag{11}$$

Let us remember that the constant $B = \dfrac{aR}{b\delta}$ is so large that each term on the right hand side of (11) is nonnegative, thus W_m is a nondecreasing function of σ, and

$$\frac{\partial}{\partial \sigma} W_m(\omega,\sigma) = \sum_{i \in I} f_2(y_i - y_m, \sigma) \left[2B + v_i^2 + \sum_{j \neq i} U(x_i - x_j) \right] \tag{12}$$

is also nonnegative. Let ω_t denote a solution of (1), then differentiating W with respect to t, and exploiting $U'(x) = -U'(-x)$ and the equations of motion, an easy calculation yields

$$\frac{\partial}{\partial t} W_m(\omega_t,\sigma) = \sum_{i \in I} \left[\frac{\partial}{\partial t} f(y_i - y_m, \sigma) \right] \left[2B + v_i^2 + \sum_{j \neq i} U(x_i - x_j) \right]$$
$$+ \frac{1}{2} \sum_{i \in I} \sum_{j \neq j} [f(y_j - y_m, \sigma) - f(y_i - y_m, \sigma)] (v_i + v_j) U'(x_i - x_j). \tag{13}$$

It is very important that the sequence φ_n of approximate solutions has been constructed in such a way that (13) holds even if $\omega_t = \varphi_n(t,\omega)$. In case of a finite system we can put $f = 1$ for each x, then (13) reduces to $\dfrac{\partial W}{\partial t} = 0$ that is the law of energy conservation.

Let us remark that the first and second sum in (13), respectively, are just the intensity of transport and transfer of W, further $V \dfrac{\partial}{\partial \sigma} W_m(\omega_t,\sigma)$ is the intensity of transport that results from compressing $[x_m - \sigma, x_m + \sigma]$ with velocity V. Therefore Lemma 1 describes how transport overbalances transfer.

Lemma 1. *Let ω_t denote either a solution or an approximate solution $\varphi_n(t,\omega_0)$ defined in (2), and suppose that we are given such a bound V that*

$$\left| \frac{d}{dt} x_i(\omega_t) \right| \leq V \quad \text{if} \quad |y_i - y_m| \leq \sigma + 3R$$

holds for some m, t, σ; then

$$\frac{\partial}{\partial t} W_m(\omega_t,\sigma) \leq 2(b+1) V \frac{\partial}{\partial \sigma} W_m(\omega_t,\sigma).$$

Proof. Using property (iii) to estimate the first sum in (13) and (iv) for the second one we obtain that

$$\frac{\partial}{\partial t} W_m(\omega_t, \sigma) \leq 2 \cdot V \frac{\partial}{\partial \sigma} W_m(\omega_t, \sigma)$$

$$+ V \sum_{i \in I} \sum_{j \neq i} [f_2(y_j - y_m, \sigma) + f_2(y_i - y_m, \sigma)] |(y_j - y_i) U'(x_i - x_j)|$$

$$= 2V \left[\frac{\partial}{\partial \sigma} W_m + \sum_{i \in I} f_2(y_i - y_m, \sigma) \sum_{j \neq i} |y_i - y_j| |U'(x_i - x_j)| \right]$$

$$\leq 2V \left[\frac{\partial}{\partial \sigma} W_m + \sum_{i \in I} f_2(y_i - y_m, \sigma) \left[\frac{2aR}{\delta} + b \sum_{j \neq i} U(x_i - x_j) \right] \right];$$

at the last step (E) and $|y_i - y_j| \leq |x_i - x_j| - \delta$ were used. Since $B = \dfrac{aR}{b\delta}$, this inequality proves the statement.

The next step is to show that fluctuations of W are of the same order as that of Q. Set

$$\bar{W}_g(\omega) = \sup_{m \in I} \sup_{\sigma \geq g(x_m)} \frac{1}{2 \cdot \sigma} W_m(\omega, \sigma) \tag{14}$$

for each MI-function g and $\omega \in \Omega$; if $g(x) = g(0) + |x|$ then the simplified notation $\bar{W}(\omega)$ will be used. It is plain that $2\bar{Q}_g(\omega) \leq \bar{W}_g(\omega)$, i.e. $\bar{W}_g(\omega) < \infty$ implies $\bar{Q}_g(\omega) < \infty$; the converse statement is less trivial.

Lemma 2. *There exists an increasing function $w = w(q)$, $q \geq 0$ depending only on U such that $\bar{Q}_g(\omega) \leq q < \infty$ implies $\bar{W}_g(\omega) \leq w(q) < \infty$.*

Proof. Let $\sigma \geq g(x_m)$ and

$$\varrho = \max_i \{|x_i - x_m| ; |y_i - y_m| \leq \sigma + 2R\}, \tag{15}$$

then $W_m(\omega, \sigma) \leq 2Q(\omega, x_m, \varrho)$, thus

$$\frac{1}{2 \cdot \sigma} W_m(\omega, \sigma) \leq 2q \max \left\{ 1, \frac{\varrho}{\sigma} \right\}. \tag{16}$$

We may assume without any loss of generality that $\varrho \geq 2\sigma + 4R$ and the maximum in (15) is attained at $i = m + k$, $k > 0$, then $\varrho \leq \sigma + 2R + k\delta$, whence $\varrho \leq 2k\delta$ follows directly, thus from (D) we obtain that

$$q \geq \frac{1}{2 \cdot \varrho} Q(\omega, x_m, \varrho) \geq \frac{1}{4k\delta} \sum_{i=m+1}^{m+k} U(x_i - x_{i-1}).$$

On the other hand, let $V = V(x)$, $x \geq 0$ denote the supremum of all convex functions $h = h(x)$, $x \geq 0$ such that $h(x) \leq U(x)$ if $x \geq 0$. As a supremum of convex functions, V is convex again for $x > \delta$, and $V(\delta + 0) = +\infty$ as $U(\delta + 0) = +\infty$. Thus by Jensen's Inequality we have $4\delta q \geq V\left(\dfrac{\varrho}{k}\right)$ as $\varrho = x_{m+k} - x_m$, and this is possible only if there exists an $\varepsilon = \varepsilon(q) > 0$ such that $\varrho \geq k(\delta + \varepsilon)$. Since $\varrho \leq \sigma + 2R + k\delta$, it follows that

$k \leqq \frac{1}{\varepsilon}(\sigma + 2R)$, i.e. $\varrho \leqq \left(1 + \frac{\delta}{\varepsilon}\right)(\sigma + 2R)$. Therefore $\frac{\varrho}{\sigma}$ remains bounded if q is fixed, thus (16) implies the statement.

We are now in a position to deduce the a priori bound implying existence of solutions.

Proposition 2. *Let ω_t denote either a g-tempered solution of* (1) *or an approximate solution* $\omega_t = \varphi_n(t, \omega_0)$, *and suppose that* $\bar{Q}_g(\omega_0) \leqq q < +\infty$; *$g$ is an arbitrary MI-function. Then*

$$\bar{Q}_g(\omega_t) \leqq w(q) \exp(2t \sqrt{w(q)}) \quad \text{for} \quad t \geqq 0 \tag{Q}$$

with $w = w(q)$ as specified in Lemma 2. Further

$$\sigma_m(\omega_t) = \max_{0 \leqq s \leqq t} |x_m(\omega_s) - x_m(\omega_0)|$$

satisfies the inequality

$$[g(|x_m(\omega_0)| + \sigma_m(\omega_t))]^{1/2} \leqq [g(x_m(\omega_0)]^{1/2} + g'(|x_m(\omega_0)| + 0)(e^{t\sqrt{w(q)}} - 1) \tag{X}$$

for each m and $t \geqq 0$; $g'(u + 0)$ denotes the right hand side derivative of g.

Proof. Observe first that in view of $v_i^2 \leqq W_i(\omega_t, R)$,

$$V = V_m(\omega_t, \sigma) = [2 \bar{W}_g(\omega_t) g(|x_m(\omega_t)| + \sigma + 3R)]^{1/2}$$

satisfies the requirement of Lemma 1. Further, any approximate solution is a g-tempered one, i.e. $\bar{W}_g(\omega_t)$ is bounded in finite intervals of time; consequently the integral equation

$$r(t) = \sigma + 2(b+1) \int_t^T V_m(\omega_s, r(s)) \, ds, \quad 0 \leqq t \leqq T, \tag{17}$$

has a unique solution $r = r(t)$ for each $\sigma > 0$, $T > 0$ such that $r(T) = \sigma$ and $r(t)$ decreases so rapidly that the partial differential inequality of Lemma 1 turns into $\frac{d}{dt} W_m(\omega_t, r(t)) \leqq 0$, i.e.

$$W_m(\omega_T, \sigma) \leqq W_m(\omega_0, r(0)). \tag{18}$$

Suppose now that $\sigma \geqq g(x_m(\omega_T))$; since $\left|\frac{d}{dt} x_m(\omega_t)\right| \leqq -r'(t)$,

$$|x_m(\omega_t)| \leqq |x_m(\omega_T)| + r(t) - \sigma \quad \text{if} \quad 0 \leqq t \leqq T. \tag{19}$$

On the other hand, any MI-function satisfies

$$g(x + y) \leqq g(x) + g'(x + 0)y \leqq g(x) + y \quad \text{if} \quad x, y \geqq 0, \tag{20}$$

whence $g(x_m(\omega_0)) \leqq g(x_m(\omega_T)) + r(0) - \sigma \leqq r(0)$. Therefore $W_m(\omega_0, r(0)) \leqq 2r(0) \bar{W}_g(\omega_0)$ so that dividing by 2σ and taking the supremum of both sides in (18) we get

$$\bar{W}_g(\omega_T) \leqq \bar{W}_g(\omega_0) \sup_\sigma \frac{r(0)}{\sigma}, \tag{21}$$

where $r(0)$ and σ are related by (17).

From (19), (20) and $r(t) \leq r(0)$ we obtain that

$$g(|x_m(\omega_t)| + r(t) + 3R) \leq g(x_m(\omega_T)) + 2r(0) - \sigma + 3R$$
$$\leq 2r(0) + 3R \quad \text{if} \quad 0 \leq t \leq T,$$

thus (17) implies that

$$r(0) \leq \sigma + 4(b+1)u(T)\sqrt{r(0) + \tfrac{3}{2}R}, \tag{22}$$

where

$$u(t) = \int\limits_0^t \sqrt{\bar{W}_g(\omega_s)}\, ds.$$

Since $r(0) \geq \sigma \geq g(0) = 3R + 16(b+1)^2$, we have

$$\sqrt{r(0)} \leq \sqrt{\sigma} + 4(b+1)u(T)\left[1 + \frac{3R}{2g(0)}\right]^{1/2},$$

whence

$$\sqrt{\frac{r(0)}{\sigma}} \leq 1 + u(T) \tag{23}$$

follows by an easy calculation. Therefore (21) turns into

$$\bar{W}_g(\omega_t) \leq \bar{W}_g(\omega_0)\left[1 + \int\limits_0^t \sqrt{\bar{W}_g(\omega_s)}\, ds\right]^2, \quad t \geq 0, \tag{24}$$

where $\bar{W}_g(\omega_0) \leq w(q)$ by Lemma 2. This means that $v(t) = \sqrt{\bar{W}_g(\omega_t)}$ satisfies the integral inequality

$$v(t) \leq \sqrt{w(q)}\left(1 + \int\limits_0^t v(s)\, ds\right)$$

for each $t \geq 0$. The maximal solution of this inequality is just $q(t)$
$= \sqrt{w(q)}\exp(t\sqrt{w(q)})$ (see a variant of Gronvall's lemma in Section 4.4 of [10]), thus (Q) follows from the trivial relation $\bar{Q}_g(\omega) \leq \bar{W}_g(\omega)$.

To prove (X) observe that (V) implies

$$|x_m(\omega_t) - x_m(\omega_0)| \leq \int\limits_0^t [4\bar{Q}_g(\omega_s)g(x_m(\omega_s))]^{1/2}\, ds, \tag{25}$$

thus $\sigma_m(\omega_t)$ satisfies

$$\sigma_m(\omega_t) \leq 2\int\limits_0^t [\bar{Q}_g(\omega_s)g(|x_m(\omega_0)| + \sigma_m(\omega_s))]^{1/2}\, ds,$$

whence by Bihari's inequality (see [11] or Section 4.5. in [10]) and by (Q) we obtain that

$$\int\limits_0^{\sigma_m(\omega_t)} [g(|x_m(\omega_0)| + u)]^{-1/2}\, du \leq 2\int\limits_0^t [\bar{Q}_g(\omega_s)]^{1/2}\, ds \leq 2(e^{t\sqrt{w(q)}} - 1).$$

Multiplying this inequality by $g'(|x_m(\omega_0)|+0)$ and taking into account that $0 < g'(|x_m(\omega_0)|+u+0) \leq g'(|x_m(\omega_0)|+0)$ if $u \geq 0$ we get

$$\int_0^{\sigma_m(\omega_t)} \frac{1}{2} g'(|x_m(\omega_0)|+u+0) [g(|x_m(\omega_0)|+u)]^{-1/2} du$$

$$\leq g'(|x_m(\omega_0)|+0)(e^{t\sqrt{w(q)}}-1);$$

the integral on the left can be calculated explicitly, the result is just (X).

In the proof of uniqueness of tempered solutions the following assertion will be needed.

Proposition 3. A tempered solution ω_t is g-tempered with an MI-function g if and only if $\omega_0 \in \bar{\Omega}_g$.

Proof. $g(x) \leq g(0)+|x|$ implies that any g-tempered solution is tempered. Let ω_t denote a tempered solution with $\bar{Q}_g(\omega_0)=q<+\infty$ and define the sequence g_n, $n=1,2,\ldots$ of even functions by $g_n(x)=g(x)$ if $|x| \leq n-1$, $g_n(x)=g(n-1)+g'(n-1+0)(|x|-n+1)$ if $|x|>n-1$; each g_n is an MI-function such that $g(x) \leq g_n(x)$, whence $\bar{Q}_{g_n}(\omega_0) \leq q$ follows by the definition of $\bar{Q}$. On the other hand, for each n there exists a constant $\varrho_n < \infty$ such that $g(0)+|x| \leq \varrho_n g_n(x)$, thus the monotonicity of Q as a function of σ implies that $\bar{Q}_{g_n}(\omega_t) \leq \varrho_n \bar{Q}(\omega_t)$ for each n and $t \geq 0$; i.e. ω_t is g_n-tempered. Therefore (Q) of Proposition 2 yields

$$\frac{1}{2 \cdot \sigma} Q(\omega_t, x_m(\omega_t), \sigma) \leq \bar{Q}_{g_n}(\omega_t) \leq w(q) \exp(2t\sqrt{w(q)}) \tag{26}$$

for $t \geq 0$, $m \in I$ and $\sigma \geq g_n(x_m(\omega_t))$. Since $Q(\omega, \mu, \sigma) = \inf_{\varrho > \sigma} Q(\omega, \mu, \varrho)$ and $g(x) = \inf_n g_n(x)$, (26) holds even if $\sigma \geq g(x_m(\omega_t))$, which proves the statement.

6. Proof of Existence

Let us rewrite (1) into the concise form

$$\frac{d}{dt} v_i(\omega_t) = F_i(\omega_t), \quad \frac{d}{dt} x_i(\omega_t) = v_i(\omega_t), \quad i \in I, \tag{1'}$$

where

$$F_i(\omega) = -\sum_{j \neq i} U'[x_i(\omega) - x_j(\omega)],$$

and observe that (Q) and (X) yield such a bound

$$Q(\varphi_n(t, \omega), x_i(\varphi_n(t, \omega)), R) \leq q_i(t, \omega), \omega \in \bar{\Omega}, t \geq 0 \tag{27}$$

that q_i is a continuous function of t and does not depend on n. Hence it follows by (V) that the sequence $\varphi_n(t, \omega)$ of approximate solutions contains a subsequence converging in the topology of $\Omega[0, \infty)$ for each fixed initial configuration $\omega \in \bar{\Omega}$. Indeed, the Arzela-Ascoli theorem on the compactness of sets of continuous functions implies that for each $\omega \in \bar{\Omega}$, $i \in I$ and $T>0$ there exists such a subsequence

n_k that $x_i(\varphi_{n_k}(t,\omega))$ converges uniformly in $[0,T]$. Applying the diagonal method first for $T_n = n$ with ω and i fixed, and then for $i \in I$ with ω fixed, we can select a subsequence $\bar{n}_k$ such that each co-ordinate $x_i(\varphi_{\bar{n}_k}(t,\omega))$ converges uniformly in any of the intervals $[0,T]$, $T > 0$.

On the other hand, (26) and (D) show that interparticle distances are bounded away from δ, thus the approximate solutions remain in such a domain of Ω where each F_i is continuous. Therefore substituting into

$$v_i(\omega_t) = v_i(\omega_0) + \int_0^t F_i(\omega_s)\, ds$$

we obtain that also $v_i(\varphi_{\bar{n}_k}(t,\omega))$ converges uniformly in finite intervals of time for each i; i.e. $\varphi_{\bar{n}_k}(t,\omega)$ converges in the topology of $\Omega[0,\infty)$. Of course, $\bar{n}_k$ may depend on ω. Finally, a similar argument shows that the above limit is a solution; g-temperedness of limiting solutions with initial data $\omega_0 \in \bar{\Omega}_g$ is a direct consequence of the lower semicontinuity of $\bar{Q}_g(\omega)$ as a function of ω, cf. (Q). Thus Theorem 1 is proven.

Proof of Theorem 2. The first part of Theorem 2 is just Proposition 3. To prove the second one assume that ω_t is a tempered solution and $\omega_t \in \bar{\Omega}_g$ for $t = s > 0$; we have to show that $\omega_0 \in \bar{\Omega}_g$. Indeed, $\bar{\omega}_t = \omega_{s-t}^+$ is a solution for $0 \leq t \leq s$, and $\bar{\omega}_s = \omega_0^+ \in \bar{\Omega}$, thus in view of (27) and of Theorem 1, $\bar{\omega}_t$ can be continued as a tempered solution for $t > s$, too. Further, $\bar{\omega}_0 = \omega_s^+ \in \bar{\Omega}_g$, whence $\bar{\omega}_s \in \bar{\Omega}_g$ follows by Proposition 3, consequently $\omega_0 = \bar{\omega}_s^+$ also belongs to $\bar{\Omega}_g$.

7. Dependence of Solutions on Initial Data

In this section we derive a bound for the deviation of g-tempered solutions from each other in terms of their initial deviation. Let λ_n denote an increasing sequence of positive numbers and define $d_n(\omega,\bar{\omega})$ as

$$d_n(\omega,\bar{\omega}) = \sum_{i \in I} f(x_i, 3Rn)\, f(\bar{x}_i, 3Rn)\, [\lambda_n |x_i - \bar{x}_i| + |v_i - \bar{v}_i|]\,, \tag{28}$$

where x_i, v_i, $\bar{x}_i$, $\bar{v}_i$ denote the corresponding positions and velocities of the i-th particle in the configurations ω and $\bar{\omega}$, while $f = f(x,\sigma)$ is the function described in Section 5 by (i)–(iv). We may assume that f satisfies the additional condition that

$$|f_1(x,\sigma)| \leq K < +\infty \quad \text{for each } x \text{ and } \quad \sigma > 0. \tag{v}$$

Of course, d_n depends on the choice of correspondence for particles from ω and $\bar{\omega}$. (V) implies a trivial bound for d_n, namely if $\max\{\bar{Q}_g(\omega), \bar{Q}_g(\bar{\omega})\} \leq q < +\infty$, then an easy calculation results in

$$d_n(\omega,\bar{\omega}) \leq \frac{1}{\delta}(\delta + 6Rn + 4R)\,[\lambda_n(6Rn + 4R) + 2\sqrt{2g(3Rn + 2R)q}]\,. \tag{29}$$

A measure $D_p(\omega,\bar{\omega})$ for the deviation of ω and $\bar{\omega}$ from each other can be defined as follows; p is an arbitrary positive number, $M_n = \lambda_1 \lambda_2 ... \lambda_{n-1}$ if $n > 1$, $M_1 = 1$, and

the sequence λ_n will be specified in such a way that $\lim_n n^{-1} \lambda_n = 0$, i.e. $\lambda_n = o(n)$. Then (29) implies that

$$D_p(\omega, \bar{\omega}) = \sum_{n=1}^{\infty} \frac{p^n}{n!} M_n d_n(\omega, \bar{\omega}), \quad \omega, \bar{\omega} \in \bar{\Omega} \tag{30}$$

is finite, and if ω_n and ω belong to $\bar{\Omega}$ then $\lim_n \omega_n = \omega$ holds in the weak topology of Ω if and only if $\lim_n D_p(\omega_n, \omega) = 0$ with some $p > 0$.

Proposition 4. *Suppose that the right hand side of* (1) *satisfies a g-Lipschitz condition* (U) *with an MI-function g. Then for each $T > 0$ and $q < +\infty$ there exist a sequence $\lambda_n = \lambda_n(q, T) = o(n)$ and a positive constant $p = p(q, T)$ such that*

$$\sup_{0 \leq t \leq T} D_1(\omega_t, \bar{\omega}_t) \leq D_p(\omega_0, \bar{\omega}_0)$$

holds for any couple $\omega_t, \bar{\omega}_t$ of g-tempered solutions satisfying $\max \{Q_g(\omega_0), Q_g(\bar{\omega}_0)\} \leq q$.

Proof. Let $q(t) = w(q) \exp(2t \sqrt{w(q)})$ and suppose that $\max \{|x_i(\omega_t)|, |x_i(\bar{\omega}_t)|\} \leq (3n+2)R$, then in view of (Q) of Proposition 2 we have

$$|U'[x_i(\omega_t) - x_j(\omega_t)] - U'[x_i(\bar{\omega}_t) - x_j(\bar{\omega}_t)]|$$
$$\leq L(2q(t) g(3Rn + 3R))(|x_i(\omega_t) - x_i(\bar{\omega}_t)| + |x_j(\omega_t) - x_j(\bar{\omega}_t)|), \tag{31}$$

where $L = L(u)$ is the Lipschitz-function defined before (1). Therefore if

$$\lambda_n = \max \{[L(2q(T) g(3Rn + 3R))]^{1/2}, [g(3Rn + 3R)]^{1/2}\}, \tag{32}$$

then $\lambda_n = o(n)$ in view of (U), and for $0 \leq t \leq T$,

$$|F_i(\omega_t) - F_i(\bar{\omega}_t)| \leq \frac{2}{\delta} R \lambda_n^2 \sum |x_j(\omega_t) - x_j(\bar{\omega}_t)|, \tag{33}$$

where the sum is over such values of j that $|x_i - x_j| \leq R$ or $|\bar{x}_i - \bar{x}_j| \leq R$. Observe now that $d_n(t) = d_n(\omega_t, \bar{\omega}_t)$ is an absolutely continuous function of t and, using the properties (i), (ii), (v) of f, its derivative can be estimated almost everywhere as

$$d_n'(t) \leq 2K d_{n+1}(t) [2q(t) g(3Rn + 3R)]^{1/2}$$
$$+ \sum_{i \in I} f(x_i, 3Rn) f(\bar{x}_i, 3Rn) [\lambda_n |v_i - \bar{v}_i| + |F_i(\omega_t) - F_i(\bar{\omega}_t)|]$$
$$\leq \left[2K [2q(T) g(3Rn + 3R)]^{1/2} + \lambda_n \left(\frac{2}{\delta}\right)^2 2R^2\right] d_{n+1}(t)$$
$$\leq C \lambda_n d_{n+1}(t) \text{ a.e. in } (0, T), \tag{34}$$

where, because of $g(3Rn + 3R) \leq \lambda_n^2$, C is a positive finite constant depending on T, q, R, K, δ, and g, but C does not depend on n. Hence a.e. in $(0, T)$ we have

$$\frac{M_n}{n!} d_n'(s)(1 + Ct - Cs)^n \leq -\frac{M_{n+1}}{(n+1)!} d_{n+1}(s) \frac{d}{ds}(1 + Ct - Cs)^{n+1}$$

if $0 \leq s \leq t \leq T$ so that

$$\sum_{n=1}^{\infty} \frac{d}{ds} \left[\frac{M_n}{n!} d_n(s)(1+Ct-Cs)^n \right] \leq -Cd_1(s)$$

$$+ \sum_{n=1}^{\infty} \frac{M_n}{n!} d_n'(s)(1+Ct-Cs)^n$$

$$+ \sum_{n=1}^{\infty} \frac{M_{n+1}}{(n+1)!} d_{n+1}(s) \frac{d}{ds}(1+Ct-Cs)^{n+1} \leq 0 \tag{35}$$

for almost every s in $(0, t)$. Therefore, integrating from 0 to t we obtain that

$$D_1(\omega_t, \bar\omega_t) \leq \sum_{n=1}^{\infty} \frac{M_n}{n!}(1+Ct)^n d_n(\omega_0, \bar\omega_0) \leq D_p(\omega_0, \bar\omega_0) \tag{36}$$

if $0 \leq t \leq T$ and $p = 1 + CT$, which proves the Proposition.

Proof of Theorem 3. Since $D_p(\omega, \bar\omega) = 0$ if and only if $\omega = \bar\omega$, Proposition 4 implies uniqueness of g-tempered solutions, whence uniqueness of tempered solutions follows by Proposition 3. On the other hand, any limiting solution ω_t is g-tempered if ω_0 belongs to $\bar\Omega_g$, thus any subsequence of $\varphi_n(t, \omega)$ converges to the same limit, i.e. there exists the limit $U_t \omega = \lim \varphi_n(t, \omega)$ if $\omega \in \bar\Omega_g$. This relation implies that U_t is a reversible semigroup in $\bar\Omega_g$, the continuity of $U_t \omega$ as a function of ω follows from the bound given in Proposition 4.

8. Potentials without Hard Core

Here we discuss briefly the case of $\delta = 0$, a more detailed study of this problem will be given in [9]. The only modification of notation used in previous sections is that we put $B = 0$ in the Definition (5) of Q. Let Ω_g denote the set of such configurations $\omega \in \bar\Omega_g$ that

$$\bar N_g(\omega) = \sup_m \sup_{\sigma \geq g(x_m(\omega))} \frac{1}{2 \cdot \sigma} N(\omega, x_m(\omega), \sigma) < +\infty; \tag{37}$$

for superstable potentials $\bar Q_g(\omega) < +\infty$ implies that also $\bar N_g(\omega) < +\infty$, see [12] or [14]. In this definition g is an MI-function, but only the particular case $g(x) = g_0(x) = g(0) + \log(1+|x|)$ will be discussed; the corresponding symbols are Ω_0 and $\bar N_0(\omega)$.

Theorem 4. *For each $\omega \in \Omega_0$ there exists a limiting solution ω_t such that $\omega_0 = \omega$.*

Proof. The proof of Theorem 1 is repeated with some modifications. Let $\omega_t = \varphi_n(t, \omega)$ denote an approximate solution corresponding to $\omega \in \Omega_0$, instead of W_m the quantity

$$Z_m(\omega, \sigma) = W_m(\omega, \sigma) + \frac{a}{b} \sum_{i \in I} f(x_i - x_m, \sigma) \sum_{j \neq i} f(x_j - x_m, \sigma + 3R) \tag{38}$$

will be used. Of course, $y_i(\omega) = x_i(\omega)$ as $\delta = 0$ in this case, a and b denote the constants specified in (E), thus Z_m is a nonnegative and nondecreasing function of σ.

Further, $v_i^2(\omega) \leqq Z_i(\omega, \sigma)$ and $U(x_i(\omega) - x_j(\omega)) \leqq Z_i(\omega, \sigma)$ if $\sigma > R$, consequently an a priori bound of type (Q) for

$$\bar{Z}_g(\omega) = \sup_m \sup_{\sigma \geqq g(x_m(\omega))} \sigma^{-2} Z_m(\omega, \sigma) \tag{39}$$

is sufficient to deduce existence of limiting solutions.

Suppose now that for some t, m, and σ we have a bound

$$\left| \frac{d}{dt} x_i(\omega_t) \right| \leqq V \quad \text{if} \quad |x_i(\omega_t) - x_m(\omega_t)| \leqq \sigma + 5R; \tag{40}$$

then following the proof of Lemma 1 we obtain that

$$\frac{\partial}{\partial t} Z_m(\omega_t, \sigma) \leqq 2(b+1) V \frac{\partial}{\partial \sigma} Z_m(\omega_t, \sigma). \tag{41}$$

Let $v(t) = \int_0^t [\bar{Z}_g(\omega_s)]^{1/2} ds$, g is an MI-function. Since ω_t is an approximate solution, $v(t)$ and $v'(t)$ are continuous functions. Set $\varrho(s) = |x_m(\omega_s)| + r(s) + 5R$ for $0 \leqq s \leqq T$, $T > 0$ and observe that the solution $r(t)$, $0 \leqq t \leqq T$ of the integral equation

$$r(t) = \sigma + 2(b+1) \int_t^T g(\varrho(s)) v'(s) ds \tag{42}$$

decreases so rapidly that $\frac{d}{dt} Z_m(\omega_t, r(t)) \leqq 0$ if $0 \leqq t \leqq T$, thus $Z_m(\omega_T, \sigma) \leqq Z_m(\omega_0, r(0))$, whence

$$\bar{Z}_g(\omega_T) \leqq \bar{Z}_g(\omega_0) \sup_\sigma \left(\frac{r(0)}{\sigma} \right)^2 \tag{43}$$

follows as $\sigma \geqq g(x_m(\omega_T))$ implies $r(0) \geqq g(x_m(\omega_0))$ in the same way as in the proof of Proposition 1.

From now on we specify g as $g(x) = g_0(x) = g(0) + \log(1 + |x|)$, the corresponding quantity $\bar{Z}_g(\omega)$ will be denoted by $\bar{Z}_0(\omega)$. Since we consider approximate solutions corresponding to the same initial value problem and this initial configuration ω belongs to Ω_0, we have a finite constant $q = |\bar{Z}_0(\omega)|^{1/2}$ independent of n in $\omega_t = \varphi_n(t, \omega)$, (43) turns into

$$v'(t) \leqq q \sup_\sigma \frac{r(0)}{\sigma}. \tag{44}$$

On the other hand, $\varrho(s) \leqq |x_m(\omega_T)| + r(0) + 5R$ and $g_0(x+y) \leqq g_0(x) + g_0(y)$ if $x > 0$, $y > 0$, whence

$$g_0(\varrho(s)) \leqq g_0(x_m(\omega_T)) + g_0(r(0) + 5R) \leqq \sigma + 5R + g_0(r(0)) \leqq 3\sigma + g_0(r(0))$$

follows directly, thus

$$r(0) \leqq \sigma + 2(b+1) v(T)(3\sigma + g_0(r(0))). \tag{45}$$

Further, $\log r \leqq \sigma \log \frac{r}{\sigma} + \sigma$ if $r \geqq \sigma \geqq 1$ so that an easy calculation yields

$$\frac{r(0)}{\sigma} \leqq 1 + cv(T) \left(1 + \log \frac{r(0)}{\sigma} \right) \leqq h(v(T)), \tag{46}$$

where c is a large enough constant depending only on b, while $h = h(v), v \geq 0$ denotes the inverse function of the strictly increasing function $z(x) = \dfrac{x-1}{c+c\log x}$, $x \geq 1$. Consequently, (44) reduces to

$$v'(t) \leq qh(v(t)), \tag{47}$$

whence

$$\int_0^{v(t)} \frac{1}{h(v)} dv \leq qt. \tag{48}$$

Substituting $v = z(x)$ into this integral and taking into account that $\dfrac{1}{x\log x}$ is not integrable in $(1, +\infty)$, it follows immediately that the maximal solution of (47) with initial condition $v(0) = 0$ is bounded in finite intervals of time. This means that we have an a priori bound for $\bar{Z}_0(\varphi_n(t, \omega))$ so that the proof can be completed in the same way as that of Theorem 1.

9. Concluding Remarks

1. We can not construct the semigroup of motion without the quasi-Lipschitz condition. In this general case it is not known whether any tempered solution is a limiting one.

2. Estimating $\dfrac{r(0)}{\sigma}$ more carefully (e. g. as in the proof of Theorem 4) Proposition 2 can be improved. For example, if $g = g_\lambda(x)$ as defined before Theorem 3, then it follows that $\bar{Q}_g(\omega_t)$ increases only as fast as a power of t depending on $\lambda < 1$.

3. Theorem 4 holds for initial configurations satisfying $\bar{Z}_0(\omega) < +\infty$, this set is larger than Ω_0. Similarily, if $\lambda > 0$ and $\varepsilon > 0$ are small enough, then Theorem 1 can be extended to such initial configurations for which

$$\sup_m \sup_{\sigma \geq g_\lambda(x_m(\omega))} \sigma^{-1-\varepsilon} Q(\omega, x_m(\omega), \sigma) < +\infty.$$

4. Instead of (E) it is enough to assume that

$$|(x-\delta) U'(x)| \leq a + bU(x)\log(e+|U(x)|).$$

5. Conservation laws for the specific values of energy, particle number, impulse and entropy as well can be derived in a similar way as done by Lanford [2], Part II. Moreover, certain mixing properties of initial probability distributions are also preserved by the semigroup of motion.

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