Commun. math. Phys. 55, 1—28 (1977)

Communications in
Mathematical
Physics

All Unitary Ray Representations of the Conformal Group SU(2,2) with Positive Energy

G. Mack

II. Institut für Theoretische Physik der Universität, D-2000 Hamburg 50,
Federal Republic of Germany

Abstract. We find all those unitary irreducible representations of the ∞-sheeted covering group $\tilde{G}$ of the conformal group $SU(2,2)/\mathbb{Z}_4$ which have positive energy $P^0 \geqq 0$. They are all finite component field representations and are labelled by dimension d and a finite dimensional irreducible representation (j_1, j_2) of the Lorentz group SL(2$\mathbb{C}$). They all decompose into a finite number of unitary irreducible representations of the Poincaré subgroup with dilations.

1. Summary and Introduction

The conformal group of 4-dimensional space time is locally isomorphic to $G = SU(2,2)$; its universal covering group $\tilde{G}$ is an infinite sheeted covering of G. Both G and $\tilde{G}$ contain the quantum mechanical Poincaré group ISL(2$\mathbb{C}$). It is of physical interest to have a complete list of all unitary irreducible representations (UIR's) of $\tilde{G}$ with positive energy $P^0 \geqq 0$. They are at the same time unitary ray representations of G. In the present paper we shall give such a complete list. We show that all the UIR of $\tilde{G}$ with positive energy are finite component field representations in the terminology of [1]. They are labelled by a real number d, called the dimension, and a finite dimensional irreducible representation (j_1, j_2) of the quantum mechanical (q.m.) Lorentz group SL(2$\mathbb{C}$). Thus, $2j_1, 2j_2$ are non-negative integers. There are 5 classes of representations. They differ in their Poincaré content $[m, s]$, m = mass, s = spin resp. helicity as follows:

(1) trivial 1-dimensional representation $d = j_1 = j_2 = 0$.
(2) $j_1 \neq 0, j_2 \neq 0, d > j_1 + j_2 + 2$ contains $m > 0$, $s = |j_1 - j_2| \ldots j_1 + j_2$ (integer steps)
(3) $j_1 j_2 = 0$, $d > j_1 + j_2 + 1$ contains $m > 0$, $s = j_1 + j_2$.
(4) $j_1 \neq 0, j_2 \neq 0, d = j_1 + j_2 + 2$ contains $m > 0$, $s = j_1 + j_2$.
(5) $j_1 j_2 = 0$, $d = j_1 + j_2 + 1$ contains $m = 0$, helicity $j_1 - j_2$.

The proof of these results proceeds in several steps.

We start from the observation [2, 3] that positive energy $P^0 \geqq 0$ implies that also $H \geqq 0$, where $H = \frac{1}{2}(P^0 + K^0)$ is the "conformal Hamiltonian", K^0 a generator of special conformal transformations. Next we point out that any UIR of $\tilde{G}$ with positive energy is very much like a finite dimensional representation in that it possesses a lowest weight vector and is determined up to unitary equivalence by its lowest weight $\lambda=(d, -j_1, -j_2)$. In particular there is an algorithm for computing the scalar product of any two "K-finite" vectors.

We then derive (necessary) inequalities for the dimension d from the condition that the unique candidate for the scalar product is indeed positive semidefinite. They come out as $d \geqq j_1+j_2+2$ if $j_1 j_2 \neq 0$, and $d \geqq j_1+j_2+1$ if $j_1 j_2=0$, except for the trivial 1-dimensional representation which has $d=j_1=j_2=0$.

In the last step we construct a unitary irreducible representation of $\tilde{G}$ for every weight λ satisfying these constraints. Practically all of them have been investigated in more or less detail before, [4—6]. In particular, a careful study of the representations with $d>j_1+j_2+3$ has been carried out in Rühls work [5]. The (massless) representations with $d=j_1+j_2+1$ have been investigated by Todorov and the author [6]. For the remaining representations there remained some open questions concerning either positivity or global realization. In particular, for practical applications one needs a clean construction as an induced representation on Minkowski space. This requires particular attention to the center Γ of $\tilde{G}$.

Our representation spaces consist of vector valued functions $\varphi(x)$ on Minkowski space M^4 with values in a finite dimensional irreducible representation space of the q.m. Lorentzgroup SL(2$\mathbb{C}$). They transform under g in $\tilde{G}$ like an induced representation

$$(T(\mathsf{g})\varphi)(x)=S(\mathsf{g},x)\varphi(\mathsf{g}^{-1}x) \quad \text{for} \quad \mathsf{g}\in\tilde{G}, x\in M^4. \tag{1.1}$$

The multiplier S is a matrix with the property that $S(\mathsf{n},0)=1$ (unit matrix) for special conformal transformations n. Thus the representations are of Type Ia in the terminology of [1]. The scalar product is constructed with the help of an intertwining operator ("2-point function"). 2-point functions have also been studied in [18, 23].

The result of this paper will be used elsewhere in the nonperturbative analysis of the axioms of quantum field theory with conformal invariance [7, 8][1]. In particular it is crucial in the demonstration that in such theories operator product expansions applied to the vacuum are convergent.

2.A. The Lie Algebra

The group $G\simeq \mathrm{SU}(2,2)$ consists of all complex 4×4 matrices g which satisfy the two conditions

$$\det \mathsf{g}=1, \quad \mathsf{g}^{-1}\beta=\beta \mathsf{g}^* \quad \text{for} \quad \beta=\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1}\end{pmatrix}. \tag{2.1}$$

$\mathbb{1}$ is the unit 2×2 matrix. Let $\mathfrak{g}$ the real Lie algebra of G.

---

[1] Notations: In the present paper, the elements of Minkowski space are denoted by x, while x stands for a translation by x. The translation group is called X. In Ref. [8] a different notation is used, viz. x, $n_x^{\sim}$, $N^{\sim}$ in place of x, x, X

For a neighborhood of the identity in G we may write $\mathsf{g}=e^X$, $X\in\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ consists therefore of all complex 4×4 matrices X satisfying the two conditions

$$\mathrm{tr}X=0, \qquad -X\beta=\beta X^*. \tag{2.2}$$

The maximal compact subgroup of G is $K\simeq S(U(2)\times U(2))$. It consists of matrices of the form

$$\mathsf{k}=\begin{pmatrix} k_1 & 0\\ 0 & k_2\end{pmatrix}, \quad k_i\in U(2), \quad \det k_1k_2=1. \tag{2.3}$$

$U(2)$ is the group of all unitary 2×2 matrices. The Lie algebra $\mathfrak{k}$ of K consists of matrices such that $X=-X^*$, whence $X\beta=\beta X^*$. (2.4)

Following Cartan, the Lie algebra may be split into a compact and a noncompact part as

$$\mathfrak{g}=\mathfrak{k}+\mathfrak{p}, \tag{2.5}$$

where $X\in\mathfrak{p}$ if $X\beta=-\beta X$, and $X\in\mathfrak{k}$ if $X\beta=+\beta X$. Explicitly, $\mathfrak{p}$ consists of matrices of the form

$$X\in\mathfrak{p} \quad \text{iff} \quad X=\begin{pmatrix} 0 & z\\ z^* & 0\end{pmatrix} \quad \text{with a complex } 2\times 2 \text{ matrix } z. \tag{2.6}$$

We denote the complexification of $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ by $\mathfrak{g}_c$, $\mathfrak{k}_c$, $\mathfrak{p}_c$ respectively. $\mathfrak{g}_c$ consists of complex linear combinations of elements of $\mathfrak{g}$ etc.

We choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which consists of all diagonal matrices in $\mathfrak{g}$. It is simultaneously a Cartan subalgebra of $\mathfrak{g}$ and of $\mathfrak{k}$. We may then decompose

$$\mathfrak{g}_c=\mathfrak{h}_c+\mathfrak{n}^++\mathfrak{n}^-=\mathfrak{k}_c+\mathfrak{n}^+\cap\mathfrak{p}_c+\mathfrak{n}^-\cap\mathfrak{p}_c, \tag{2.7}$$

where $\mathfrak{n}^+(\mathfrak{n}^-)$ consists of upper (lower) triangular 4×4 matrices in $\mathfrak{g}_c$. In particular

$$X^+\in\mathfrak{n}^+\cap\mathfrak{p}_c \quad \text{iff} \quad X^+=\begin{pmatrix} 0 & z\\ 0 & 0\end{pmatrix}$$

with a complex 2×2 matrix z. (2.7′)

For such X^+ the adjoint action of $\mathsf{k}\in K$ of the form (2.3) is given by

$$\mathrm{ad}(\mathsf{k})\cdot X^+\equiv \mathsf{k}X^+\mathsf{k}^{-1}=\begin{pmatrix} 0 & k_1zk_2^{-1}\\ 0 & 0\end{pmatrix}. \tag{2.8}$$

We see that $\mathfrak{p}_c\cap\mathfrak{n}^+$ transforms under an irreducible representation of K which restricts to the UIR $(\frac{1}{2},\frac{1}{2})$ of $\mathrm{SU}(2)\times\mathrm{SU}(2)$.

We may select a basis of $\mathfrak{g}_c$ which is diagonal under the adjoint action of $\mathfrak{h}$, this gives us the commutation relations of $\mathfrak{g}_c$ in Cartan normal form.

Let us choose a basis of $\mathfrak{h}_R\equiv i\mathfrak{h}$ consisting of

$$H_0=\tfrac{1}{2}\begin{pmatrix} \mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}, \quad H_1=\tfrac{1}{2}\begin{pmatrix} \sigma^3 & 0\\ 0 & 0\end{pmatrix}, \quad H_2=\tfrac{1}{2}\begin{pmatrix} 0 & 0\\ 0 & \sigma^3\end{pmatrix}. \tag{2.9}$$

σ^3 is the third Pauli-matrix, $\sigma^3=\mathrm{diag}(+1,-1)$.

The possible eigenvalues of $H_{1,2}$ are $\pm\frac{1}{2}$ for eigenvectors in $\mathfrak{n}^+\cap\mathfrak{p}_c$. We will use them to label the basis X_{jk}; $j,k=\pm\frac{1}{2}$ of $\mathfrak{n}^+\cap\mathfrak{p}_c$.

Thus

$$[H_0, X^{\pm}_{jk}] = \pm X^{\pm}_{jk}\,; \qquad [H_1, X^{\pm}_{jk}] = jX^{\pm}_{jk}\,, \qquad [H_2, X^{\pm}_{jk}] = kX^{\pm}_{jk} \tag{2.10}$$

for the upper sign $+$. A basis for $\mathfrak{n}^-\cap\mathfrak{p}_c$ can be chosen as $X^-_{jk}=(X^+_{-j-k})^*$; this gives CR. (2.10) for the lower signs $-$.

The compact subalgebra $\mathfrak{k}$ transforms of course according to the adjoint representation $(0,1)+(1,0)$ of $\mathrm{SU}(2)\times\mathrm{SU}(2)$.

Therefore we may choose $X^0_{jk}\in(\mathfrak{n}^++\mathfrak{n}^-)\cap\mathfrak{k}_c$ with $(j,k)=(0,\pm1)$, $(\pm1,0)$ such that

$$\begin{aligned}&[H_0, X^0_{jk}]=0\,, \qquad [H_1, X^0_{jk}]=jX^0_{jk}\,, \qquad [H_2, X^0_{jk}]=kX^0_{jk}\,,\\ &(j,k)=(0,\pm1) \quad \text{or} \quad (\pm1,0)\,.\end{aligned} \tag{2.11}$$

Explicitly the matrices X^l_{jk} may be chosen as follows: Let us label the rows and columns of a 2×2 matrix by $\frac{1}{2}$, $-\frac{1}{2}$ from top to bottom and from left to right. Let e_{jk} the 2×2 matrix with 1 in the jk-position, and 0 otherwise. Thus

$$\begin{aligned}&e_{\frac{1}{2}\frac{1}{2}}=\begin{pmatrix}1&0\\0&0\end{pmatrix}, \qquad e_{-\frac{1}{2}-\frac{1}{2}}=\begin{pmatrix}0&0\\0&1\end{pmatrix},\\ &e_{\frac{1}{2}-\frac{1}{2}}=\begin{pmatrix}0&1\\0&0\end{pmatrix}, \qquad e_{-\frac{1}{2}\frac{1}{2}}=\begin{pmatrix}0&0\\1&0\end{pmatrix}.\end{aligned} \tag{2.12}$$

We also introduce Pauli matrices σ^k, in particular $\sigma^3=e_{\frac{1}{2}\frac{1}{2}}-e_{-\frac{1}{2}-\frac{1}{2}}$.

The multiplication law of these auxiliary 2×2 matrices is:

$$\begin{aligned}&e_{ij}e_{kl}=\delta_{jk}e_{il}\,, \qquad \sigma_3 e_{ij}=\delta_{\frac{1}{2}i}e_{\frac{1}{2}j}-\delta_{-\frac{1}{2}i}e_{-\frac{1}{2}j}\,;\\ &e_{ij}\sigma^3=\delta_{j\frac{1}{2}}e_{i\frac{1}{2}}-\delta_{j-\frac{1}{2}}e_{i-\frac{1}{2}}\,,\end{aligned} \tag{2.13}$$

with δ_{ij} the Kronecker-δ. Define

$$\begin{aligned}&X^+_{jk}=\begin{pmatrix}0&e_{j-k}\\0&0\end{pmatrix}, \qquad X^-_{jk}=(X^+_{-j-k})^*=\begin{pmatrix}0&0\\e_{k-j}&0\end{pmatrix}\\ &X^0_{2k,0}=\begin{pmatrix}e_{k-k}&0\\0&0\end{pmatrix}, \qquad X^0_{0,2k}=-\begin{pmatrix}0&0\\0&e_{k-k}\end{pmatrix}\end{aligned} \tag{2.14}$$

and H_0, H_1, H_2 as in (2.9). The matrices H_m, X^l_{jk} given thereby form a complete basis for $\mathfrak{g}_c$. Their CR. may be worked out by explicit computation using multiplication law (2.13). One verifies in this way the CR. (2.10), (2.11); in addition one finds

$$\begin{aligned}&[X^-_{-k-l}, X^+_{kl}]=H_{kl}\equiv H_0+2kH_1+2lH_2\\ &[X^-_{ij}, X^+_{kl}]=-\delta_{j,-l}X^0_{k+i,0}-\delta_{i,-k}X^0_{0,j+l} \quad \text{for} \quad (i,j)\neq(-k,-l)\\ &[X^0_{0,2k}, X^{\pm}_{ij}]=\pm\delta_{k,-j}\,X^{\pm}_{ik}\,; \qquad [X^0_{2k,0}, X^{\pm}_{ij}]=\pm\delta_{k,-i}X^{\pm}_{kj}\\ &[X^0_{0,-1}, X^0_{0,1}]=2H_2\,; \qquad [X^0_{-1,0}, X^0_{1,0}]=2H_1\,; \qquad [X^0_{2k,0}, X^0_{0,2j}]=0\,.\end{aligned} \tag{2.15}$$

Equations (2.10), (2.11), (2.15) are the CR. of $\mathfrak{g}_c$ in Cartan normal form relative to the compact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. The generators $-iH_0$, $-iH_1$, and $-iH_2$ of $\mathfrak{h}$ commute of course.

The real Lie algebra $\mathfrak{g}$ is spanned by the generators

$$\begin{aligned}
&\mathfrak{p}: X^+_{jk}+X^-_{-j-k}; \quad i(X^+_{jk}-X^-_{-j-k}) \quad (j=\pm\tfrac{1}{2}, k=\pm\tfrac{1}{2})\\
&\mathfrak{k}: -iH_m(m=0,1,2); \quad X^0_{1,0}-X^0_{-1,0}, \quad i(X^0_{1,0}+X^0_{-1,0});\\
&\quad X^0_{0,1}-X^0_{0,-1}, i(X^0_{0,1}+X^0_{0,-1}).
\end{aligned} \tag{2.16}$$

Besides the compact Cartan subgroup $\exp i\mathfrak{h}_R$ generated by H_0, H_1, H_2, the group G also possesses two noncompact ones. The most noncompact Cartan subgroup can be exhibited as follows. We make a basis transformation,

$$\hat{\mathfrak{g}}=U\mathfrak{g}U^{-1} \quad \text{with} \quad U=\frac{1}{\sqrt{2}}\begin{pmatrix}\mathbb{1} & -\mathbb{1}\\ \mathbb{1} & \mathbb{1}\end{pmatrix}. \tag{2.17}$$

The group G may be identified with the set of all complex 4×4 matrices satisfying the constraints

$$\det\hat{\mathfrak{g}}=1, \quad \hat{\mathfrak{g}}^{-1}\hat{\beta}=\hat{\beta}\mathfrak{g}^* \quad \text{with} \quad \hat{\beta}=U\beta U^{-1}=\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}. \tag{2.18}$$

The set of all diagonal matrices satisfying these constraints forms a noncompact Cartan subgroup of G. Furthermore we may now exhibit in a convenient form several important subgroups of G. To every 4-vector (x^μ) we associate hermitean 2×2 matrices $\underset{\sim}{x}$ and $\tilde{x}$ as follows (σ^k are Pauli matrices)

$$\underset{\sim}{x}=x^0\mathbb{1}+\Sigma x^k\sigma^k, \quad \tilde{x}=x^0\mathbb{1}-\Sigma x^k\sigma^k. \tag{2.19}$$

To every $A\in\mathrm{SL}(2\mathbb{C})$ there is associated a Lorentz transformation such that

$$A\underset{\sim}{x}A^*=\underset{\sim}{x}', \quad A^{*-1}\tilde{x}A^{-1}=\tilde{x}' \quad \text{with} \quad x'^\mu=\Lambda(A)^\mu_\nu x^\nu. \tag{2.20}$$

With this notation, we introduce subgroups of G as follows (They are all at the same time subgroups of $\tilde{G}$, s. below.) We omit the ^ henceforth.

M: Lorentztransformations

$$\mathsf{m}=\begin{pmatrix}A & 0\\ 0 & A^{*-1}\end{pmatrix}, \quad A\in\mathrm{SL}(2\mathbb{C})$$

A: dilations

$$\mathsf{a}=\begin{pmatrix}|a|^{1/2}\mathbb{1} & 0\\ 0 & |a|^{-1/2}\mathbb{1}\end{pmatrix}, \quad |a|>0$$

N: special conf. transformations

$$\mathsf{n}=\begin{pmatrix}\mathbb{1} & 0\\ i\tilde{n} & \mathbb{1}\end{pmatrix}, \quad n^\mu \text{ real}$$

X: translations

$$\mathsf{x}=\begin{pmatrix}\mathbb{1} & i\underset{\sim}{x}\\ 0 & \mathbb{1}\end{pmatrix}, \quad x^\mu \text{ real}. \tag{2.21}$$

The generators of M, A, N, X are denoted by $M^{\mu\nu}$, D, K^μ, and P^μ respectively (after dividing by $\sqrt{-1}$ as is costumary in physics). The reader may work out for himself the connection with the generators introduced before. One has in particular

$$H_0 = \tfrac{1}{2}(P_0 + K_0).$$

2.B. The Lie Groups

Let us now turn to the universal covering group $\tilde{G}$ of G. It is an infinite sheeted covering and is given by a standard construction (cp. text books, e.g. [9]): $\tilde{G}$ consists of equivalence classes of directed paths on G starting at the identity. Two paths are equivalent if they have the same end point and can be continuously deformed one into the other. By the group action in G a path may be transported such that it starts at any given point. Using this, group multiplication in $\tilde{G}$ may be defined by juxtaposition of paths.

The structure of G is best understood in terms of its Iwasawa decomposition (cp. text books, e.g. [10]). Let $M \simeq UA_\mathrm{m}N_\mathrm{m}$ the Iwasawa decomposition of the q.m. Lorentz group M. $U \simeq \mathrm{SU}(2)$ is the maximal compact subgroup of M, A_m consists of Lorentz boosts in the z-direction and N_m is the two-dimensional abelian group which is contained in Wigners little group [11] of a lightlike vector p pointing in z-direction. The Iwasawa decomposition of G is then [12]

$$G \simeq KA_\mathrm{p}N_\mathrm{p} \quad \text{with} \quad A_\mathrm{p} = A_\mathrm{m}A, \qquad N_\mathrm{p} = N_\mathrm{m}N,$$

A, N as in (2.21). The subgroup $A_\mathrm{p}N_\mathrm{p}$ is simply connected, therefore any two paths on $A_\mathrm{p}N_\mathrm{p}$ with the same end points can be continuously deformed into each other. Thus

$$\tilde{G} = \tilde{K}A_\mathrm{p}N_\mathrm{p}, \quad \tilde{K} = \text{universal covering of } K.$$

Explicitly $\tilde{K} \simeq \mathbb{R} \times (\mathrm{SU}(2) \times \mathrm{SU}(2))$. Here $\mathbb{R}$ is the additive group of real numbers, $\times$ denotes the direct product. The center Γ of $\tilde{G}$ is contained in $\tilde{K}$. It suffices then to consider K and its coverings. This gives the chain of isomorphisms

$$\begin{pmatrix}\text{conf. group of} \\ \text{Minkowski space}\end{pmatrix} \simeq \mathrm{SO}(4,2)/\mathbb{Z}_2 \simeq \mathrm{SU}(2,2)/\mathbb{Z}_4 \simeq \tilde{G}/\mathbb{Z}_2 \times \mathbb{Z}.$$

The conformal group of Minkowski space has trivial center. The center Γ of $\tilde{G}$ is thus isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}$ and has two generating elements γ_1 and γ_2, with $\gamma_1^2 = e$.

$$\Gamma = \{\gamma_1^{n_1}\gamma_2^{n_2}\,;\, n_1 = 0, 1, \quad n_2 = 0, \pm 1, \ldots\} \equiv \Gamma_1\Gamma_2.$$

γ_1 is the rotation by 2π contained in $\mathrm{SL}(2\mathbb{C})$. An explicit formula for γ_2 will be given in the next section.

Finally, $\tilde{G}$ is also a covering of G, viz $G \simeq \tilde{G}/\Gamma'$. $\Gamma' \subset \Gamma$ is given by $\Gamma' = \{(\gamma_1\gamma_2^2)^n$, $n = 0, \pm 1, \ldots\}$. The image Γ/Γ' of Γ in G is the center of G, it consists of the elements $i^m I$, $m = 0 \ldots 3$, $I = 4 \times 4$ unit matrix, $i = \sqrt{-1}$.

3. Representations with Positive Energy

Let T a unitary irreducible representation of $\tilde{G}$ by operators $T(g)$ on a Hilbert space $\mathscr{H}$. Suppose that it has positive energy, $T(P^0) \geqq 0$. There exists an element $\mathscr{R}$ of $\tilde{G}$ such that $\mathscr{R}P^0\mathscr{R}^{-1} = K^0$. Explicitly $\mathscr{R} = \exp 2\pi i H_2$. [$\mathscr{R}$ acts on compactified Minkowski space like a reciprocal radius transformation followed by a space reflection. It has been pointed out by Kastrup long ago that this is an element of the identity component of the conformal group.]

Positivity of energy $T(P^0) \geqq 0$ means that $(\Psi, T(P_0)\Psi) \geqq 0$ for arbitrary states Ψ in the $\tilde{G}$-invariant domain of $T(P^0)$. Consider

$$\begin{aligned}(\Psi, T(H_0)\Psi) &= \tfrac{1}{2}(\Psi, T(P^0)\Psi) + \tfrac{1}{2}(\Psi, T(K^0)\Psi)\\ &= \tfrac{1}{2}(\Psi, T(P^0)\Psi) + \tfrac{1}{2}(\Psi', T(P^0)\Psi') \geqq 0\end{aligned}$$

with $\Psi' = T(\mathscr{R}^{-1})\Psi$. Therefore we have the

Lemma 1. *$T(P^0) \geqq 0$ implies $T(H_0) \geqq 0$ for the conformal Hamiltonian $H_0 = \frac{1}{2}(P^0 + K^0)$.*

This result was known before [2, 3], the proof given here is a modification, due to Lüscher, of Segal's argument.

Consider next the action of the center Γ of $\tilde{G}$. It consists of elements of the form

$$\Gamma : \gamma = \gamma_1^{n_1}\gamma_2^{n_2}, \quad \gamma_2 = \mathscr{R}\exp i\pi H_0, \quad \gamma_1^2 = 1.$$

Since the UIR T is irreducible

$$T(\gamma)\Psi = \omega(\gamma)\Psi \quad \text{for all } \Psi \text{ in } \mathscr{H} \tag{3.1}$$

with

$$\omega(\gamma) = \exp 2\pi i n d \quad \text{for} \quad \gamma = \gamma_2^{2n} = \exp 2\pi i n H_0.$$

d is some real number which is determined up to an integer.

It follows then from the spectral theorem for the selfadjoint generator $T(H_0)$ that all its spectral values are of the form $d+m$, m some integer. Since $T(H_0) \geqq 0$ by Lemma 1, the spectral values $d+m \geqq 0$. We may therefore fix the integer part of d such that the lowest spectral value is d. This gives

Lemma 2. *In a* UIR *T of $\tilde{G}$ with positive energy, the generator $T(H_0)$ has a discrete spectrum. It contains a lowest eigenvalue d, and all the other eigenvalues are of the form $d+m$, m positive integer.*

4. Lowest Weights

By a vector space V we shall mean a linear space with a finite or countable basis such that the elements of V can all be written as finite linear sums of basis vectors.

Consider an irreducible representation of the Lie algebra $\mathfrak{g}_c$ (resp. $\mathfrak{k}_c$) by linear operators $T(X)$ on a complex, possibly ∞-dimensional vector space V. Irreducibility means that there exists no invariant subspace of V. We say that the representation T possesses a lowest weight vector $\Omega \in V$ with weight λ if

$$T(X)\Omega = 0 \quad \text{for all } X \in \mathfrak{n}^- \quad (\text{resp. } X \in \mathfrak{n}^- \cap \mathfrak{k}_c),$$

and

$$T(H)\Omega = \lambda(H)\Omega \quad \text{for all } H \in \mathfrak{h}_c . \tag{4.1}$$

The weight λ is a linear form on $\mathfrak{h}_c$, viz $\lambda \in \mathfrak{h}_c^*$. λ is specified by the three numbers

$$\lambda_i = \lambda(H_i). \quad \text{We write } \lambda = (\lambda_0 ; \lambda_1 ; \lambda_2).$$

A classic result says that every finite dimensional representation of $\mathfrak{g}_c$ resp. $\mathfrak{k}_c$ has a lowest weight. In particular, finite dimensional representations of $\mathfrak{k}_c$ have a lowest weight of the form

$$\lambda = (\lambda_0 ; -j_1 ; -j_2) \quad \text{with } 2j_1, 2j_2 \text{ nonnegative integers.} \tag{4.2}$$

Infinite dimensional representations of $\mathfrak{g}_c$ need not possess a lowest weight. We will however prove below that representations T of $\mathfrak{g}_c$ which are obtained from a UIR of $\tilde{G}$ with positive energy possess a lowest weight.

Consider a unitary irreducible representation T of $\tilde{G}$ on a Hilbertspace $\mathscr{H}$. It restricts to a (reducible) representation of $\tilde{K}$. $\tilde{K}$ is a direct product of an abelian factor isomorphic to $\mathbb{R}$ which is generated by H_0, and a compact Lie group K_1.

$$\tilde{K} = \mathbb{R} \times K_1 , \quad K_1 \simeq \mathrm{SU}(2) \times \mathrm{SU}(2), \quad \mathbb{R} = \{\exp i\alpha H_0, \alpha \text{ real}\}. \tag{4.3}$$

Since $T(H_0)$ has a discrete spectrum, $\mathscr{H}$ decomposes into a Hilbert sum

$$\mathscr{H} = \bigoplus_{\mu} V^{\mu} \quad (\text{Hilbert sum}), \tag{4.4}$$

where V^{μ} is a Hilbert space that decomposes into copies of one and the same UIR of $\tilde{K}$ with lowest weight μ. By Lemma 2, all the weights μ appearing in (4.4) are of the form

$$\mu = (d+N, -J_1, -J_2), N, 2J_1, 2J_2 \text{ nonnegative integers.} \tag{4.5}$$

Let us introduce the algebraic sum V of the subspaces V^{μ}

$$V = \sum_{\mu} V^{\mu} \quad (\text{algebraic sum})$$

it consists of finite linear combinations of elements of the V^{μ}.

It is a standard result in the general representation theory of semi-simple Lie groups with a finite center that all the V^{μ} are finite dimensional when we decompose with respect to the maximal compact subgroup [13]. Consequently, V is a vector space. Furthermore V is a common dense domain (of essential selfadjointness) for all the generators X of $\mathfrak{g}$. Thus there is associated with the UIR T of the group an irreducible representation of its Lie algebra by linear operators $T(X)$ on the vector space V. Conversely, any representation of $\mathfrak{g}$ by skew-hermitean operators on V can be integrated to a UIR of the group, and so infinitesimal equivalence implies unitary equivalence ([13], Theorems 4.5 and 5.3).

We will take it for granted that all this remains true for the representations of our group $\tilde{G}$ which we wish to study here, even though $\tilde{G}$ does not have finite center Γ, and the covering $\tilde{K}$ of the maximal compact subgroup $\tilde{K}/\Gamma$ of $\tilde{G}/\Gamma$ is no longer compact[2]. The vector space V will be called the "space of $\tilde{K}$-finite vectors". We say that the UIR T of $\tilde{G}$ possesses a lowest weight if the associated representation of its complexified Lie algebra $\mathfrak{g}_c$ on V possesses a lowest weight.

Let d the lowest eigenvalue of $T(H_0)$. Then there must occur among the weights μ in (4.5) at least one weight λ of the form

$$\lambda=(d;-j_1,-j_2) \tag{4.6}$$

with some integers $2j_1, 2j_2$. There exists then in V^λ a common eigenvector Ω of $T(H_i)$, $i=0, 1, 2$, to eigenvalues d, $-j_1$, $-j_2$, viz.

$$T(H_0)\Omega=d\Omega\,, \qquad T(H_k)\Omega=-j_k\Omega \quad (k=1,2)\,. \tag{4.7}$$

We claim that this is a lowest weight vector.

We have to verify that $T(X)\Omega=0$ for all $X\in\mathfrak{n}^-$. Now $\mathfrak{n}^-$ is spanned by $X_{kl}^-(k,l=\pm\frac{1}{2})$, $X^0_{-1,0}$, $X^0_{0,-1}$.

Consider then the vector $T(X_{kl}^-)\Omega$. We have

$$\begin{aligned}T(H_0)T(X_{kl}^-)\Omega&=T([H_0X_{kl}^-])\Omega+T(X_{kl}^-)T(H_0)\Omega\\&=(d-1)T(X_{kl}^-)\Omega\end{aligned}$$

by C.R. (2.10) Since d is the lowest eigenvalue of $T(H_0)$ by hypothesis, it follows that $T(X_{kl}^-)\Omega=0$.

Consider next $T(X^0_{-1,0})\Omega$. We find from the C.R. (2.10) as above that this is an eigenvector of $T(H_1)$ to eigenvalue $-j_1-1$. Since $X^0_{-1,0}\in\mathfrak{k}_c$, the vector $T(X^0_{-1,0})\Omega$ will lie in V^λ. But since V^λ consists of copies of one and the same UIR of $\tilde{K}$ with lowest weight λ, the only possible eigenvalues of $T(H_1)$ are $-j_1$, $-j_1+1,\ldots,j_1$. Therefore $-j_1-1$ is not a possible eigenvalue, hence $T(X^0_{-1,0})\Omega=0$. One shows in the same way that $T(X^0_{0,-1})\Omega=0$.

We have proven part of the following

Proposition. *Let T a unitary irreducible representation of $\tilde{G}$ with positive energy. Then T possesses a unique lowest weight. Any two such representations with the same lowest weight are unitarily equivalent.*

Proof. Let T_1, T_2 two representations of the Lie algebra $\mathfrak{g}_c$ on vector spaces V_1, V_2. We call them (linearly) equivalent if there exists a bijective map between V_1 and V_2 which commutes with the action of $\mathfrak{g}_c$.

We know already that any UIR T of G with positive energy possesses a lowest weight. Consider the associated representation of the complex Lie algebra $\mathfrak{g}_c$ on the vector space V. A standard theorem ([14], Theorem 4.4.5) asserts the following:

The lowest weight of an irreducible representation of $\mathfrak{g}_c$ on V is unique if it exists. Let Ω the lowest weight vector and $\{X_i\}_{i=1\ldots6}$ a basis for $\mathfrak{n}^+$. Then V is spanned by vectors of the form $T(X_i)^{n_1}\ldots T(X_6)^{n_6}\Omega$, n_i nonnegative integers. Finally, any two irreducible representations of $\mathfrak{g}_c$ with the same lowest weight are linearly equivalent.

---

2 A proof is given by M. Lüscher in [22]

[It follows from this also that the eigenspace V^λ of $T(H_0)$ to the lowest eigenvalue d carries an irreducible representation of $\mathfrak{k}$.]

Uniqueness of the lowest weight is thereby proven. As for unitary equivalence it suffices to show that a $\mathfrak{g}$-invariant scalar product on V is unique if it exists, cp. the discussion after (4.5). By a $\mathfrak{g}$-invariant scalar product we mean a scalar product such that $T(X)$ is skew-hermitean for X in the real Lie algebra $\mathfrak{g}$ of $\tilde{G}$.

Skew hermiticity of operators $T(X)$ for $X \in \mathfrak{g}$ implies that

$$T(Z)^* = T(\beta Z^* \beta^{-1}) \quad \text{for} \quad Z \in \mathfrak{g}_c \tag{4..8}$$

since every element Z of $\mathfrak{g}_c$ is of the form $Z = X + iY$; X, Y in $\mathfrak{g}$.

Let $\{X_i\}$ the basis of $\mathfrak{n}^+ \subset \mathfrak{g}_c$ introduced before, and consider vectors in V of the form

$$\Psi_{\{n\}} = T(X_1)^{n_1} \ldots T(X_6)^{n_6} \Omega . \tag{4.9}$$

They span V. It may happen that $\Psi_{\{n\}} = 0$. The scalar product of two such vectors must then be of the form

$$(\Psi_{\{n'\}}, \Psi_{\{n\}}) = (\Omega, T(\beta X_6^* \beta^{-1})^{n'_6} \ldots T(\beta X_1^* \beta^{-1})^{n'_1} T(X_1)^{n_1} \ldots T(X_6)^{n_6} \Omega) . \tag{4.10}$$

If $X_i \in \mathfrak{n}^+$ then $\beta X_i^* \beta^{-1} \in \mathfrak{n}^-$; hence $T(\beta X_i^* \beta^{-1})\Omega = 0$. We may therefore use the C.R. of the Lie algebra (Section 2) and hermiticity condition (3.8) to rewrite the left hand side of (3.10) as a sum of terms of the form

$$(\Omega, T(H_0)^{m_0} T(H_1)^{m_1} T(H_2)^{m_2} \Omega) = d^{m_0} (-j_1)^{m_1} (-j_2)^{m_2} (\Omega, \Omega) .$$

To this end one needs only switch all the operators $T(\beta X_i^* \beta^{-1})$ to the right and operators $T(X_i)$ to the left until they anihilate Ω.

In conclusion, there exists an algorithm for computing the scalar product of arbitrary vectors in V [= finite linear span of vectors of the form (4.9)] if it exists. Therefore the scalar product is unique up to normalization and Proposition 3 is proven. Moreover, a scalar product can only exist if the bilinear form computed by the above algorithm gives a positive semidefinite norm squared $\|\Psi\|^2 = (\Psi, \Psi)$ to all the vectors Ψ of the form (4.10).

5. Necessary Conditions for Unitarity

Having established uniqueness, we now turn to the question of existence: What are the conditions on $\lambda = (d; -j_1, -j_2)$ that λ is lowest weight of some UIR of $\tilde{G}$. We know already that

$$\begin{aligned} &\lambda = (d; -j_1, -j_2) \quad \text{with} \quad 2j_1, 2j_2 \quad \text{nonnegative integers,} \\ &d \geqq 0 . \end{aligned} \tag{5.1}$$

The last condition comes from the requirement (Lemma 1) that $T(H_0) \geqq 0$, which implies that the lowest eigenvalue d of $T(H_0)$ is nonnegative.

We shall derive sharper inequalities on d. They come from the requirement stated at the end of the last section: The bilinear form computed by the algorithm of Section 4 must assign positive semidefinite norm to vectors Ψ of the form (4.9).

Let us introduce the vectors (in V^λ) defined by

$$\Omega_{m_1 m_2} = \left\{ \frac{(j_1-m_1)!\,(j_2-m_2)!}{2j_1!(j_1+m_1)!2j_2!(j_2+m_2)!} \right\}^{\frac{1}{2}} T(X^0_{1,0})^{j_1+m_1} T(X^0_{0,1})^{j_2+m_2} \Omega . \tag{5.2}$$

One knows from the theory of angular momentum that they are normalized if $(\Omega,\Omega)=1$ as we assume. Moreover the generators of $\tilde{K}$ act on them as follows:

$$\begin{aligned} &T(H_0)\Omega_{m_1 m_2} = d\Omega_{m_1 m_2}\,; \qquad T(H_k)\Omega_{m_1 m_2} = m_k \Omega_{m_1 m_2} (k=1,2)\\ &T(X^0_{\pm 1,0})\Omega_{m_1 m_2} = [(j_1 \mp m_1)(j_1 \pm m_1 + 1)]^{\frac{1}{2}} \Omega_{m_1 \pm 1, m_2}\\ &T(X^0_{0,\pm 1})\Omega_{m_1 m_2} = [(j_2 \mp m_2)(j_2 \pm m_2 + 1)]^{\frac{1}{2}} \Omega_{m_1, m_2 \pm 1}\,. \end{aligned} \tag{5.3}$$

We shall distinguish 3 types of lowest weights $\lambda = (d;\ -j_1,\ -j_2)$.
1*st Case*: $j_1 \neq 0$, $j_2 \neq 0$. Consider the vectors

$$\begin{aligned} \Psi^{j_1-\frac{1}{2},j_2-\frac{1}{2}}_{M_1 M_2} = \sum_{m_1 m_2} & C(j_1,\tfrac{1}{2},j_1-\tfrac{1}{2};M_1-m_1,m_1)C(j_2,\tfrac{1}{2},j_2-\tfrac{1}{2};M_2-m_2,m_2)\\ &\cdot T(X^+_{m_1 m_2})\Omega_{M_1-m_1,M_2-m_2}\,. \end{aligned}$$

Herein C are vector coupling coefficients in the notation of Rose [15]. We remark that this vector transforms according to the representation of $\tilde{K}$ with the lowest weight $(d+1;\ -j_1+\frac{1}{2},\ -j_2+\frac{1}{2})$.

Since $T(X^-_{m_1 m_2})\Omega = 0$, the norm of this vector is

$$\begin{aligned} (\Psi^{j_1-\frac{1}{2},j_2-\frac{1}{2}}_{M_1 M_2}, \Psi^{j_1-\frac{1}{2},j_2-\frac{1}{2}}_{M_1 M_2}) = -\sum_{m_1 m_2}\sum_{m_1' m_2'} &(CG\text{-coefficients})\\ &\cdot(\Omega_{M_1-m_1',M_2-m_2'}, [T(X^-_{m_1' m_2'}), T(X^+_{m_1 m_2})]\Omega_{M_1-m_1,M_2-m_2})\,. \end{aligned}$$

We insert commutation relations (2.15) and evaluate the resulting matrix elements with (5.3). With the vector coupling coefficients (B.1) of Appendix B we obtain the final result

$$(\Psi^{j_1-\frac{1}{2},j_2-\frac{1}{2}}_{M_1 M_2}, \Psi^{j_1-\frac{1}{2},j_2-\frac{1}{2}}_{M_1 M_2}) = d - j_1 - j_2 - 2\,.$$

This must not be negative; we obtain therefore the condition

$$d \geqq j_1 + j_2 + 2 \quad \text{if} \quad j_1 \neq 0, j_2 \neq 0\,. \tag{5.4a}$$

2*nd Case*: $j_1 \neq 0$, $j_2 = 0$. We consider the vectors

$$\Psi^{j_1-\frac{1}{2},\frac{1}{2}}_{M_1 M_2} = \sum_m C(j_1,\tfrac{1}{2},j_1-\tfrac{1}{2};M_1-m,m)T(X^+_{mM_2})\Omega_{M_1-m,0}\,.$$

The norm squared of these vectors is computed in the same way as above to be

$$(\Psi^{j_1-\frac{1}{2},\frac{1}{2}}_{M_1 M_2}, \Psi^{j_1-\frac{1}{2},\frac{1}{2}}_{M_1 M_2}) = d - j_1 - 1\,.$$

This must not be negative; we obtain therefore the condition

$$d \geqq j_1 + 1 \quad \text{if} \quad j_1 \neq 0, j_2 = 0\,. \tag{5.4b}$$

3rd Case: $j_1=0$, $j_2\neq 0$. This case is just like the 2nd case, one finds the condition

$$d \geqq j_2+1 \quad \text{if} \quad j_1=0, j_2 \neq 0. \tag{5.4c}$$

4th Case: $j_1=j_2=0$. We consider the vector

$$\Psi=\sum_{m_1 m_2} T(X^+_{m_1 m_2})T(X^+_{-m_1-m_2})\Omega_{00}$$

We remark that it transforms according to the representation of $\tilde{K}$ with lowest weight $(d+2; 0, 0)$. The norm squared is computed in the same way as before. One finds

$$(\Psi, \Psi)=8d(d-1)$$

This must not be negative, we obtain therefore the condition

$$d=0 \quad \text{or} \quad d \geqq 1 \quad \text{if} \quad j_1=j_2=0 \tag{5.4d}$$

By uniqueness, the special case $d=j_1=j_2=0$ corresponds to the trivial 1-dimensional representation which is indeed unitary.

Conditions (4.4) are necessary for the existence of a UIR of $\tilde{G}$ with lowest weight $\lambda=(d; -j_1, -j_2)$. We shall see below that they are also sufficient.

6. Induced Representations on Minkowski Space

Let $\tilde{G}$ the universal covering group of $G \simeq \mathrm{SU}(2,2)$. As we know, the center Γ of $\tilde{G}$ is $\Gamma=\Gamma_1\Gamma_2$ with $\Gamma_1 \simeq \mathbb{Z}_2$, $\Gamma_2 \simeq \mathbb{Z}$.

It is well known that Minkowski space $M^4=\{y^\mu\}$ can be compactified in such a way that it becomes a homogeneous space for G, and therefore also for $\tilde{G}$. The conformal group of (compactified) Minkowski space is isomorphic to $SO_e(4,2)/\mathbb{Z}_2 \simeq G/\mathbb{Z}_4 \simeq \tilde{G}/\Gamma$. It is compounded from the following subgroups

M/Γ_1 Lorentz transformations
$y^\mu \to \Lambda^\mu_\nu y^\nu, \Lambda \in SO_e(3,1)$

A dilatations
$y^\mu \to |a| y^\mu, |a|>0$

N special conformal transformations
$y^\mu \to \sigma(y)^{-1}(y^\mu - n^\mu y^2)$,
with
n^μ real, $\sigma(y)=1-2ny+n^2y^2$

X translations
$y^\mu \to y^\mu + x^\mu, x^\mu$ real (6.1)

The need for considering a compactified Minkowski space M^4_c arises from the fact that special conformal transformations can take points to infinity.

The little group in $\tilde{G}/\Gamma$ of the point $x=0$ consists of Lorentz transformations, dilations and special conformal transformations. Thus $M_c^4 \simeq (\tilde{G}/\Gamma_2\Gamma_1)/(MAN/\Gamma_1)$, or

$$M_c^4 \simeq \tilde{G}/\Gamma_2 MAN. \tag{6.2}$$

This is meaningful since MAN is simply connected and therefore contained both in G and in $\tilde{G}$. Here and in the following we denote by M the quantum mechanical Lorentzgroup, it contains the factor Γ_1 of the center of $\tilde{G}$. On the other hand $\Gamma_2 \simeq \mathbb{Z}$ has a generating element γ_2 as we know (Secs. 2B, 3)

$$\Gamma_2 = \{\gamma_2^N, N=0, \pm 1, \ldots\}, \qquad \gamma_2 = \mathscr{R} \exp i\pi H_0\,; \qquad \mathscr{R} = \exp 2\pi i H_2\,. \tag{6.3}$$

We leave it to the reader to verify that the parametrization (2.21) of $G \simeq \tilde{G}/\Gamma'$ induces the transformation law (6.1) on cosets.

Let us now turn to induced representations on M_c^4. To every $\lambda = (d; -j_1, -j_2)$ we associate a finitedimensional representation of $\Gamma_2 MAN$ by

$$D^\lambda(\gamma\mathsf{man}) = |a|^c e^{i\pi Nc} D^{j_2 j_1}(\mathsf{m}) \quad \text{with} \quad c = d-2, \quad \text{for} \quad \gamma = \gamma_2^N. \tag{6.4}$$

Here $D^{j_2 j_1}$ is the familiar spinor representation (j_2, j_1) of $M \simeq \mathrm{SL}(2\mathbb{C})$, viz. $D^{j_2 j_1}(\mathsf{m}) \equiv D^{j_2 j_1}(A)$ for m of the form (2.21). It acts on a $(2j_1+1)(2j_2+1)$-dimensional vector space E^λ. We equip E^λ with the natural scalar product $<,>$ which is such that

$$D^{j_2 j_1}(\mathsf{m}^*) = D^{j_2 j_1}(\mathsf{m})^* \quad \text{for } \mathsf{m} \in M \text{ as in (2.21)} \tag{6.4'}$$

Consider the space $\mathscr{E}_\lambda$ of all infinitely differentiable functions φ on $\tilde{G}$ with values in E^λ which have the covariance property

$$\varphi(\mathsf{g}\gamma\mathsf{man}) = |a|^2 D^\lambda(\gamma\mathsf{man})^{-1} \varphi(\mathsf{g}) \tag{6.5}$$

We make $\mathscr{E}_\lambda$ into a representation space for $\tilde{G}$ by imposing the transformation law

$$(T(\mathsf{g})\varphi)(\mathsf{g}') = \varphi(\mathsf{g}^{-1}\mathsf{g}') \tag{6.6}$$

Since translations act transitively on the dense subspace $M^4 \subset M_c^4 \simeq \tilde{G}/\Gamma_2 MAN$, almost every element g of $\tilde{G}$ may be decomposed uniquely in the form

$$\mathsf{g} = \mathsf{x}\gamma\mathsf{man}, \qquad \mathsf{x} \in X, \qquad \gamma\mathsf{man} \in \Gamma_2 MAN \tag{6.7}$$

Therefore functions φ in $\mathscr{E}_\lambda$ are completely determined by their values on X. Let x' and $\gamma\mathsf{man}$ determined by x, g through the unique decomposition

$$\mathsf{g}^{-1}\mathsf{x} = \mathsf{x}'\gamma\mathsf{man}, \qquad \mathsf{g} \in \tilde{G}; \qquad \mathsf{x}, \mathsf{x}' \in X; \qquad \gamma\mathsf{man} \in \Gamma_2 MAN. \tag{6.8}$$

The transformation law (6.6) becomes then by virtue of the covariance property (6.5)

$$(T(\mathsf{g})\varphi)(\mathsf{x}) = |a|^2 D^\lambda(\gamma\mathsf{man})^{-1} \varphi(\mathsf{x}') \tag{6.9}$$

Note: translations $\mathsf{x} \in X$ are in one to one correspondence with cosets $x = \mathsf{x}\Gamma_2 MAN$. Both may be parametrized by Minkowskian coordinates x^μ, $\mu = 0 \ldots 3$. Functions φ may thus be considered as functions on Minkowski space $\{x^\mu\}$ with values in the finite dimensional irreducible representation space E^λ of the q.m. Lorentz group M.

We call them "finite component wave functions (or fields)". Equation (6.9) is the typical transformation law for an induced representation on Minkowski space, induced by a finite dimensional nonunitary representation of the (nonminimal parabolic) subgroup of stability $\Gamma_2 MAN$. Equation (6.8) says that x'^{μ} is determined by x^{μ} by the usual action on cosets, $x' = \mathsf{g}^{-1}x$, which is explicitly given by (6.1).

A. Intertwining Operator

As a prerequisite for writing down an invariant scalar product on $\mathscr{E}_\lambda$ we shall first define a map (or operator)

$$\Delta_+^\lambda : \mathscr{E}_\lambda \to \mathscr{F}_\lambda$$

where $\mathscr{F}_\lambda$ is a space of generalized functions Φ on $\tilde{G}$ with values in E^λ having covariance property

$$\Phi(\mathsf{g}\gamma\mathsf{man}) = |a|^2 D^\lambda(\gamma\mathsf{man})^* \Phi(\mathsf{g}) \quad \text{for} \quad \mathsf{g} \in \tilde{G}, \gamma\mathsf{man} \in \Gamma_2 MAN \tag{6.10}$$

It is made into a representation space for $\tilde{G}$ by imposing the transformation law

$$(T(\mathsf{g})\Phi)(\mathsf{g}') = \Phi(\mathsf{g}^{-1}\mathsf{g}') \tag{6.11}$$

The map Δ_+^λ will be required to commute with the action of the group, viz.

$$\Delta_+^\lambda T(\mathsf{g})\varphi = T(\mathsf{g})\Delta_+^\lambda \varphi \quad \text{for} \quad \varphi \text{ in } \mathscr{E}_\lambda \tag{6.12}$$

Because of this property, Δ_+^λ is called an intertwining operator. The construction of Δ_+^λ parallels to a large extent the construction of the intertwining operator for the Euclidean conformal group as described by Koller [17, see also 18].

Consider the special element $\mathscr{R}$ of $\tilde{G}$ introduced in Section 2. It has the following properties:

$$\mathscr{R}^2 = e; \quad \mathscr{R}N\mathscr{R}^{-1} = X, \quad \mathscr{R}\mathsf{m}\mathscr{R}^{-1} \equiv \bar{\mathsf{m}} \in M \quad \text{for} \quad \mathsf{m} \in M,$$
$$\mathscr{R}\mathsf{a}\mathscr{R}^{-1} = \mathsf{a}^{-1} \quad \text{for} \quad \mathsf{a} \in A \tag{6.13}$$

Working with the parametrization (2.21) of M one has $\bar{\mathsf{m}} = (\mathsf{m}^*)^{-1}$, therefore

$$D^{j_2 j_1}(\bar{\mathsf{m}})^* = D^{j_2 j_1}(\mathsf{m})^{-1} \tag{6.14}$$

We define the map Δ_+^λ by a generalized Kunze Stein formula [19]

$$\Phi(\mathsf{g}) = \Delta_+^\lambda \varphi(\mathsf{g}) = n_+(\lambda) \int_X d\mathsf{x}\varphi(\mathsf{g}\mathscr{R}\mathsf{x}) \tag{6.15}$$

n_+ is a normalization constant. Integration is over the subgroup of translations, with Haar measure $d\mathsf{x} = dx^\circ \dots dx^3$. One may ask under what conditions the integral makes sense (it may need regularization). This is a difficult question which we postpone. For the moment we proceed formally.

Let us verify that Φ has covariance property (6.10).

$$\Phi(\mathsf{g}\gamma\mathsf{man}) = n_+ \int_X d\mathsf{x}'\varphi(\mathsf{g}\gamma\mathsf{man}\mathscr{R}\mathsf{x}') = n_+ \int d\mathsf{x}'\varphi(\mathsf{g}\,\mathscr{R}\gamma\bar{\mathsf{m}}\mathsf{a}^{-1}\mathsf{x}\mathsf{x}')$$

with $\mathsf{x}=\mathscr{R}\mathsf{n}\mathscr{R}^{-1}\in X$. We introduce new variables of integration

$$\mathsf{x}''=\bar{\mathsf{m}}\mathsf{a}^{-1}\mathsf{x}\mathsf{x}'\mathsf{a}\bar{\mathsf{m}}^{-1}\,; \qquad d\mathsf{x}''=|a|^{-4}d\mathsf{x}'$$

This gives

$$\begin{aligned}\Phi(\mathsf{g}\gamma\mathsf{man})&=n_+|a|^4\int d\mathsf{x}''\varphi(\mathsf{g}\mathscr{R}\mathsf{x}''\gamma\bar{\mathsf{m}}\mathsf{a}^{-1})\\&=n_+|a|^4|a|^{-2}D^\lambda(\gamma\bar{\mathsf{m}}\mathsf{a}^{-1})^{-1}\int d\mathsf{x}''\varphi(\mathsf{g}\mathscr{R}\mathsf{x}'')\\&=n_+|a|^2D^\lambda(\gamma\mathsf{man})^*\Phi(\mathsf{g}) \qquad \text{q.e.d.}\end{aligned}$$

In the second line we used covariance property (6.5) and in the third line we used (6.14) and the definition (6.4) of D^λ.

Let us next express the map Δ_+^λ in terms of the restriction of functions φ to X. We have

$$\Phi(\mathsf{x})=n_+(\lambda)\int_X d\mathsf{x}'\varphi(\mathsf{x}\mathscr{R}\mathsf{x}')$$

Using the decomposition (6.7) we may define x'', $\gamma\mathsf{man}$ as functions of x' by

$$\mathscr{R}\mathsf{x}'=\mathsf{x}''\mathsf{s}^{-1}, \qquad \mathsf{s}=\gamma\mathsf{man}\in\Gamma_2 MAN \tag{6.16}$$

The jacobian of the transformation $\mathsf{x}'\to\mathsf{x}''$ will be found below with the result (cp. (6.20b))

$$d\mathsf{x}'=|a|^4d\mathsf{x}''$$

Thus

$$\begin{aligned}\Phi(\mathsf{x})&=n_+(\lambda)\int_X d\mathsf{x}''\varphi(\mathsf{x}\mathsf{x}''(\gamma\mathsf{man})^{-1})\\&=n_+(\lambda)\int d\mathsf{x}''|a|^2D^\lambda(\gamma\mathsf{man})\varphi(\mathsf{x}\mathsf{x}'')\end{aligned} \tag{6.17}$$

Let us reinterpret (6.16) as an equation which determines x', $\mathsf{s}=\gamma\mathsf{man}$ in terms of x'', viz

$$\mathscr{R}^{-1}\mathsf{x}''=\mathsf{x}'\gamma\mathsf{man} \tag{6.18a}$$

Define the intertwining kernel $\Delta_+^\lambda(\mathsf{x})$ by

$$\Delta_+^\lambda(\mathsf{x}''^{-1})=|a|^2D^\lambda(\gamma\mathsf{man}) \tag{6.18b}$$

$\gamma\mathsf{man}$ depending on x'' through the unique decomposition (6.18a). Writing multiplication in X additively, viz. $\mathsf{x}-\mathsf{y}$ in place of $\mathsf{x}\mathsf{y}^{-1}$, Equation (6.17) becomes

$$\Phi(\mathsf{x})=n_+(\lambda)\int_X d\mathsf{y}\Delta_+^\lambda(\mathsf{x}-\mathsf{y})\varphi(\mathsf{y}) \tag{6.19}$$

Since X may be parametrized by Minkowskian coordinates $\{x^\mu\}$, the intertwining kernel $\Delta_+^\lambda(\mathsf{x})$ may be considered as a matrix-valued function on Minkowski space M^4.

Our next object will be to derive an explicit expression for the kernel (6.18b).

To this end we must evaluate $\gamma\mathsf{man}$. Write $\gamma=\gamma_2^N$, γ_2 the generating element of Γ_2 introduced before, viz. $\gamma_2=\mathscr{R}\exp i\pi H_0$.

Let us first consider Equation (6.18a) modulo Γ', i.e. as an equation between elements in $G \simeq \tilde{G}/\Gamma'$. We write x in place of x''. Using parametrization (2.12) we have

$$\mathsf{x}'\gamma\mathsf{man} = i^N \begin{pmatrix} \varrho A - \varrho^{-1} \underset{\sim}{x}' \bar{A} \tilde{n} & i\varrho^{-1} \underset{\sim}{x}' \bar{A} \\ i\varrho^{-1} \bar{A} \tilde{n} & \varrho^{-1} \bar{A} \end{pmatrix} \quad \text{where} \quad \bar{A} \equiv (A^*)^{-1}, \varrho = |a|^{\frac{1}{2}}$$

and

$$\mathscr{R}^{-1}\mathsf{x} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & i\underset{\sim}{x} \end{pmatrix}$$

The solution of the equation $\mathscr{R}^{-1}\mathsf{x} = \mathsf{x}'\gamma\mathsf{man} \pmod{\Gamma'}$ is found by comparing both expressions. From comparison of the second column we have

$$i^N \varrho^{-1} \bar{A} = i\underset{\sim}{x}\,; \qquad \mathbb{1} = i^{N+1} \varrho^{-1} \underset{\sim}{x}' \bar{A}$$

We take the determinand of the first equation and use $\det \bar{A} = 1$. This gives $\varrho^{-2} = (-)^{N-1} \det \underset{\sim}{x} > 0$. But $\tilde{x}\underset{\sim}{x} = \det x = x^\mu x_\mu \equiv x^2$. Inserting in the second equation gives the final result

$$\varrho^2 = |a| = |x^2|^{-1}\,; \qquad A^{-1} = i^{N-1} |x^2|^{-\frac{1}{2}} \underset{\sim}{x}\,; \qquad (-)^N = -\operatorname{sgn} x^2 \tag{6.20a}$$

$$\underset{\sim}{x}' = -\underset{\sim}{x}^{-1} \quad \text{viz.} \quad x'^\mu = -x_\mu / x^2\,, \qquad d\mathsf{x}' = |x^2|^{-4} d\mathsf{x} = |a|^4 d\mathsf{x} \tag{6.20b}$$

Similarly one finds from the first column

$$\tilde{n} = -\tilde{x}[x^2]^{-1} \tag{6.20c}$$

It remains to determine $\gamma = \gamma_2^N$. This is done by applying both sides of Equation (6.18a) to the identity coset in $M \simeq \tilde{G}/MAN$. The necessary computations will be done in Appendix C. The result is

$$N = N(x) = \Theta(x^2) \operatorname{sign} x^0 \equiv \operatorname{sign} x \tag{6.21a}$$

Inserting this into formula (6.18b) for the kernel we obtain

$$\Delta_+^\lambda(-\mathsf{x}) = n_+(\lambda) |x^2|^{-2-c} e^{i\pi c N(x)} D^{j_2 j_1}(i^{1-N} |x^2|^{\frac{1}{2}} \underset{\sim}{x}^{-1})$$

We extend the definition of the representation $D^{j_2 j_1}$ of SL(2C) to GL(2C) by

$$D^{j_2 j_1}(\varrho A) = \varrho^{2j_2 + 2j_1} D^{j_2 j_1}(A)$$

Using $\tilde{x} = x^2 \underset{\sim}{x}^{-1}$ we obtain the final result $(d = 2 + c)$

$$\Delta_+^\lambda(\mathsf{x}) = n_+(\lambda)(-x^2 + i\varepsilon x^0)^{-d-j_1-j_2} D^{j_2 j_1}(i\tilde{x}) \tag{6.22}$$

The matrix elements of $D^{j_2 j_1}(i\tilde{x})$ are monomials in the coordinates x^μ.

B. Scalar Product

For functions φ in $\mathscr{E}_\lambda$ we introduce a sesquilinear form by

$$(\varphi_1, \varphi_2) = \int d\mathsf{x}_1 d\mathsf{x}_2 \langle \varphi_1(\mathsf{x}_1), \Delta_+^\lambda(\mathsf{x}_1 - \mathsf{x}_2) \varphi_2(\mathsf{x}_2) \rangle \tag{6.23}$$

Herein $\langle,\rangle$ is the scalar product on the vector space E^λ introduced with (6.4'). We note that the sesquilinear form (6.23) is formally $\tilde{G}$-invariant:

Let $\Phi_2 = \Delta_+^\lambda \varphi_2$. Because of the intertwining property (6.12) of Δ_+^λ

$$(T(\mathsf{g})\varphi_1, T(\mathsf{g})\varphi_2) = \int d\mathsf{x}_1 \langle (T(\mathsf{g})\varphi_1)(\mathsf{x}_1), (T(\mathsf{g})\Phi_2)(\mathsf{x}_1)\rangle$$

Let $\mathsf{g}^{-1}\mathsf{x}_1 = \mathsf{x}\gamma\mathsf{man}$, whence $d\mathsf{x}_1 = |a|^{-4} d\mathsf{x}$. Then this is

$$= \int d\mathsf{x} \langle D^\lambda(\gamma\mathsf{man})^{-1}\varphi_1(\mathsf{x}), D^\lambda(\gamma\mathsf{man})^* \Phi_2(\mathsf{x})\rangle$$

$$= \int d\mathsf{x} \langle \varphi_1(\mathsf{x}), \Phi_2(\mathsf{x})\rangle = (\varphi_1, \varphi_2) \qquad \text{q.e.d}$$

It remains to investigate the question under what conditions on λ the candidate (6.23) for a scalar product is well-defined and positive semi-definite (for suitable choice of $n_+(\lambda)$).

Ideally, the scalar product (6.23) should be well defined and positive on all of the representation space $\mathscr{E}_\lambda$. We shall be less ambitious for the start. Functions φ in $\mathscr{E}_\lambda$ are infinitely differentiable functions on $\tilde{G}$. It is therefore clear that their restriction $\varphi(\mathsf{x})$ to X defines functions on Minkowski space $\{x^\mu\}$ that are ∞ differentiable in the coordinates x^μ. We shall therefore also write $\varphi(x)$, $\Delta(x)$, $dx \equiv d^4x$ in place of $\varphi(\mathsf{x})$, $\Delta(\mathsf{x})$, $d\mathsf{x}$ etc. That is not all, however. In addition $\varphi(x)$ must admit certain asymptotic expansions when some or all $x^\mu \to \infty$. We will not write them down explicitly, but we note their existence. They come from the requirement that $\varphi(\mathsf{g})$ are ∞ differentiable also at those points g which map $x^\mu = 0$ into points of M_c^4 at infinity of Minkowski space M^4.

Consider now the subspace $\mathscr{S}_\lambda$ of vector-valued Schwartz test-functions on X (or M^4) with values in E^λ. They can be extended by covariance equation (6.5) to ∞ differentiable functions on $\tilde{G}$ which vanish with all their derivatives at points g in $\tilde{G}$ that map $x^\mu = 0$ into points at infinity. Thus $\mathscr{S}_\lambda \subset \mathscr{E}_\lambda$ is a proper subspace of $\mathscr{E}_\lambda$ which is not $\tilde{G}$-invariant. Indeed it is clear that $\mathscr{E}_\lambda$ is the smallest $\tilde{G}$-invariant space containing $\mathscr{S}_\lambda$. $\mathscr{S}_\lambda$ is however invariant under the Poincaré subgroup with dilations, and it is also invariant under the Lie algebra $\mathfrak{g}$ of $\tilde{G}$ which acts by differentiation with respect to g on functions $\varphi(\mathsf{g})$ on $\tilde{G}$.

Elements of $\mathscr{S}_\lambda$ possess a Fourier transform (F.T.)

$$\tilde{\varphi}(p) = \int dx e^{ipx} \varphi(x) \quad \text{with} \quad px \equiv p_\mu x^\mu \tag{6.24}$$

We see from (6.22) that the intertwining kernel is a distribution in $\mathscr{S}'_\lambda$ and possesses therefore also a Fourier transform. We are now going to determine it.

Let $\hat{p} = (E, \mathbf{0})$ and $U \simeq SU(2)$ the q.m. rotation group $U \subset M$, it leaves $\hat{p}$ invariant. The generators of U in the (j_2, j_1) representation of M will be denoted by $\boldsymbol{J} = (J^1 J^2 J^3)$. We may decompose the vector space E^λ into irreducible subspaces with respect to U

$$E^\lambda = \sum_{s=|j_1-j_2|}^{j_1+j_2} \hat{\Pi}^s E^\lambda \qquad \text{so that } \boldsymbol{J}^2 \hat{\Pi}^s E^\lambda = s(s+1)\hat{\Pi}^s E^\lambda \tag{6.25}$$

$\hat{\Pi}^s$ are projection operators that project on the irreducible subspace of E^λ which transforms according to the $2s+1$-dimensional representation of U.

$$\hat{\Pi}^s = \hat{\Pi}^{s*}, \qquad \hat{\Pi}^s \hat{\Pi}^t = \delta_{st} \hat{\Pi}^s \tag{6.26}$$

For p in V_+, the open forward light cone, define $\Pi^s(p)$ by

$$\Pi^s(\Lambda(\mathsf{m})\hat{p}) = D^{j_2 j_1}(\mathsf{m}^{-1})^* \Pi^s D^{j_2 j_1}(\mathsf{m}^{-1}) \quad \text{for} \quad \mathsf{m} \in M, \hat{p} = (E, \mathbf{0}). \tag{6.27}$$

For reasons of dilational and Lorentz-invariance, the Fourier transform of the intertwining kernel (6.22) will be of the form [$\lambda = (d, -j_1 - j_2)$ as usual]:

$$\tilde{\Delta}_+^\lambda(p) = \int dx e^{ipx} \Delta_+^\lambda(x) = \Gamma(d - j_1 - j_2 - 1)^{-1} \sum_{s=|j_1+j_2|}^{j_1+j_2} \alpha_s(\lambda) \Pi^s(p) (p^2)_+^{-2+d}$$

where

$$(p^2)_+^{-2+d} = \Theta(p^2)\Theta(p_0)(p^2)^{-2+d} \quad \text{for} \quad d > j_1 + j_2 + 1 \tag{6.28}$$

$(p^2)^{j_1+j_2}\Pi^s(p)$ are polynomials in p_μ; $\tilde{\Delta}_+^\lambda(p)$ is therefore an integrable function for the indicated range of d. We will fix the normalization factor $n_+(\lambda)$ in the intertwining kernel by imposing the

$$\text{normalization convention } \alpha_{j_1+j_2} = 1 \tag{6.29a}$$

The c-number coefficients $\alpha_s(\lambda)$ will be determined in Appendix D, the result is

$$\alpha_s(\lambda) = \frac{(d - j_1 - j_2 - 2) \dots (d - s - 1)}{(d + j_1 + j_2 - 2) \dots (d + s - 1)} \quad \text{for} \quad s = j_1 + j_2, j_1 + j_2 - 1, \dots, |j_1 - j_2|$$
$$\lambda = (d; -j_1, -j_2) \tag{6.29b}$$

The sesquilinear form (6.23) becomes now

$$(\varphi_1, \varphi_2) = \Gamma(d - j_1 - j_2 - 1)^{-1} \sum_{s=|j_1-j_2|}^{j_1+j_2} \alpha_s(\lambda) \int_{V_+} d^4p (p^2)^{-2+d} \cdot \langle \tilde{\varphi}_1(p), \Pi^s(p) \tilde{\varphi}_2(p) \rangle \tag{6.30}$$

The boosted projection operators $\Pi^s(p)$ are positive and the integral exists for $d > j_1 + j_2 + 1$. Equation (6.30) will therefore define a positive semi-definite scalar product for d in this range if all $\alpha_s(\lambda) \geqq 0$. From the explicit expression (6.29) we see that this will be so in the following cases

$$(\varphi, \varphi) \geqq 0 \quad \text{for all } \varphi \in \mathscr{S}_\lambda \quad \text{if} \tag{6.31}$$

either

$$j_1 \neq 0, \quad j_2 \neq 0, \quad d \geqq j_1 + j_2 + 2$$

or

$$j_1 = 0 \text{ and/or } j_2 = 0, \quad d > j_1 + j_2 + 1$$

In the second case there is only one term in the sum over s in (6.30).

It remains to investigate the limiting cases $j_2 = 0$, $d = j_1 + 1$ and $j_1 = 0$, $d = j_2 + 1$. Suppose $j_2 = 0$. Then $\hat{\Pi}^{j_1} = 1$ and

$$(p^2)^{j_1} \Pi^{j_1}(p) = D^{0 j_1}(\tilde{p}) \to \Pi^{j_1}_{\text{hel}}(p) \quad \text{as} \quad p^2 \to 0 \tag{6.32}$$

through V_+.

Here Π^j_{hel} is the covariantly normalized projection operator on the unique eigenstate (1-dim. subspace) in E^λ of the helicity $\boldsymbol{Jp}/p_0$ to eigenvalue j_1. It is normalized according to

$$\Pi^j_{\text{hel}}(p)\Pi^j_{\text{hel}}(p) = 2p_0\Pi^j_{\text{hel}}(p)$$

To verify the first of Equations (6.32) take m of the form (2.21) with $A = (\tilde{p}/\sqrt{p^2})^{\frac{1}{2}}$ and use the fundamental formula (2.20) of spinor calculus, viz. $A^{*-1}\tilde{p}A^{-1} = (\Lambda(A)p)^{\sim}$. The second assertion of (6.32) is well known from the theory of massless particles [11].

The second case $j_1 = 0$ is analogous. To take the limit in (6.25) we use a standard formula for the δ-function [16] and insert (6.32). The result is

$$\begin{aligned} &\Delta^\lambda_+(p) = \theta(p_0)\Pi^{j_1-j_2}_{\text{hel}}(p)\delta(p^2) \quad \text{for} \quad \lambda = (d, -j_1, -j_2) \\ &\qquad d = j_1 + j_2 + 1; \quad j_1 = 0 \quad \text{or} \quad j_2 = 0. \end{aligned} \tag{6.33a}$$

The scalar product becomes then

$$\begin{aligned} &(\varphi_1, \varphi_2) = \int_{p_0>0} d^4p\,\delta(p^2)\langle\tilde{\varphi}_1(p), \Pi^{j_1-j_2}_{\text{hel}}(p)\tilde{\varphi}_2(p)\rangle \geqq 0 \\ &\text{for} \quad d = j_1 + j_2 + 1, \quad j_1 = 0 \quad \text{or} \quad j_2 = 0. \end{aligned} \tag{6.33b}$$

It is positive semidefinite since also $\Pi^j_{\text{hel}}(p)$ is a positive operator.

C. Poincaré—Content and Irreducibility

Using the positive semidefinite scalar product (φ_1, φ_2) introduced in the last subsection we can complete $\mathscr{S}_\lambda$ to a Hilbertspace $\mathscr{H}_\lambda$ after dividing out zero norm vectors. The elements of $\mathscr{H}_\lambda$ will be equivalence classes of functions, the equivalence relation will be denoted by $\sim$ and will be explicitly given below.

To exhibit the Poincaré content of $\mathscr{H}_\lambda$ let us define to every p in the forward lightcone V_+ a boost $L(p) \in \mathrm{SL}(2\mathbb{C})$ which takes $\hat{p} = (\sqrt{p^2}, \mathbf{0})$ to p. Explicitly we may take

$$L(p) = (\underset{\sim}{p}/\sqrt{p^2})^{\frac{1}{2}} \quad \text{since then} \quad L(p)\underset{\sim}{\hat{p}}L(p)^* = \underset{\sim}{p} \tag{6.34}$$

by the fundamental formula of spinor calculus (2.20).

To every $\varphi \in \mathscr{S}_\lambda$ we associate a Wigner wave function $\Psi(p)$ with values in E^λ defined for $p \in V_+$ by

$$\Psi(p) = D^{j_2 j_1}(L(p))^{-1}\tilde{\varphi}(p) \tag{6.35}$$

Let us introduce a basis e_{sm} in E^λ which consists of orthonormal simultaneous eigenvectors of $\boldsymbol{J}^2$ and J^3 ($\boldsymbol{J}$ = generators of the rotation group) to eigenvalues $s(s+1)$ and m respectively. We may then expand

$$\Psi(p) = \sum_{s=|j_1-j_2|}^{j_1+j_2} \Psi^{sm}(p)e_{sm} \tag{6.35'}$$

with complex functions Ψ_{sm}. They transform under homogeneous Lorentz-transformations in the Wigner way,

$$(T(\mathsf{m})\Psi)^{sm'}(p)=\sum_{m''}D^s_{m'm''}(L^{-1}(p)AL(\Lambda^{-1}p))\Psi^{sm''}(\Lambda^{-1}p)$$

$$\text{for}\quad \mathsf{m}=\begin{pmatrix}A & 0\\ 0 & A^{*-1}\end{pmatrix}\in M\,;$$

$$\Lambda^{\mu}_{\nu}\equiv\Lambda(\mathsf{m})^{\mu}_{\nu}=\tfrac{1}{2}\mathrm{tr}\,\sigma^{\mu}A\sigma^{\nu}A^*\,;\qquad p\in V_+\,. \tag{6.36}$$

D^s is the $(2s+1)$-dimensional representation of the q.m. rotation group SU(2). We leave it to the reader as an exercise to rederive (6.36) from the transformation law (6.9) with $\mathsf{g}^{-1}=\mathsf{m}\in M$. The label s has the physical significance of Lorentz-invariant spin.

We can reexpress the scalar product (6.30) in terms of the Wigner wave functions $\Psi(p)$. Since $\hat{\Pi}^t e_{sm}=\delta_{st}e_{sm}$ we obtain for the norm

$$(\varphi,\varphi)=\Gamma(d-j_1-j_2-1)^{-1}\sum_{s=|j_1-j_2|}^{j_1+j_2}\alpha_s(\lambda)\int_{V_+}d^4p(p^2)^{-2+d}\cdot\sum_m|\Psi^{sm}(p)|^2 \tag{6.37}$$

Consider first the case when $d>j_1+j_2+2$ or $j_1j_2=0$, $d>j_1+j_2+1$. Then all $\alpha_s(\lambda)>0$. Thus $(\varphi,\varphi)=0$ if and only if all $\Psi^{sm}(p)=0$ for $p\in V_+$. Translated back to wave functions φ, this means that the Hilbert space $\mathscr{H}_\lambda$ consists of equivalence classes of functions with equivalence relation $\sim$ as follows:

$\mathscr{H}_\lambda:\varphi_1\sim 0$ iff $\tilde{\varphi}_1(p)=0$ for all $p\in V_+$

provided $\lambda=(d;-j_1,-j_2)$ with $d>j_1+j_2+2$ or $j_1j_2=0$, $d>j_1+j_2+1$.

If $j_1j_2\neq 0$ and $d=j_1+j_2+2$ then $\alpha_{j_1+j_2}=1$ but $\alpha_s=0$ for $s<j_1+j_2$. Thus $(\varphi,\varphi)=0$ iff $\hat{\Pi}^{j_1+j_2}\Psi(p)=0$. Translated back this means that $\mathscr{H}_\lambda$ consists of equivalence classes of functions as follows

$\mathscr{H}_\lambda:\varphi\sim 0$ iff $\Pi^{j_1+j_2}(p)\tilde{\varphi}(p)=0$ for all $p\in V_+$

in the case $j_1\neq 0$, $j_2\neq 0$, $d=j_1+j_2+1$.

Lastly consider the case $d=j_1+j_2+1, j_1j_2=0$. We see from (6.33) that $\mathscr{H}_\lambda$ consists of equivalence classes of functions

$\mathscr{H}_\lambda:\varphi\sim 0$ iff $\Pi^{j_1-j_2}_{\mathrm{hel}}(p)\tilde{\varphi}(p)=0$ for $p^2=0, p_0>0$

in the case $j_1j_2=0$, $d=j_1+j_2+1$

From Equation (6.37) resp. (6.33) we can also read off the Poincaré content of the representation space $\mathscr{H}_\lambda$. The result is as indicated in Section 1.

Let us next turn to the question of irreducibility. If either $j_1j_2=0$ or $d=j_1+j_2+2$ irreducibility of $\mathscr{H}_\lambda$ is obvious since the representation restricts to an irreducible representation of the Poincaré group with dilations. It remains to investigate the case $d>j_1+j_2+2, j_1j_2\neq 0$.

We start from the infinitesimal form of the transformation law (6.9). We denote the conformal generators obtained from $T(\mathsf{g})$ by K^μ, P^μ, $M^{\mu\nu}$, D as usual; while the generators in the finite dimensional representation $D^{j_2j_1}$ of the Lorentzgroup will be denoted by $\Sigma^{\mu\nu}$—they act in the vector space E^λ.

The infinitesimal form of the transformation law (6.9) reads then as follows $(\partial_\mu=\partial/\partial x^\mu)$

$$\begin{aligned}&P^\mu\varphi(x)=i\partial^\mu\varphi(x)\,; \qquad M^{\mu\nu}\varphi(x)=i(x^\mu\partial^\nu-x^\mu\partial^\nu-i\Sigma^{\mu\nu})\varphi(x)\\ &D\varphi(x)=i(4-d+x_\nu\partial^\nu)\varphi(x)\\ &K^\mu\varphi(x)=i([8-2d]x^\mu+2x^\mu x^\nu\partial_\nu-x^2\partial^\mu-2ix_\nu\Sigma^{\mu\nu})\varphi(x)\,.\end{aligned} \tag{6.37'}$$

In view of the general result of [1] it suffices to check validity at $x^\mu=0$ (identity in X), everything else follows then from covariance. We have from (6.9) and (6.4)

$$\begin{aligned}&(T(\mathsf{m})\varphi)(0)=D^{j_1j_2}(\mathsf{m})\varphi(0) \quad \text{for} \quad \mathsf{m}\in M\\ &(T(\mathsf{a})\varphi)(0)=|a|^{4-d}\varphi(0) \quad \text{for} \quad \mathsf{a}\in A\,;\\ &(T(\mathsf{n})\varphi)(0)=\varphi(0) \quad \text{for} \quad \mathsf{n}\in N\,.\end{aligned} \tag{6.38}$$

for Lorentztransformations m, dilatations a and special conformal transformations n, respectively. The infinitesimal form of this is (6.37') with $x^\mu=0$.

Let us introduce matrices $(J^1, J^2, J^3)=\boldsymbol{J}$, $(N^1, N^2, N^3)=\boldsymbol{N}$

$$J^i=\tfrac{1}{2}\varepsilon_{ijk}\Sigma^{jk}\,, \qquad N^k=\Sigma^{0k} \quad \text{(sum over repeated indices, } \varepsilon_{123}=1)$$

We wish to derive from (6.37') the action of infinitesimal special conformal transformations $\hat{K}^\mu$ on Wigner wave functions $\Psi(p)$. It is defined in terms of the action (6.37) of K^μ by

$$K^\mu D^{j_2j_1}(L(p))\Psi(p)=D^{j_2j_1}(L(p))\hat{K}^\mu\Psi(p)$$

We have

$$L(p)=\exp-i\theta\frac{\boldsymbol{p}}{|\boldsymbol{p}|}\boldsymbol{N}=1-im^{-1}\boldsymbol{p}\boldsymbol{N}-(2m^2)^{-1}(\boldsymbol{p}\boldsymbol{N})^2+\dots$$

where

$$p=(p_0,\boldsymbol{p})\,, \qquad m=\sqrt{p^2}\,, \qquad \sinh\theta=|\boldsymbol{p}|^2/m\,.$$

A straightforward computation leads from the Fouriertransform of (6.37') to

$$\begin{aligned}&\hat{K}^0\Psi(\boldsymbol{p}=0)=\{-2d\partial^0-2p^\nu\partial_\nu\partial^0+p^0\Box+m^{-1}\boldsymbol{N}^2\}\Psi(\boldsymbol{p}=0)\\ &\hat{\boldsymbol{K}}\Psi(\boldsymbol{p}=0)=\{-2d\boldsymbol{\partial}-2p_\nu\partial^\nu\boldsymbol{\partial}-2i(\boldsymbol{J}\times\boldsymbol{\partial})\\ &\qquad\qquad+2m^{-1}[i(d-1)\boldsymbol{N}-\boldsymbol{J}\times\boldsymbol{N}]\}\Psi(\boldsymbol{p}=0)\,.\end{aligned} \tag{6.39}$$

It suffices to have the tranformation law at $\boldsymbol{p}=0$ since K^μ transforms as a 4-vector, viz.

$$T(\mathsf{m})K^\mu T(\mathsf{m})^{-1}=\Lambda(\mathsf{m})^\mu_\nu K^\nu$$

for Lorentz transformations $\mathsf{m}\in M$. (6.40)

And we know from Equation (6.36) that Lorentz transformations do not make transitions between spin states. Neither do dilatations nor translations.

We insert the expansion in basis vectors (6.35′) and make use of the explicitly known action of the generators $\boldsymbol{J}$, $\boldsymbol{N}$ on basis vectors e_{sm} of E^λ (cp. Appendix A). As a result we obtain

$$\begin{aligned}\hat{K}^3\Psi(\boldsymbol{p}=0) &= \hat{K}^3 \sum_{s,m} e_{s,m}\Psi^{sm}(\boldsymbol{p}=0)\\ &= \frac{-2i}{\sqrt{p^2}}\sum_{s,m}\{(2-d-s)(s-m)^{\frac{1}{2}}(s+m)^{\frac{1}{2}}\\ &\quad\cdot C_s e_{s-1,m}-(3-d+s)(s+m+1)^{\frac{1}{2}}(s-m+1)^{\frac{1}{2}}\\ &\quad\cdot C_{s+1}e_{s+1,m}+\ldots\}\Psi^{sm}(\boldsymbol{p}=0)\end{aligned} \tag{6.41}$$

where the dots stand for terms proportional to $e_{s,m}$, and $C_s=C_s^{j_2j_1}$ are the constants given by Equation (A.1) of Appendix A.

We see that K^3 makes transitions between states with different s. The coefficients of $e_{s-1,m}$ and $e_{s+1,m}$ do not vanish (identically in m) for $d>j_1+j_2+2$ unless

$$s=s_{\min}=|j_1-j_2| \quad \text{resp. } s=s_{\max}=j_1+j_2\,.$$

Therefore there exists no invariant subspace and the representation is irreducible.

D. Integrability

So far we have demonstrated existence and positivity of the scalar product (φ_1,φ_2) only for Schwartz test functions φ in $\mathscr{S}_\lambda$. But unfortunately $\mathscr{S}_\lambda$ is invariant only under the action of the Lie algebra $\mathfrak{g}$ of $\tilde{G}$ but not under the group $\tilde{G}$ itself (cp. Sec. 6B). Therefore we are faced with the question whether our representation of the Lie algebra is integrable to a unitary representation of the group $\tilde{G}$. [It follows then a posteriori that the scalar product is defined and positive for functions φ in $\mathscr{E}_\lambda$, since $\mathscr{E}_\lambda$ is the smallest $\tilde{G}$-invariant space containing $\mathscr{S}_\lambda$]. This problem is solved by the

Lemma 3. *Suppose the scalar product*

$$(\varphi_1,\varphi_2)=(2\pi)^{-4}\int d^4p\langle\tilde{\bar{\varphi}}_1(p),\tilde{\Delta}_+^\lambda(p)\tilde{\varphi}_2(p)\rangle$$

exists and is positive for functions φ *such that*

$$\tilde{\varphi}(p)=\int_{s>0}ds\int d^3x e^{-p_0s+i\boldsymbol{px}}\chi(s,\boldsymbol{x}) \quad \textit{for} \quad p^2\geqq 0, p_0>0\,. \tag{6.42}$$

χ *an infinitely differentiable function with values in* E^λ *and compact support contained in the half plane* $s>0$. *Then the representation of* $\mathfrak{g}$ *is integrable to a unitary representation of* $\tilde{G}$.

This lemma is a corrolary of the theorem of Lüscher and the author on analytic continuation of contractive Lie semigroup representations (generalized Hille Yosida theorem) [3]. A proof of the lemma is implicit in Section 4 of Ref. [7].

Remark. In purely group theoretical language what is involved here is this: Functions of the form (6.42) with suppχ in a given compact subset of the upper half-plane $s>0$ form a dense set of equi-analytic vectors for the hermitean generators of $\tilde{G}$. Integrability follows then from a classic result of Nelson's [13, 21].

It is evident from the explicit form (6.28), (6.33a) of the intertwining kernel $\tilde{\Delta}_+^\lambda$ that the hypothesis of the lemma is fullfilled. We have thus constructed unitary representations of the universal covering group $\tilde{G}$ of SU(2, 2).

E. Another Realization

Let $\mathscr{F}_\lambda$ the space of (generalized) functions of the form

$$\Phi(x)=\int dy\Delta_+^\lambda(x-y)\varphi(y), \qquad \varphi\in\mathscr{E}_\lambda$$

$\mathscr{E}_\lambda$ is the function space introduced at the beginning of this section. $\mathscr{F}_\lambda$ is a representation space for $\tilde{G}$. Since the F.T. $\tilde{\Delta}_+^\lambda(p)$ has support concentrated in $\bar{V}_+$, the closed forward lightcone, $\Phi(x)$ are boundary values of holomorphic functions in the field theoretic tube domain. In the limiting cases $j_1j_2 \neq 0$, $d=j_1+j_2+2$ and $j_1j_2=0$, $d=j_1+j_2+1$ they satisfy in addition certain differential equations. For instance

$$[\boldsymbol{J}\cdot\partial+(j_1-j_2)\partial^0]\Phi(x)=0 \quad \text{if} \quad j_1j_2=0, \qquad d=j_1+j_2+1 \tag{6.43}$$

Since Φ fixes uniquely the equivalence class of φ in $\mathscr{H}_\lambda$, the scalar product (6.23) makes $\mathscr{F}_\lambda$ into a Hilbertspace which carries the same unitary representation of $\tilde{G}$ constructed before. In practical applications it can be useful to deal with the space $\mathscr{F}_\lambda$ of generalized functions instead of the spaces of equivalence classes of functions in $\mathscr{E}_\lambda$. Rühl's work deals with functions in $\mathscr{F}_\lambda$.

As our last task we should show that the UIR's of $\tilde{G}$ in the Hilbertspaces $\mathscr{H}_\lambda$ constructed so far have lowest weights λ. If so, it follows by the uniqueness theorem of Section 4, that we have constructed all the inequivalent UIR's of $\tilde{G}$ with positive energy. We shall instead refer to Rühl's work [5]. It follows from his results (and the remarks above) that all our representations constructed so far are (linearly) equivalent to analytic representations that have explicitly known lowest weight vectors (viz. constant functions) with the right weight λ.

We mention one last result without detailed proof. A UIR of a semi-simple Lie group G is said to belong to the discrete series if (and only if) its matrix elements are square integrable on the group. It is known that the discrete series is nonempty iff G has finite center Γ and possesses a compact Cartan subgroup [13]. Quotient groups $\tilde{G}/\Gamma''$ with $\Gamma''\subset\Gamma$ of our group $\tilde{G}$ possess these properties if their center Γ/Γ'' is finite. This motivates the

Definition. A unitary irreducible representation T of the semi-simple Lie group $\tilde{G}$ with denumerable center Γ is said to belong to the interpolated discrete series iff

$$\int_{\tilde{G}/\Gamma} dg|(\Psi, T(g)\Phi)|^2<\infty$$

for some nonzero vectors Ψ, Φ in the representation space. (dg is Haar measure on the group $\tilde{G}/\Gamma$).

We note that the definition is meaningful since the integrand is invariant under $\mathsf{g} \to \mathsf{g}\gamma$ for $\mathsf{g} \in \tilde{G}$, $\gamma \in \Gamma$ (cp. Sec. 3). It can therefore be considered as a function on $\tilde{G}/\Gamma$.

The representations of $\tilde{G}$ constructed in this paper belong to the interpolated discrete series if and only if

$$d > j_1 + j_2 + 3 \tag{6.44}$$

Sketch of Proof. There is a canonical way of reconstructing unitary irreducible representations as (irreducible parts of) induced representations on $\tilde{G}/\tilde{K}$. [Here we may consider the space of functions $f_m^{\Psi}(g) = (\Omega_m, T(g^{-1})\Psi)$, $m = (m_1 m_2)$; cp. Sec. 5]. Representations with lowest weight give rise to analytic representations in this way. Square integrability furnishes a scalar product on this function space. Rühl has constructed the analytic representations on $\tilde{G}/\tilde{K}$ and has found the condition (6.44) for the scalar product in question to converge [5].

Acknowledgement. The author is indebted to M. Lüscher for discussions.

Appendix A: Finitedimensional Representations of SL(2ℂ)

Let $\boldsymbol{J}$ and $\boldsymbol{N}$ the generators of rotations and Lorentz boosts respectively. They satisfy the usual commutation relations

$$[J^1, J^2] = iJ^3, \qquad [N^1, N^2] = -iJ^3, \qquad [J^1, N^2] = iN^3 \quad \text{and cyclic.}$$

Write

$$J_\pm = J^1 \pm iJ^2; \qquad N_\pm = N^1 \pm iN^2.$$

Finite dimensional representations of SL(2ℂ) are labelled by (j_1, j_2); $2j_1$, $2j_2$ nonnegative integers. A basis in the representation space may be labelled by s, m, with $s(s+1)$ the eigenvalue of $\boldsymbol{J}^2$, and m the eigenvalue of J^3: $s = |j_1 - j_2| \ldots j_1 + j_2$, $m = -s \ldots s$ in integer steps.

According to Naimark [20] the action of the generators on the basis vectors $e_{s,m}$ is

$$J_\pm e_{s,m} = [(s \mp m)(s \pm m + 1)]^{\frac{1}{2}} e_{s,m\pm 1}; \qquad J^3 e_{s,m} = m e_{s,m}$$

and for the boosts

$$\begin{aligned} N_\pm e_{s,m} = &\pm [(s \mp m)(s \mp m - 1)]^{\frac{1}{2}} C_s e_{s-1,m\pm 1} \\ &- [(s \mp m)(s \pm m + 1)]^{\frac{1}{2}} A_s e_{s,m\pm 1} \\ &\pm [(s \pm m + 1)(s \pm m + 2)]^{\frac{1}{2}} C_{s+1} e_{s+1,m\pm 1}. \end{aligned}$$

$$\begin{aligned} N^3 e_{s,m} = &[(s-m)(s+m)]^{\frac{1}{2}} C_s e_{s-1,m} \\ &- m A_s e_{s,m} - [(s+m+1)(s-m+1)]^{\frac{1}{2}} C_{s+1} e_{s+1,m}. \end{aligned}$$

with

$$A_s = ikc/s(s+1), \qquad C_s = (i/s)\{(s^2 - k^2)(s^2 - c^2)/(4s^2 - 1)\}^{\frac{1}{2}} \tag{A.1}$$

$c = j_1 + j_2 + 1$, $k = j_1 - j_2$, $s = |k| \ldots c - 1$ in integer steps.

The sign of the square root in C_s is a matter of phase conventions. It is costumary to have the generators N^k, and therefore also C_s, change sign when one interchanges $(j_1,j_2)\rightarrow(j_2,j_1)$.

Examples:

$$(j_1,j_2)=(\tfrac{1}{2},0): \quad \boldsymbol{J}=\tfrac{1}{2}\boldsymbol{\sigma}, \boldsymbol{N}=-\frac{i}{2}\boldsymbol{\sigma}$$

$$(j_1,j_2)=(0,\tfrac{1}{2}): \quad \boldsymbol{J}=\tfrac{1}{2}\boldsymbol{\sigma}, \boldsymbol{N}=\frac{i}{2}\boldsymbol{\sigma}.$$

Appendix B: Clebsch Gordan Coefficients for SU (2)

The vector coupling coefficients $C(j_1,\frac{1}{2},j_1-\frac{1}{2};m-m_2,m_2)$ in the notation (and phase convention) of Rose are given by [15]

$$C(j_1,\tfrac{1}{2},j_1-\tfrac{1}{2};m\mp\tfrac{1}{2},\pm\tfrac{1}{2})=\mp[(j_1\mp m+\tfrac{1}{2})/(2j_1+1)]^{\frac{1}{2}}. \tag{B.1}$$

Appendix C: The Homogeneous Space $M=\tilde{G}/MAN$

Let MAN the (nonminimal) parabolic subgroup of G consisting of Lorentztransformations $\mathsf{m}\in M\simeq \mathrm{SL}(2\mathbb{C})$, dilations $\mathsf{a}\in A$ and special conformal transformations $\mathsf{n}\in N$. MAN is simply connected and therefore also contained in $\tilde{G}$. Consider the Iwasawa decompositions $\tilde{G}\simeq\tilde{K}A_{\mathrm{p}}N_{\mathrm{p}}$ and $M\simeq UA_{\mathrm{m}}N_{\mathrm{m}}$ with $A_{\mathrm{p}}=A_{\mathrm{m}}A$; $N_{\mathrm{p}}=N_{\mathrm{m}}N$ (see Sec. 2). It follows that the homogeneous space

$$\boldsymbol{M}=\tilde{G}/MAN\simeq\tilde{K}/U\simeq\mathbb{R}\times S^3$$

S^3 the unit sphere in $\mathbb{R}^4$. Thus $\boldsymbol{M}$ may be parametrized as

$$\boldsymbol{M}=\{(\tau,\boldsymbol{\varepsilon}), -\infty<\tau<\infty, \boldsymbol{\varepsilon}=(\varepsilon^1\varepsilon^2\varepsilon^3,\varepsilon^5) \text{ a unit 4-vector}\}.$$

Elements of $\tilde{K}\simeq\mathbb{R}\times K_1$, act on $\boldsymbol{M}$ as translations of τ and rotations of ε. In particular

$$e^{i\sigma H_0}: \tau\rightarrow\tau+\sigma, \qquad \boldsymbol{\varepsilon}\rightarrow\boldsymbol{\varepsilon}$$

$$\mathscr{R}: \tau\rightarrow\tau \qquad \boldsymbol{\varepsilon}\rightarrow-\boldsymbol{\varepsilon}.$$

The center $\Gamma=\Gamma_1\Gamma_2$ of $\tilde{G}$ acts therefore on $\boldsymbol{M}$ as follows: Γ_1 acts trivially, while Γ_2 consists of elements of the form γ_2^N

$$\gamma_2=\mathscr{R}e^{i\pi H_0} \quad \text{takes} \quad \tau\rightarrow\tau+\pi, \qquad \boldsymbol{\varepsilon}\rightarrow-\boldsymbol{\varepsilon}$$

A domain F contained in $\boldsymbol{M}$ is called a fundamental domain (with respect to the discrete subgroup Γ_2) if

$$\boldsymbol{M}=\overline{\bigcup_{\gamma\in\Gamma_2}\gamma F}, \qquad F\cap\gamma F=\emptyset \quad \text{for} \quad \gamma\neq e \quad \text{in} \quad \Gamma_2.$$

A fundamental domain F may be chosen as follows:

$$F=\{(\tau,\boldsymbol{\varepsilon})\in \boldsymbol{M},\ -\pi<\tau<\pi,\ \boldsymbol{\varepsilon}^5>-\cos\tau\}$$

It may be identified with Minkowski space $\boldsymbol{M}^4$ through the reparametrization

$$x^0=\frac{\sin\tau}{\cos\tau+\varepsilon^5};\qquad x^i=\frac{\varepsilon^i}{\cos\tau+\varepsilon^5}\quad (i=123)$$

translations $\mathsf{x}\in X$ map F into itself. They translate coordinates x^μ. For further details see e.g. Section 7 of Ref. [3].

Consider now the equation encountered in Section 6A.

$$\mathscr{R}^{-1}\mathsf{x}=\mathsf{x}'\gamma\mathsf{man};\qquad \mathsf{x},\mathsf{x}'\ \text{in}\ X,\qquad \mathsf{man}\in MAN,\qquad \gamma=\gamma_2^N\in\Gamma_2$$

We wish to determine N as a function of x. Apply both sides of the equation to the identity coset $\dot{\mathsf{e}}=(0,\hat{\boldsymbol{\varepsilon}})$ $\hat{\boldsymbol{\varepsilon}}=(000,1)$. Evidently, by what has been said above

$$\mathsf{x}'\gamma\mathsf{man}\dot{\mathsf{e}}\in\gamma_2^N F$$

Since we know that the integer N is a Lorentz-invariant, it suffices to consider 3 cases for the right hand side

$x^\mu x_\mu<0$: take $x^0=0$ then $\mathsf{x}\dot{e}=(0,\boldsymbol{\varepsilon})$ with $\varepsilon^5<1$

therefore $\mathscr{R}^{-1}\mathsf{x}\dot{\mathsf{e}}=(0,-\boldsymbol{\varepsilon})$ with $-\varepsilon^5>-1=-\cos 0$.
Thus $\mathscr{R}^{-1}\mathsf{x}\dot{\mathsf{e}}\in F$ whence $N=0$.

$x^\mu x_\mu>0$, $x^0>0$: take $\boldsymbol{x}=0$, $x^0>0$. Then $\mathsf{x}\dot{\mathsf{e}}=(\tau,\hat{\boldsymbol{\varepsilon}})$ with $0<\tau<\pi$.

therefore $\mathscr{R}^{-1}\mathsf{x}\dot{e}=(\tau,-\boldsymbol{\varepsilon})$ with $0<\tau<\pi$, $\hat{\varepsilon}^5=-(-\hat{\varepsilon}^5)=1$.

Thus $\mathscr{R}^{-1}\mathsf{x}\dot{e}\in\gamma_2 F$ whence $N=1$.

$x^\mu x_\mu>0$, $x^0<0$: In the same way one finds $N=-1$.

Appendix D: Fouriertransform of the Intertwining Kernel

Our task is to determine the intertwining kernel $\tilde{\Delta}_+^\lambda(p)$ in momentum space. We know already that it will be of the form (6.28). Consider

$$\begin{aligned}\tilde{\Delta}_+^\lambda(p)&=D^{j_2j_1}(L(p))^*\hat{\Delta}_+^\lambda(p)D^{j_2j_1}(L(p))\\&=\Gamma(d-j_1-j_2-1)^{-1}\sum_s\alpha_s(\lambda)\hat{\Pi}^s(p^2)_+^{-2+d}\end{aligned}\tag{D.1}$$

Instead of working out the Fourier transform of (6.22) it is easier to work out the coefficients α_s from the requirements of infinitesimal conformal invariance. In particular, we must have

$$\hat{K}^{3\prime}\tilde{\Delta}_+^\lambda(p)\Psi(p)=\tilde{\Delta}_+^\lambda(p)\hat{K}^3\Psi(p)\tag{D.2}$$

for arbitrary Wigner wave functions $\Psi(p)=\sum e_{s,m}\Psi^{sm}$.

$\hat{K}_3$ is given by Equation (6.39) or (6.41), and $\hat{K}^{3'}$ is obtained from it by substituting $d \to 4-d$ and reversing the sign of boost-generators $\boldsymbol{N}$. This is in accordance with the transformation law (6.10) of $\Phi = \Delta_+^\lambda \varphi \in \mathscr{F}_\lambda$ which differs from (6.9) for $\varphi \in \mathscr{E}_\lambda$.

The projection operators

$\hat{\Pi}^t e_{s,m} = \delta_{st} e_{s,m}$

From Equation (6.41) we find

$$\tilde{\Delta}_+^\lambda(p)\hat{K}^3 \Psi(\boldsymbol{p}=0)$$
$$= -2i(p^2)_+^{d-5/2} \sum_{s,m} \{\alpha_{s-1}(2-d-s)\,[(s-m)(s+m)]^{\frac{1}{2}} C_s e_{s-1,m}$$
$$-\alpha_{s+1}(3-d+s)\,[(s+m+1)(s-m+1)]^{\frac{1}{2}} C_{s+1} e_{s+1,m} + \ldots\} \Psi^{sm},$$

while

$$\hat{K}^{3'} \tilde{\Delta}_+^\lambda(p) \Psi(\boldsymbol{p}=0)$$
$$= -2i(p^2)_+^{d-5/2} \sum_{s,m} \alpha_s\{-(d-2-s)\,[(s-m)(s+m)]^{\frac{1}{2}} C_s e_{s-1,m}$$
$$+(d-1+s)\,[(s+m+1)(s-m+1)]^{\frac{1}{2}} C_{s+1} e_{s+1,m} + \ldots\} \Psi^{sm}.$$

The dots stand in each case for terms proportional $e_{s,m}$. C_s are the constants [for the $(j_2 j_1)$ representation] given in Appendix A. By comparison we find two identical conditions on α_s, viz.

$$\alpha_{s-1} = \frac{d-2-s}{d-2+s}\alpha_s \quad \text{for} \quad s = |j_1 - j_2| + 1 \ldots j_1 + j_2$$

This is a recursion relation whose solution is

$$\alpha_s = \frac{(d-2-j_1-j_2)\ldots(d-s-1)}{(d-2+j_1+j_2)\ldots(d+s-1)}\alpha_{j_1+j_2}; \qquad s = |j_1 - j_2| \ldots j_1 + j_2. \tag{D.3}$$

References

1. Mack, G., Abdus Salam: Ann. Phys. (N.Y.) **53**, 174 (1969)
2. Segal, I.: MIT preprint
3. Lüscher, M., Mack, G.: Commun. math. Phys. **41**, 203 (1975)
4. Graev, M.L.: Dokl. Acad. Nauk SSR **98**, 517 (1954)
   Castell, L.: Nucl. Phys. **B4**, 343 (1967)
   Yao, T.: J. Math. Phys. **8**, 1931 (1967); **9**, 1615 (1968);
   Sternheimer, D.: J. Math. Pure Appl. **47**, 289 (1969)
   and references cited in 1
5. Rühl, W.: Commun. math. Phys. **30**, 287 (1973); **34**, 149 (1973); The canonical dimension of fields as the limit of noncanonical dimensions, preprint Kaiserslautern (March 1973)
6. Mack, G., Todorov, I.T.: J. Math. Phys. **10**, 2078 (1969)
7. Mack, G.: Osterwalder-Schrader positivity in conformal invariant quantum field theory. In: Lecture notes in physics, Vol. 37, (ed. H. Rollnik, K. Dietz), p. 66. Berlin-Heidelberg-New York: Springer 1975
8. Mack, G.: Commun. math. Phys. **53**, 155 (1977); Nucl. Phys. **B118**, 445 (1977)
9. Dieudonné, I.: Treatise on analysis, Vol. III. New York: Academic Press 1972

10. Hermann, R.: Lie groups for physicists, Chap. 6, 7. New York: W. A. Benjamin 1966
11. Wigner, E.: Ann math. **40**, 149 (1939);
Joos, H.: Forschr. Physik **10**, 65 (1965);
Weinberg, S.: Phys. Rev. **133**, B 1318 (1964), **134**, B 882 (1964)
12. Kihlberg, A., Müller, V. F., Halbwachs, F.: Commun. math. Phys. **3**, 194 (1966)
13. Warner, G.: Harmonic analysis on semi-simple Lie groups, Vols. I, II. Berlin-Heidelberg-New York: Springer 1972
14. Wallach, N. R.: Harmonic analysis on homogeneous spaces. New York: Marcel Dekker 1973
15. Rose, M. E.: Elementary theory of angular momentum, Appendix I. New York: John Wiley 1957
16. Gelfand, I. M., Shilov, G. E.: Generalized functions, Vol. I. New York: Academic Press
17. Koller, K.: Commun. math. Phys. **40**, 15 (1975)
18. Dobrev, V. K., Mack, G., Petkova, V. B., Petrova, S. G., Todorov, I. T.: Elementary representations and intertwining operators for the generalized Lorentz group. Lecture notes in physics, Vol. 63. Berlin-Heidelberg-New York: Springer 1977
19. Kunze, R., Stein, E.: Amer. J. Math. **82**, 1 (1960); **83**, 723 (1961); **89**, 385 (1967)
Knapp, A., Stein, E.: Ann. Math. **93**, 489 (1971);
Schiffmann, G.: Bull. Soc. Math. France **99**, 3 (1971)
20. Neumark, M. A.: Lineare Darstellungen der Lorentzgruppe, §8, Satz 2, p. 110. Berlin: VEB dt. Verlag der Wissenschaften 1963
21. Nelson, E.: Analytic vectors, Ann. Math. **70**, 572 (1959)
22. Lüscher, M.: Analytic representations of simple Lie groups and their continuation to contractive representations of holomorphic Lie semi-groups, DESY 75/71 (1975)
23. Ferrara, S., Gatto, R., Grillo, A.: Phys. Rev. **D 9**, 3564 (1975);
Zaikov, R. P.: Bulg. J. Phys. **2**, 2 (1975)

Communicated by H. Araki

Received October 4, 1976