Commun. math. Phys. 54, 193—218 (1977)

Communications in Mathematical Physics

The Euclidean Loop Expansion for Massive $\lambda\Phi_4^4$: Through One Loop

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Abstract. As an application of the theory of solutions of the classical, Euclidean field equation, we prove the existence of solutions to the renormalized functional field equation, for the $\lambda\Phi^4$ interaction in four Euclidean space dimensions, with non-negative λ and nonzero mass, through order $\hbar c$. That is, we prove that the functional derivative of the connected generating functional is in the Schwartz space $\mathrm{Re}\mathscr{S}(R^4)$, when evaluated at external sources in $\mathrm{Re}\mathscr{S}$, through order $\hbar c$. We also prove the existence of all functional derivatives of the connected generating functional through the same order. All quantities of interest are analytic in the coupling constant at $0 \leqq \lambda < \infty$, and continuous in the external source.

I. Introduction

A large number of formal, and several exact results, already exist for the loop expansion of the generating functional for connected, time-ordered vacuum expectation values of scalar field operators over Minkowski space. In this paper, we begin to develop the Euclidean version of the loop expansion for the massive scalar field with $\lambda\Phi^4$ interaction, $\lambda \geqq 0$, in four Euclidean dimensions, by proving the existence of the renormalized theory through order $\hbar c$ (one loop). We do that by studying the functional form of the renormalized Euclidean field equation. The techniques of linear and nonlinear functional analysis have matured to the point where this becomes a "standard" calculation, and we think it reasonable to hope that the same is true to all orders in the loop expansion.

Some motivating remarks follow:

(i) In the Minkowski version, Jackiw [1] gives a systematic treatment of the effective potential in the loop expansion, and he discusses the renormalization of one and two loops for $\lambda(\Phi^4)_{1+3}$ in some detail. We are interested in the Euclidean version because it is somewhat easier to state and prove rigorous theorems. We say "somewhat", because the classical field equation in the presence of an external source plays a central role; and the mathematics for the classical field equation in the Minkowski $\lambda\Phi^4$ theory is well developed, albeit in the absence of external sources [2, 3]. One could hope that no new ideas would be needed to handle external sources, and that the Minkowski parallel to the discussion in this paper would go through. But the Euclidean version is certainly less involved, and we are encouraged by the Osterwalder-Schrader [4] and Nelson [5], Euclidean to Minkowski reconstruction theorems to believe that there is no loss of generality in considering it.

(ii) The principles of renormalization have not yet been formulated or proved as definitive formal power series (fps) statements *in* $\hbar c$, in contrast to the fps expansion in the coupling constant. Rather complete expositions of the latter situation can be found in the articles of Hepp [6] and Epstein and Glaser [7]. Whether there might be combinatoric or other advantages to renormalizing the loop expansion, over the coupling constant expansion, thus remains unknown.

(iii) A treatment of sufficiently high orders[1] might conceivably suggest nonperturbative techniques of renormalization different in flavor from those currently used in constructive field theory. Unfortunately, the superficially nonperturbative direction of the loop relative to the coupling constant expansion cannot be expected to give direct information about things like bound states [1] or phase transitions, for the interaction treated here, because the terms in the loop expansion, through one loop at least, turn out to be analytic in the coupling constant. We are currently studying two or more loops.

(iv) Four dimensions is an upper bound for the techniques of this paper. Three dimensions admits the Φ^6 interaction, and two dimensions admits any power, subject to positivity constraints. The three-dimensional Euclidean theory might be considered warm-up practice for the richer existence theory of static, finite energy soliton solutions in gauge field theories [8].

(v) The immediate practical motivation is that a sufficiently complete treatment of the Euclidean classical field equation (CFE), which controls a large chunk of the nonlinearity in the problem, now exists, due to a somewhat one-sided[2] collaboration between J. Rauch and myself [9].

We devote the remainder of this introduction largely to a discussion of the functional form of the Euclidean quantum field equation, which is our starting point.

We imagine the following Euclidean generating functional to exist:

$$\mathscr{E}(f) = \frac{\langle \Omega_0, \exp(-V/\hbar c)\exp(\Phi(f)/\hbar c)\Omega_0\rangle}{\langle \Omega_0, \exp(-V/\hbar c)\Omega_0\rangle} \tag{1}$$

where Ω_0 is the vacuum for the free, Euclidean scalar field of mass $m>0$, f is in $\mathrm{Re}\mathscr{S}(R^4)$, $\Phi(f)$ is the smeared, Euclidean free field, and

$$V = \int [2^{-2}\lambda(1+C)\Phi^4 + 2^{-1}A\mu^2\Phi^2 - 2^{-1}B\Phi\Delta\Phi]dx \tag{2}$$

with $\lambda \geqq 0$ *and* $\mu = mc/\hbar$ both renormalized quantities. There is no normal ordering.

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[1] Jackiw [1] emphasizes that less than two loops is structurally too simple

[2] Although I was able to contribute a few parallel arguments, the style of the collaboration was mainly that J. Rauch explained to me what he regarded as the standard analysis of the questions I posed, which I then digested as what seemed to me powerful and interesting applications of unfamiliar techniques

The renormalization constants are regarded as fps in $\hbar c$ (they may equally well be regarded as fps in λ):

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \sum_{n=1}^{\infty} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} (\hbar c \lambda)^n, \tag{3}$$

with dimensionless coefficients a_n, b_n, and c_n. The infinite parts of these coefficients are to be chosen in n-th order to make the solution of the field equation finite in that order. We may leave the finite parts unspecified, corresponding to finite renormalizations.

At the formal level, we think of $\mathscr{E}(f)$ as the Laplace transform of the interacting, Euclidean path space measure. The reason for studying the Laplace rather than the Fourier transform is that it gives c-number fields that are real, with the correct sign of the coupling constant in the Euclidean field equation.

The field equation can be derived by a formal integration by parts on the free Gaussian measure:

$$0 = \langle \Omega_0, [(-\Delta + \mu^2)\Phi + \delta V/\delta\Phi - f] \exp(-V/\hbar c) \exp \Phi(f)/\hbar c \, \Omega_0 \rangle, \tag{4}$$

where we have dropped an infinite normalization factor that turns out to be irrelevant. If we express $\mathscr{E}(f)$ in terms of the connected generating functional,

$$\mathscr{E}(f) = \exp L(f)/\hbar c, \tag{5}$$

it is well known that a fps expansion of $L(f)$ in $\hbar c$ is the loop expansion of connected Feynman graphs [10], the term of order $(\hbar c)^n$ (the term with n loops) being an infinite fps expansion in λ. The field equation takes the functional form

$$\begin{aligned} f = & -(1+B)\Delta\Phi_c + \mu^2(1+A)\Phi_c \\ & + \lambda(1+C)[\Phi_c^3 + 3\hbar c \Phi_c \delta\Phi_c + (\hbar c)^2 \delta^2 \Phi_c], \end{aligned} \tag{6}$$

where

$$\begin{aligned} \Phi_c \equiv \delta L/\delta f(x), \qquad \delta\Phi_c \equiv \delta\Phi_c(x)/\delta f(x), \\ \delta^2\Phi_c \equiv \delta^2\Phi_c(x)/\delta f(x)\delta f(x). \end{aligned} \tag{7}$$

Infinite renormalization is required because of the singularities in the functional derivatives at equal arguments. We conjecture that the c-number function Φ_c belongs to the real Schwartz space of test functions $\mathrm{Re}\mathscr{S}(R^4)$ to all orders in $\hbar c$, if f is in $\mathrm{Re}\mathscr{S}(R^4)$. We prove that through order $\hbar c$. The functional derivatives of Φ_c, at independent arguments, give the connected n-point functions, when evaluated at $f=0$. Finite orders in $\hbar c$ for these objects will also be finite sums in λ, so the standard application of perturbative renormalization makes them finite. We show that these functional derivatives are tempered distributions before putting $f=0$. Infinite fps in λ are being summed here, and we are getting thereby the loop expansion of the Laplace transform of moments of the putative Euclidean measure. The smeared functional derivatives of Φ_c belong to $\mathrm{Re}\mathscr{S}$ through order $\hbar c$, just as Φ_c does.

The connected generating functional $L(f)$ may itself be computed in terms of these results, through order $\hbar c$; and we do so.

To set up the detailed part of our discussion, let

$$\Phi_c = \sum_{n=0}^{\infty} \varphi_n (\hbar c)^n . \tag{8}$$

The field equations for order $n=0$ and 1 are:

$n=0$ *(tree approximation):*

$$K\varphi_0 + \lambda\varphi_0^3 = f, \qquad K \equiv -\Delta + \mu^2 . \tag{9}$$

This is the classical field equation (CFE), which we discuss in Section III.

$n=1$ *(one loop correction):* Let

$$K_\lambda \equiv K + 3\lambda\varphi_0^2 . \tag{10}$$

Then

$$\begin{aligned} K_\lambda \varphi_1 &= -3\lambda\varphi_0 \delta\varphi_0 - c_1\lambda^2\varphi_0^3 - a_1\lambda\mu^2\varphi_0 + b_1\lambda\Delta\varphi_0 \\ &\equiv -3\lambda\varphi_0\delta\varphi_{0,R} + b_1\lambda\Delta\varphi_0 . \end{aligned} \tag{11}$$

In Section IV, we show that the infinite parts of a_1 and c_1 can be chosen to renormalize $\delta\varphi_0$ into

$$\delta\varphi_{0,R} = \delta\varphi_0 + \tfrac{1}{3}c_1\lambda\varphi_0^2 + \tfrac{1}{3}\mu^2 a_1 \in \operatorname{Re} O_M(R^4), \tag{12}$$

where O_M is the set of infinitely differentiable functions with polynomial bounded derivatives. We shall see that this puts φ_1 in $\operatorname{Re}\mathscr{S}$. The constant b_1 is finite, as usual.

The n-th order correction obeys an equation of the form

$$K_\lambda \varphi_n = R_{n-1} , \tag{13}$$

where the r.h.s. still requires renormalization, but depends only on φ_s and its functional derivatives for $s \leqq n-1$.

In Section II, we review some relevant properties of Sobolev spaces. Although in some sense the natural arena for our discussion is $\operatorname{Re}\mathscr{S}$, only the Sobolev norms get much use.

In Section III, we state the basic theorem on solutions of the CFE; and we show enough continuity of the solutions in the external source to let us prove that the generating functional in the tree approximation has functional derivatives of all orders, which are analytic in the coupling constant.

We renormalize the one loop correction in Section IV, show that the generating functional is well-defined, and prove the existence and analyticity of its functional derivatives.

Several appendices contain the proofs of certain lemmas.

The reader may survey our main results by taking a look at Theorems 1, 12, and 20.

Acknowledgments. I should like to thank Jeffrey Rauch for introducing me to some of the techniques of partial differential equations, and Paul Federbush, Ira Herbst, and Rudolph Seiler for their encouragement.

II. Technical Preliminaries

We want to review what we need to know about Sobolev spaces and Sobolev estimates in four dimensions. Let $\mathscr{S}(R^4)$ be the Schwartz space of infinitely differentiable functions of rapid decrease. The notation H_n stands for the completion of the pre-Hilbert space $\mathscr{S}(R^4)$ with inner product

$$\langle f, f\rangle_n = \langle f, (-\Delta + \mu^2)^n f\rangle, \tag{14}$$

where $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_0$ is the $L_2(R^4)$ inner product. The Sobolev norm is

$$\|f\|_{2,n} = (\langle f, f\rangle_n)^{1/2}. \tag{15}$$

Most important are the norms for $n = -1, 0, 1, 2, \ldots$.

The injections induced by the inclusions $H_{n+1} \subset H_n$ are continuous relative to the respective norms. Indeed,

$$\|f\|_{2,n+1} \geqq \mu \|f\|_{2,n}. \tag{16}$$

The Sobolev inequalities for four dimensions correspond to the continuous inclusions

$$\begin{aligned} &\mathrm{H}_1 \subset \mathrm{L}_p, \quad 2 \leqq p \leqq 4;\\ &H_2 \subset L_p, \quad 2 \leqq p < \infty. \end{aligned} \tag{17}$$

The inequalities are the statement of the boundedness of continuous linear maps, such as the injections induced above, between Banach spaces.

We also recall that

$$H_n \subset L_\infty, \quad n > 2, \tag{18}$$

for four dimensions, again a continuous inclusion. In this case, $f \in H_n$ has an L_1 Fourier transform, and so is absolutely continuous and zero at infinity.

Our notation for the L_p norms in four dimensions is $\|\cdot\|_p$.

We use the common multi-index notation $l = (l_1, \ldots l_4)$,

$$D^l = (\partial/\partial x_1)^{l_1} \ldots (\partial/\partial x_4)^{l_4}, \quad |l| = l_1 + \ldots + l_4; \quad x^l = x_1^{l_1} \ldots x_4^{l_4}. \tag{19}$$

For bounded operator norms on H_n, we use the notation

$$|O|_n = |K^{n/2} O K^{-n/2}|, \tag{20}$$

where $|\cdot|$ is the L_2, bounded operator norm. We denote the normed space of bounded, linear operators on H_n by $B(H_n)$.

In case the operator O is multiplication by a function h, we can estimate its norm, for non-negative integers n, by

$$|h|_n \leqq C_n \max[\|h\|_\infty, \|h\|_{2,n+1}]. \tag{21}$$

The argument for this is easy and presumably known, and we are just ignorant about whom to quote[3]. Nevertheless, we present it in Appendix I. Note that the H_{n+1} norm controls unless $n=0$ or 1.

[3] Reed and Simon give a similar result for general dimensions that is not quite as sharp as that in Equation (21) for four dimensions. See Proposition 2 on page 51 of [2]

We use the familiar notation $F=0(\varepsilon)$ to mean $F \leqq C|\varepsilon|$, where C is uniformly bounded in ε near $\varepsilon=0$. We use the same notation for the absolute value of a number as for the $B(L_2)$ norm.

We shall be dealing with functionals $F(f)$, where f is in $\mathrm{Re}\mathscr{S}(R^4)$ and F has values in the complex numbers, H_n, or $B(H_n)$. Continuity of F at f_0 in $\mathrm{Re}\mathscr{S}$, relative to the norm appropriate for the values of F, is typically achieved by having F be norm continuous as $f \to f_0$ in H_n for some non-negative n. Since these H_n norms are part of a complete set of seminorms for the topology of $\mathscr{S}$, if $f \to f_0$ in $\mathrm{Re}\mathscr{S}$, then $f \to f_0$ in H_n, and hence $F(f) \to F(f_0)$ in norm.

III. The Tree Approximation

Our starting point is a theorem on solutions of the CFE:

Theorem 1. *Let $K\varphi+\lambda\varphi^3=f$ be the* CFE, *where $\mu^2>0$ and $\lambda \geqq 0$. Then for each f in $\mathrm{Re}\,\mathscr{S}(R^4)$, there is a unique solution φ in $\mathrm{Re}\mathscr{S}$. The solution is analytic at all points $\lambda \in [0, \infty)$ in all of the H_n norms.*

The proof is given in [9], for a general class of interactions and dimensions one through four.

As terminology, we sometimes shorten "φ is a solution" of the CFE for f in $\mathrm{Re}\mathscr{S}$ to "φ is a solution".

The map $\varphi \mapsto f$ is trivially continuous from $\mathrm{Re}\mathscr{S}$ to $\mathrm{Re}\mathscr{S}$. We need continuity for the inverse map $f \mapsto \varphi$ in the H_n norms, and that results from the next two lemmas.

Lemma 2. *Let φ_1 and φ_2 be solutions corresponding to f_1 and f_2. Let $n \geqq 1$ be an integer. Then if there exists a polynomial P_2 such that*

$$\|\varphi_1-\varphi_2\|_{2,n} \leqq \|f_1-f_2\|_{2,n-2} P_2(\|f_1\|_{2,n-2}, \|f\|_{2,n-2}) \tag{22}$$

for all f_1 and f_2 in $\mathrm{Re}\mathscr{S}$, it follows that there is a polynomial P_1 such that

$$\|\varphi\|_{2,n+1} \leqq \|f\|_{2,n-1} P_1(\|f\|_{2,n-1}). \tag{23}$$

Proof. We learned the basic argument from J. Rauch. Let $\Delta_a\varphi(x)=\varphi(x+a)-\varphi(x)$. Since the CFE is translation invariant, and has unique solutions, $f^a(x) \equiv f(x+a) \mapsto \varphi^a(x)$. Thus,

$$\|\Delta_a\varphi\|_{2,n} \leqq \|\Delta_a f\|_{2,n-2} P_2(\|f\|_{2,n-2}, \|f\|_{2,n-2}), \tag{24}$$

where the two arguments of P_2 are the same because of the translation invariance of the norm; and

$$\|\Delta_a\varphi\|_{2,n}/|a| \leqq (\|\Delta_a f\|_{2,n-2}/|a|) P_2. \tag{25}$$

A standard theorem says that $\Delta_a/|a|$ is uniformly bounded as a linear operator from H_n to H_{n-1}, for $0<|a| \leqq 1$. Passing to the limit along fixed directions, we find

$$\begin{aligned}\|\nabla\varphi\|_{2,n} &\leqq \|\nabla f\|_{2,n-2} P_2 \\ &\leqq \|f\|_{2,n-1} P_2.\end{aligned} \tag{26}$$

Since this is true for any directional derivative, and since we may put $f_2=\varphi_2=0$ in the hypothesis of the lemma to eventually get a similar bound on $\|\varphi\|_{2,n}$, there is a constant C such that

$$\|\varphi\|_{2,n+1}\leqq C\|f\|_{2,n-1}P_2\,. \tag{27}$$

The H_{n-2} norm in P_2 may be replaced by the larger (up to a factor μ^{-1}) H_{n-1} norm. □

Lemma 3. *Let φ_1 and φ_2 be solutions. Then for each integer $n\geqq 1$ there is a polynomial P_2 such that*

$$\|\varphi_1-\varphi_2\|_{2,n}\leqq\|f_1-f_2\|_{2,n-2}P_2(\|f_1\|_{2,n-2},\|f_2\|_{2,n-2})\,. \tag{28}$$

For $n=1$, we learned from J. Rauch that $f\mapsto\varphi$ is in fact a contraction from H_{-1} to H_1. We repeat his argument, and give our own induction proof for $n>1$ in Appendix II. The proof is straightforward, given the $B(H_n)$ bound in Equation (21). It is valid for a large class of interactions, including any polynomial obeying certain positivity laws [9].

Remark 1. Lemma 3 states a stronger condition than we actually need. For example, it would be enough to know that *some* Sobolev norm occurs on the r.h.s., not necessarily the one for $n-2$. We have taken some pains to get the $n-2$ norm because in the theory of the CFE there is a natural correspondence: $f\in H_n\Rightarrow\varphi\in H_{n+2}$.

In the next lemma we collect some useful facts about the linear operator $K_\lambda=K+3\lambda\varphi^2$, corresponding to any fixed $f\in\operatorname{Re}\mathscr{S}$.

Lemma 4. (i) *As an operator on L_2, K_λ for $\lambda\geqq 0$ is strictly positive and self-adjoint on the domain of K, $\mathscr{D}(K)=H_2\subset L_2$.*

(ii) *The inverse operator K_λ^{-1} is bounded on L_2 and maps H_n into H_{n+2} for every integer $n\geqq 0$.*

(iii) *K_λ and K_λ^{-1} map $\mathscr{S}$ and $\operatorname{Re}\mathscr{S}$ continuously onto themselves.*

(iv) *The operators KK_λ^{-1} and $K_\lambda^{-1}K$ are bounded on H_n for all integers $n\geqq 0$.*

(v) *The same operators are analytic at $\lambda\in[0,\infty)$ in the bounded operator norm on H_n for all integers $n\geqq 0$.*

The proof is in Appendix III.

We want to study functional derivatives of solutions of the CFE. In doing so, functionals $F(f)$ will arise that have values in the complex numbers, in H_n, or in $B(H_n)$. The functional derivative with respect to f in the g direction, for $g\in\operatorname{Re}\mathscr{S}$, is

$$\begin{aligned}\delta_f(g)F(f)&=\lim_{\varepsilon\to 0}\Delta_f(\varepsilon g)F(f)/\varepsilon\\ &=(d/d\varepsilon)F(f+\varepsilon g)|_{\varepsilon=0}\,,\end{aligned} \tag{29}$$

where we define the difference operator with respect to f by

$$\Delta_f(h)F(f)=F(f+h)-F(f)\,, \tag{30}$$

and where the convergence is in the norm appropriate to the image space of F.

We make the index f explicit in these notations, because later we are going to want to consider functional derivatives and finite differences with respect to φ.

In the case of interest, $\delta_f(g)F$ is a continuous linear functional for g in $\mathcal{S}$, and higher functional derivatives are multilinear and separately continuous on $\mathcal{S}^{\times m}$, so the following notation is sensible:

$$\begin{aligned}\delta_f^m(g_1, \ldots, g_m)F &= \delta_f(g_1)\ldots\delta_f(g_m)F \\ &= \int \frac{\delta^m F(f)}{\delta f(x_1)\ldots\delta f(x_m)} g_1(x_1)\ldots g_m(x_m)dx_1\ldots dx_m.\end{aligned} \tag{31}$$

The difference operator gets bounded in the next few lemmas.

Lemma 5. *Let l be any multi-index, $m \geqq 1$ any integer power, and $2 \leqq p \leqq \infty$. Then there exists an n such that*

$$\|D^l[\Delta_f(h)\varphi]^m\|_p = O(\|h\|_{2,n}^m). \tag{32}$$

Proof. First, suppose $p \neq \infty$. By its definition, the difference operation commutes with the gradient; so we can let the derivatives act, and bound the L_p norm by a sum of products of p_i norms, $p < p_i < \infty$, of the form

$$\begin{aligned}\|\Delta_f(h)D^{l_i}\varphi\|_{p_i} &= \|D^{l_i}\Delta_f(h)\varphi\|_{p_i} \\ &\leqq C\|\Delta_f(h)\varphi\|_{2,2+|l_i|} = O(\|h\|_{2,|l_i|}),\end{aligned} \tag{33}$$

where the last step is from Lemma 3.

That leaves $p = \infty$. Bound the L_∞ norm by a sum of products of the same form as the l.h.s. above, but with $p_i = \infty$. Now bound these norms by H_3 norms, etc. □

It is sometimes convenient to treat $\Delta_f(h)\varphi$ as a multiplication operator on H_n.

Lemma 6. *For any integer $n \geqq 0$, there is an n' such that*

$$|\Delta_f(h)\varphi|_n = O(\|h\|_{2,n'}). \tag{34}$$

Proof. Lemma 5 for $p = 2$, and the estimate in Equation (21). □

Lemma 7. *Consider K_λ^{-1} as a functional of f. Then for every integer $n \geqq 0$ there is an n' such that*

$$|\Delta_f(h)K_\lambda^{-1}(f)|_n = O(\|h\|_{2,n'}). \tag{35}$$

Proof. The idea is that the resolvent expansion for $K_\lambda^{-1}(f+h)$ about $K_\lambda^{-1}(f)$ converges in norm for h small in some norm, and that we may thereby bound the difference. Thus

$$\begin{aligned}|\Delta_f(h)K_\lambda^{-1}|_n &\leqq |K_\lambda^{-1}|_n \sum_{m=1}^{\infty} [|\Delta_f(h)(3\lambda\varphi^2)|_n |K_\lambda^{-1}|_n]^m \\ &\leqq a|K_\lambda^{-1}|_n^2/(1 - a|K_\lambda^{-1}|_n),\end{aligned} \tag{36}$$

where $a \equiv |\Delta_f(h)(3\lambda\varphi^2)|_n = O(\|h\|_{2,n'})$, by Lemma 6. □

These lemmas can no doubt be sharpened by computing the optimum norm for h. Which norm occurs, however, is irrelevant for us, because we typically consider limits where h scales to zero, and so goes to zero in every H_n.

With these estimates, we begin to compute functional derivatives. We use the temporary notation $\Delta_\varepsilon = \Delta_f(\varepsilon g)$. We get the first functional derivative of φ from

$$\Delta_\varepsilon \varphi = K_\lambda^{-1}\{\varepsilon g - \lambda[3\varphi(\Delta_\varepsilon\varphi)^2 + (\Delta_\varepsilon\varphi)^3]\}. \tag{37}$$

Lemma 8. *$\delta_f(g)\varphi = K_\lambda^{-1} g$ belongs to* Re$\mathscr{S}$, *exists in H_n norm for every integer $n \geqq 1$, is analytic at $\lambda \in [0, \infty)$ in those norms, and*

$$\|\Delta_f(h)\delta_f(g)\varphi\|_{2,n} = O(\|h\|_{2,n'}) \tag{38}$$

for some $n'(n)$.

Proof. To compute the functional derivative write

$$\begin{aligned} &\|\varepsilon^{-1}\Delta_\varepsilon\varphi - K_\lambda^{-1} g\|_{2,n} \\ &\leqq (\lambda/\varepsilon)\|K_\lambda^{-1}[3\varphi(\Delta_\varepsilon\varphi)^2 + (\Delta_\varepsilon\varphi)^3]\|_{2,n}. \end{aligned} \tag{39}$$

The factor K_λ^{-1} is bounded on H_n, and its norm may be factored out. Apply Lemma 5 to show that the r.h.s. goes to zero like ε. The bound on the functional derivative follows from Lemma 7. The fact that the derivative belongs to Re$\mathscr{S}$ follows from Lemma 4.iii; and analyticity in λ is a consequence of Lemma 4.v. □

Before considering higher derivatives, let us introduce the functional derivative and the difference operation relative to φ. Since the correspondence between f and φ is one-to-one, we can make a change of variables and write

$$F(f) = G(\varphi). \tag{40}$$

Then we define

$$\Delta_\varphi(h)F = G(\varphi + h) - G(\varphi) \equiv F_h - F. \tag{41}$$

We often do not bother to give F a new name in terms of the variable φ, and the notation is always $F_h = G(\varphi + h)$, unless we state otherwise.

Functional derivatives of F with respect to φ are now defined like those with respect to f, but using the difference with respect to φ instead of f. An important link between the two derivatives is the

Bounded Difference Condition. A functional $F(f)$ with values in one of the relevant normed spaces is said to obey the *bounded difference condition* relative to the norm in that space is there is an n such that

$$\|\Delta_\varphi(h)F\| = O(\|h\|_{2,n}). \tag{42}$$

Note that, by Lemma 3 and the continuity property mentioned in Section II, a functional that obeys the bounded difference condition is continuous from Re$\mathscr{S}$ to the image space. In practise, the φ differences that we shall encounter obey the condition above for complex h. And although we shall only need real φ derivatives, all objects that we consider will in fact be analytic functions of the parameter ε in the definition of the derivative, for small ε.

Lemma 9. *Let $F(f)$ have values in a normed space. Let F obey the bounded difference condition relative to the norm, and let $\delta_\varphi(g)F$ exist in norm, for all $g\in\mathrm{Re}\mathscr{S}$. Then*

$$\delta_f(g)F=\delta_\varphi(K_\lambda^{-1}g)F, \tag{43}$$

where the f derivative exists in norm.

Proof.

$$\begin{aligned}&\delta_\varphi(K_\lambda^{-1}g)F-\Delta_f(\varepsilon g)F/\varepsilon\\&=\delta_\varphi(K_\lambda^{-1}g)F-\Delta_\varphi(\varepsilon K_\lambda^{-1}g)F/\varepsilon\\&\quad+(G(\varphi+\varepsilon K_\lambda^{-1}g)-G[\varphi(f+\varepsilon g)])/\varepsilon.\end{aligned} \tag{44}$$

The difference in the first two terms on the r.h.s. converges to zero in norm by hypothesis. The second term obeys.

$$\begin{aligned}&\varepsilon^{-1}\|G(\varphi+\varepsilon K_\lambda^{-1}g)-G[\varphi(f+\varepsilon g)]\|\\&=\varepsilon^{-1}O(\|\varphi+\varepsilon K_\lambda^{-1}g-\varphi(f+\varepsilon g)\|_{2,n'})\\&=O(\|K_\lambda^{-1}g-\varepsilon^{-1}\Delta_f(\varepsilon g)\varphi\|_{2,n'}),\end{aligned} \tag{45}$$

which converges to zero by Lemma 8. □

Lemma 10. *Let $A_1,\dots,A_m$ be functionals of f with values in $B(H_n)$, and let ψ be a functional of f with values in H_n.*

(i) *If $A_1,\dots,A_m$ and ψ obey the bounded difference condition in the relevant norms, then so do the products $A_1\dots A_m$ and $A_1\dots A_m\psi$.*

(ii) *If the A's and ψ have φ derivatives in the relevant norms, then so do the products, and the answer is given by the Leibniz rule.*

Proof. (i)

$$\begin{aligned}\Delta_\varphi(h)[A_1\dots A_m\psi]=&[\Delta_\varphi(h)A_1]A_{2,h}\dots\psi_h+A_1[\Delta_\varphi(h)A_2]A_{3,h}\dots\psi_h\\&+\dots+A_1\dots A_m\Delta_\varphi(h)\psi.\end{aligned} \tag{46}$$

Given n, choose n' large enough so that each factor obeys the difference condition with $O(\|h\|_{2,n'})$. Each of the factors $A_{i,h}$ or ψ_h is thus uniformly bounded in norm for small $\|h\|_{2,n'}$, and the result follows by bounding the H_n norm of the above expression in the obvious way. The same argument works with ψ left out.

(ii) If the φ derivative is written in terms of $d/d\varepsilon$, this is a standard theorem for norm differentiable operators and vectors. □

To handle higher functional derivatives, we first study the derivative of the operator K_λ^{-1}:

Lemma 11.

$$\delta_\varphi(g)K_\lambda^{-1}=-6\lambda K_\lambda^{-1}\varphi g K_\lambda^{-1}, \tag{47.a}$$

$$\delta_f(g)K_\lambda^{-1}=-6\lambda K_\lambda^{-1}(K_\lambda^{-1}g)\varphi K_\lambda^{-1}. \tag{47.b}$$

Both derivatives exist and obey the bounded difference condition in $B(H_n)$ for every integer $n\geqq 0$.

Proof. By the resolvent formula,

$$\Delta_\varphi(h)K_\lambda^{-1} = -K_\lambda^{-1}[\Delta_\varphi(h)(3\lambda\varphi^2)]K_{\lambda,h}^{-1}. \tag{48}$$

The proof of Lemma 7 can be adapted to show that K_λ^{-1} obeys the bounded difference condition in $B(H_n)$. Thus, putting $h=\varepsilon g$, we find that $K_{\lambda,\varepsilon g}^{-1}\to K_\lambda^{-1}$ in norm as $\varepsilon\to 0$. The difference divided by ε converges in norm, because in the middle factor, the multiplication operator φ has the trivial norm, φ derivative, g. The f derivative exists in norm, by Lemma 9, because we just saw that K_λ^{-1} obeys the difference condition.

The φ derivative obeys the difference condition because K_λ^{-1} and the multiplication operator φ do, and the f derivative obeys it because the multiplication operator $\delta_f(g)\varphi = K_\lambda^{-1}g$ does. □

Higher functional derivatives with respect to f or φ may now be computed routinely, and they converge in H_n and $B(H_n)$ for positive integers n. Although there is no particular advantage in getting f derivatives through intermediate φ derivatives at the level of the tree approximation, it becomes convenient at the one loop level; so we discuss both derivatives.

The general f derivative is a sum of terms of the form

$$K_\lambda^{-1}h_1K_\lambda^{-1}\dots h_jK_\lambda^{-1}g,$$

where the h's are multiplication operators by functions of the form: a product of two, lower order f derivatives of φ (including φ itself). It is clear that this form is preserved under the operation of taking the next f derivative, because of Lemma 11, and the Leibniz rule for differentiating products. A simple induction argument, based on Lemma 4.iii, shows that the higher order f derivatives belong to $\mathrm{Re}\mathscr{S}$; and an induction based on Lemma 4.v shows that they are analytic at $\lambda\in[0,\infty)$ in the H_n norms. An induction plus Lemma 10.i shows that they obey the bounded difference condition.

The φ derivatives admit an analogous discussion.

Separate norm continuity in H_n of the f or φ derivatives as the g's vary in $\mathscr{S}$ is easy to check, as a consequence of the fact that $g\to 0$ in $\mathscr{S}$ entails $\|g\|_{2,n}\to 0$ and $|g|_n\to 0$. It is easy to verify that the derivatives are multilinear in the g's.

We summarize what we have learned so far, and complete our characterization of the tree approximation, in the following theorem:

Theorem 12. (i) *The generating functional for the tree approximation is*

$$\begin{aligned}L_0(f) &= -\int(\tfrac{1}{2}\varphi K\varphi + \tfrac{1}{4}\lambda\varphi^4 - \varphi f)dx\\ &= \int(\tfrac{1}{2}\varphi K\varphi + \tfrac{3}{4}\lambda\varphi^4)dx,\end{aligned} \tag{49}$$

where $\varphi = \delta L_0/df(x)\in\mathrm{Re}\mathscr{S}$ *is the solution of the* CFE *corresponding to* $f\in\mathrm{Re}\mathscr{S}$. L_0 *is continuous for f in* $\mathrm{Re}\mathscr{S}$ *and analytic at* $\lambda\in[0,\infty)$.

(ii) *For g's in* $\mathrm{Re}\mathscr{S}$, the smeared functional *derivatives* $\delta_f^m(g_1,\dots,g_m)\varphi$ *belong to* $\mathrm{Re}\mathscr{S}$, converge in H_n *for every n, are analytic at* $\lambda\in[0,\infty)$, *are continuous in f, and obey the bounded difference condition in those norms. Similar statements hold for the φ derivatives of any of the f derivatives.*

(iii) *The unsmeared functional derivatives,*

$$L_0(x_1, \ldots, x_m; f) = \frac{\delta}{\delta f(x_1)} \cdots \frac{\delta}{\delta f(x_m)} L_0(f), \tag{50}$$

are real, symmetric, tempered distributions.

Proof. (i) The expression for $L_0(f)$ is well known, for Minkowski fields, in the sense of an "effective potential" [1]; what is new here is the precise characterization. It is easy to check that L_0 is analytic in λ, and that

$$\Delta_\varphi(h) L_0 = O(\|h\|_{2,1}), \tag{51}$$

by bounding the L_4 norm by the H_1 norm, and by Theorem 1. It is easy to show that the φ derivative exists, and hence the f derivative does, too, by Lemma 9. The CFE gives $\delta_f(g) L_0 = \langle g, \varphi \rangle$.

(ii) This part of the theorem was already discussed.

(iii) That we get tempered distributions is a consequence of the nuclear theorem and our earlier discussion on separate continuity of the f derivatives of φ in the g's. The *symmetry* of the functional derivatives is easy to check for lower derivatives. For example, for the two-point function,

$$\delta^2 L_0 / \delta f(x_1) \delta f(x_2) = \langle x_1 | K_\lambda^{-1} | x_2 \rangle = \langle x_2 | K_\lambda^{-1} | x_1 \rangle, \tag{52}$$

it follows from the reality and Hermiticity of K_λ^{-1}. As for the higher derivatives, note that we could have evaluated them at nonzero ε by the formula

$$\begin{aligned} &\delta_f^m(g_1, \ldots, g_m) L_0(f + \varepsilon_1 g_1 + \ldots + \varepsilon_m g_m) \\ &= L_0(f + \varepsilon_1 g_1 + \ldots + \varepsilon_m g_m) / \partial\varepsilon_1 \ldots \partial\varepsilon_m . \end{aligned} \tag{53}$$

By a standard theorem from real analysis, we could conclude the symmetry of the f derivative in the g's from continuity in $(\varepsilon_1, \ldots, \varepsilon_m)$, because we know that the partial derivatives with respect to ε_i exist with any ordering; and we would then know them to be equal. But continuity, say at $(\varepsilon_1^0, \ldots, \varepsilon_m^0)$, follows from the bounded difference condition relative to f, which in turn is a consequence of that condition relative to φ, plus Lemma 3:

$$\begin{aligned} &|\Delta_f(\varepsilon'_1 g_1 + \ldots + \varepsilon'_m g_m) \langle g_m, \delta^{m-1}(g_1, \ldots, g_{m-1}) \varphi(f + \varepsilon_1^0 g_1 + \ldots + \varepsilon_m^0 g_m) \rangle| \\ &O(\|\varepsilon'_1 g_1 + \ldots + \varepsilon'_m g_m\|_{2,n'}), \end{aligned} \tag{54}$$

for some n'. □

Five remarks complete our discussion of the tree approximation.

Remark 2. $L_0(f)$ is invariant under the full, inhomogeneous Euclidean group, including reflections, because of the invariance of the CFE, the uniqueness of its solutions, and the invariance of four-dimensional Lebesgue measure dx.

Remark 3. We are not sure in precisely what way the tree functional $\mathscr{E}_0(f) = \exp L_0(f)/\hbar c$ violates the axioms of Euclidean field theory. For example, does it violate Symanzik-Nelson positivity; and if so, is it perhaps still the Laplace transform of a signed measure?

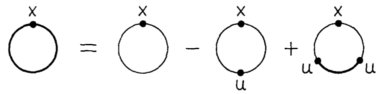

Fig. 1. Infinite parts (a) and (b), and the finite part (c) of $\delta\varphi = (a) - (b) + (c)$

Remark 4. The theory of the CFE [9] shows that $L_0(f)$ is well-defined for $f \in \mathrm{Re}\,L_2$. In that case, $\varphi \in \mathrm{Re}\,H_1$, and a Sobolev inequality says that the φ^4 term and the φf term are both finite.

Remark 5. We have made no statements about the size or uniformity of the size of the domain of analyticity in λ. We sort of expect that there will be no uniformity, neither in the number of points in the n-point function for a fixed order in the loop expansion, nor in the order of the loop expansion, for a given n-point function, nor even perhaps in f or in the choice of H_n norm in the CFE itself.

Remark 6. We certainly expect the functional derivatives of φ to exist, and be analytic in λ and continuous in f, not only in the H_n norms, but in the "rapid decrease" seminorms that remain to make a complete set for the topology of $\mathscr{S}$.

IV. The One Loop Correction

1. Solution of the Functional Field Equation

The object to be renormalized at the one loop level is $\delta\varphi = \delta\varphi(x)/\delta f(x)$, where φ obeys the CFE. The infinite parts can be isolated by expanding the kernel $\langle x|K_\lambda^{-1}|y\rangle$ through the resolvent formula, with one iteration. Let

$$u \equiv 3\lambda\varphi^2 . \tag{55}$$

Then

$$K_\lambda^{-1} = K^{-1} - K^{-1}uK^{-1} + K^{-1}uK^{-1}uK_\lambda^{-1} . \tag{56}$$

The strategy is to show that

$$\delta\varphi_N = \langle x|K^{-1}|x\rangle + 3^{-1}\mu^2 a_1 , \tag{57}$$

$$\delta\varphi_L = -\langle x|K^{-1}uK^{-1}|x\rangle + 9^{-1}c_1 u(x) , \tag{58}$$

are finite by the choice of the infinite parts of a_1 and c_1, while

$$\delta\varphi_F = \langle x|K^{-1}uK^{-1}uK_\lambda^{-1}|x\rangle \tag{59}$$

is automatically finite. The first two expressions above correspond, respectively, to the normal ordering of Φ^2, the x-space Feynman graph in Figure (1a), and the renormalization of the loop in Figure (1b). Figure (1c) represents a rearrangement of $\delta\varphi_F$.

The renormalized functional derivative at equal arguments is then written

$$\delta\varphi_R = \delta\varphi_N + \delta\varphi_L + \delta\varphi_F , \tag{60}$$

where all three pieces on the r.h.s. are finite.

The normal ordering term is diposed of in the usual way; the kernel

$$\langle x|K^{-1}|y\rangle=(2\pi)^{-4}\int e^{ik\circ(x-y)}/(k^2+\mu^2)dk \tag{61}$$

is $(\hbar c)^{-1}$ times the free, Euclidean propagator[4]. It is translation invariant, and so is constant when evaluated at $x=y$. We choose

$$a_1=\frac{-3}{\mu^2(2\pi)^4}\int\frac{dk}{k^2+\mu^2}+a_{1F}, \tag{62}$$

where a_{1F} is finite. Thus,

$$\delta\varphi_N=\mu^2 a_{1F}/3. \tag{63}$$

The loop term is also handled by the standard renormalization. Define the Fourier transform $\hat{u}$ by

$$\hat{u}(k)=(2\pi)^{-2}\int u(x)e^{-ik\circ x}dx. \tag{64}$$

Putting the infinite part of c_1 equal to the logarithmically divergent constant

$$c_{1\infty}=9(2\pi)^{-4}\int(q^2+\mu^2)^{-2}dq, \tag{65}$$

we get

$$\begin{aligned}\delta\varphi_L=&(2\pi)^{-6}\int\left\{\frac{1}{(q^2+\mu^2)^2}-\frac{1}{[(q-k)^2+\mu^2](q^2+\mu^2)}\right\}\\ &\cdot\hat{u}(k)e^{ik\circ x}dq\,dk+c_{1F}u(x)/9\\ =&(2\pi)^{-2}\int I_1(k)\hat{u}(k)e^{ik\circ x}dk+c_{1F}u(x)/9,\end{aligned} \tag{66}$$

where

$$\begin{aligned}&I_1(k)=(8\pi^2)^{-1}(\tau\log(\tau+\tfrac{1}{2})/(\tau-\tfrac{1}{2})-1),\\ &\tau=(k^2+4\mu^2)^{1/2}/2k.\end{aligned} \tag{67}$$

The renormalized loop integral I_1 is continuous, non-negative, and monotone increasing in k, starting at zero at $k=0$, and growing logarithmically at large k. Since $\hat{u}\in\mathscr{S}$, $I_1\hat{u}$ is continuous and rapidly decreasing. Taking derivatives in x-space gives $I_1\hat{u}k^l$ in k-space, which is absolutely integrable. We conclude that $\delta\varphi_L$ is smooth, with all derivatives uniformly bounded and vanishing at infinity. These functions are multipliers on $\mathscr{S}$, so $\varphi\delta\varphi_L\in\mathscr{S}$.

It is convenient to consider I_1 as a multiplication operator in k-space, and as a convolution operator in *x-space:*

$$\begin{aligned}\check{I}_1*\psi&=\int\check{I}_1(x-y)\psi(y)dy\\ &=\int I_1\hat{\psi}e^{ik\circ x}dk.\end{aligned} \tag{68}$$

Then $\check{I}_1*$ is a real, positive, self-adjoint operator on the appropriate domain in L_2, and is a bounded operator from H_n to $H_{n-\varepsilon}$ for every n and every $\varepsilon>0$.

Lemma 13. *$\varphi_{1L}=-3\lambda K_\lambda^{-1}[\varphi\delta\varphi_L]\in\mathrm{Re}\mathscr{S}$. All functional derivatives with respect to f and φ exist in H_n norm for every integer $n\geqq 0$. φ_{1L} and its functional derivatives are analytic at $\lambda\in[0,\infty)$ and obey the bounded difference condition in those norms.*

---

4 All dot products in this paper are Euclidean

Proof. We are in Re$\mathscr{S}$ because of Theorem 1 and Lemma 4.iii. The only term of concern is $\check{I}_1 * \varphi^2$. Note that it obeys the bounded difference condition in the H_n norm, because

$$\|\Delta_\varphi(h)\check{I}_1 * \varphi^2\|_{2,n} \leqq C\|\Delta_\varphi(h)\varphi^2\|_{2,n+1}, \tag{69}$$

and φ^2 obeys the condition.

Putting $h = \varepsilon g$, it is easy to check that

$$\delta_\varphi(g)\check{I}_1 * \varphi^2 = \check{I}_1 * (2\varphi g) \tag{70}$$

exists in H_n norm, and so by Lemma 9,

$$\delta_f(g)\check{I}_1 * \varphi^2 = 2\check{I}_1 * (\varphi K_\lambda^{-1} g). \tag{71}$$

This derivative clearly obeys the bounded difference condition in H_n norm. Higher functional derivatives are no problem; they can be treated by simple induction and all properties easily verified.

The expression $K_\lambda^{-1}(\varphi \check{I}_1 * \varphi^2)$ is now easy to treat as a product of factors, each of which has functional derivatives of all orders obeying the bounded difference condition and analytic in λ. That leaves $K_\lambda^{-1}(\varphi^3)$, which is even easier to treat. □

There remains the piece $\delta\varphi_F$, which can be written

$$\delta\varphi_F = \langle x|K^{-1} u K_\lambda^{-1} u K^{-1}|x\rangle, \tag{72}$$

because the resolvent formula says $K^{-1} u K_\lambda^{-1}$ is Hermitean. To treat higher functional derivatives, we want to consider expressions of the form

$$\begin{aligned} X(x) &= \langle x|K^{-1} u_1 K_\lambda^{-1} u_2 \dots K_\lambda^{-1} u_m K^{-1}|x\rangle \\ &= (2\pi)^{-8} \int \frac{\langle u_1^{k_1}, K_\lambda^{-1} u_2 \dots K_\lambda^{-1} u_m^{k_2}\rangle}{(k_1^2+\mu^2)(k_2^2+\mu^2)} e^{i(k_1-k_2)\circ x} dk_1\, dk_2, \end{aligned} \tag{73}$$

where $u_1, \dots, u_m \in \mathscr{S}$,

$$u^k(x) \equiv e^{ik\circ x} u(x) = T(k) u(x), \tag{74}$$

and $T(k)$ is the unitary, momentum translation operator on L_2. Denote the matrix element in the integrand by

$$M(k_1 - k_2, k_2) = \langle u_1^{k_1}, K_\lambda^{-1} u_2 \dots K_\lambda^{-1} u_m^{k_2}\rangle. \tag{75}$$

We begin our attack on X by studying M. We use the multi-index notation D_k^l for derivatives with respect to k.

Lemma 14. *Let $x^l \psi \in L_2$. Then, considered as a function of k,*

$$N(k) \equiv \|D_k^l K_\lambda^{-1} T(k)\psi\|_2 \in L_p(R^4) \tag{76}$$

for $2 < p \leqq \infty$. For these values of p,

$$\|N\|_p \leqq \|(k^2+\mu^2)^{-1}\|_p \, \|x^l\psi\|_2 \, |K_\lambda^{-1} K|. \tag{77}$$

Proof. The effect of the k-derivative is simply to bring down x^l on ψ. We shall therefore forget about the derivative and restore the x^l at the end of the argument. Thus

$$N(k)=\|K_\lambda^{-1}KK^{-1}T(k)\psi\|_2 \leqq |K_\lambda^{-1}K|\,\|K^{-1}T(k)\psi\|_2\,. \tag{78}$$

Next,

$$\|K^{-1}T(k)\psi\|_2^2=\int[(q+k^2)^2+\mu^2]^{-2}|\psi(q)|^2dq\,. \tag{79}$$

Applying the Young inequality, we find

$$\|N^2(\cdot)\|_p \leqq |K_\lambda^{-1}K|^2\,\|\,|\hat{\psi}|^2\|_1\|(k^2+\mu^2)^{-2}\|_p\,, \tag{80}$$

which makes sense on the r.h.s. for $1<p\leqq\infty$. Taking square roots, noting that $\|\,|\hat{\psi}|\,\|_2=\|\psi\|_2$, and restoring x^l, we get the result. □

Lemma 15. *Define M as in Equation (75), and let $k=k_1-k_2$, $q=k_2$. Let*

$$N(k,q)=k^l D_{k_1}^{l'_1}D_{k_2}^{l'_2}M(k,q)\,. \tag{81}$$

Consider N as an L_p function of q for fixed k. Then for every $2<p\leqq\infty$, there is a polynomial P, depending on p and l, but independent of k, such that

$$\begin{aligned}\|N(k,\cdot)\|_p \leqq\; & P(|K_\lambda^{-1}|\,\|u\|_{2,|l|+1})\,\|x^{l'_1}u_1\|_{2,|l|}\\ &\cdot\|u_2\|_{2,|l|+1}\cdots\|u_{m-1}\|_{2,|l|+1}\\ &\cdot\|x^{l'_2}u_m\|_{2,|l|}\,|K_\lambda^{-1}|^{m-2}|K_\lambda^{-1}K|\,.\end{aligned} \tag{82}$$

In the exceptional cases $|l|=0$ or 1, the $H_{|l|+1}$ norms should be replaced by H_3 norms.

Proof. The k-derivatives bring down powers $x^{l'_1}$ and $x^{l'_2}$, which may be put onto u_1 and u_m, respectively. We forget them for now. M is clearly a C^∞ function of k_1 and k_2. It is convenient to rewrite M as follows:

$$M(k,q)=\langle u_1, T(-k)K_\lambda^{-1}(q)u_2\ldots K_\lambda^{-1}(q)u_m\rangle\,, \tag{83}$$

where

$$\begin{aligned}K_\lambda(q)&\equiv T(-q)K_\lambda T(q)=K(q)+u\,,\\ K(q)&=T(-q)KT(q)=(-i\nabla+q)\cdot(-i\nabla+q)+\mu^2\,.\end{aligned} \tag{84}$$

The idea of the proof is the same as in the Jost-Hepp theorem [11], except that here we look at matrix elements of the momentum rather than the space translation operator, and we have a uniformity problem in the extra variable q. As in the Jost-Hepp argument, we write

$$k^l T(-k)=(\mathrm{ad}\, i\nabla)^l T(-k)\,, \tag{85}$$

where the r.h.s. is a multiple commutator of order $|l|$ of $T(-k)$ with components of the operator ∇. Any ∇'s standing to the left of $T(-k)$ get absorbed into the left-hand vector u_1; the maximum degree on u_1 is $|l|$. Those standing to the right are to be pushed through the K_λ^{-1} and u factors, so they can be absorbed by u_m.

The operator $K(q)$ is still a function of ∇, and that makes the commutators controllable:

$$[\nabla, K_\lambda^{-1}(q)]=-K_\lambda^{-1}(q)(\nabla u)K_\lambda^{-1}(q)\,. \tag{86}$$

The result after pushing the V's through is a sum of matrix elements of the same form as before, except that there may be more $K_\lambda^{-1}(q)$'s, and a certain number of factors $D^{l_i}u$ may be inserted, and $u_2, \ldots, u_m$ may get derivatives, at most of degree $|l|$. Each matrix element has a bound of the form

$$\prod_i (|D^{l_i}u|\,|K_\lambda^{-1}|) \prod_{j=2}^{m-1} (|D^{l_j}u_j|\,|K_\lambda^{-1}|)$$
$$\cdot \|u_1\|_{2,|l|}\,\|K_\lambda^{-1}\,T(q)D^{l_m}u_m\|_2\,.$$

The result follows by Lemma 14. □

Note that the above arguments and the statement of Lemma 15 are scarcely modified if we put different f's in each K_λ, or if we allow the λ's to be unequal and complex, but near the real axis. Also note that with this modification, if the u's are the usual functionals of φ, then $\Delta_\varphi(h)M$ or $\Delta_\lambda(\varepsilon)M$ (we hope the latter notation is clear) is a sum of terms with the same structure as M, with extra factors of K_λ^{-1}, $K_{\lambda,h}^{-1}$, etc., and with similar estimates that are $O(\|h\|_{2,n})$ for some n, or $O(\varepsilon)$.

Lemma 16. *Define X by Equation* (73). *Then X is in $\mathcal{S}$, and in* Re$\mathcal{S}$ *when $u_1, \ldots, u_m$ are real. $\delta\varphi_F$ and all of its functional derivatives with respect to φ and f exist, obey the bounded difference condition, and are analytic at $\lambda \in [0, \infty)$, in H_n norm for all n.*

Proof. Write

$$X = (2\pi)^{-8} \int \frac{M(k,q)e^{ik\circ x}\,dk\,dq}{[(k+q)^2+\mu^2](q^2+\mu^2)}\,. \tag{87}$$

Note that one over the denominator, along with any number of derivatives, belongs to L_p for $p>1$, in the variable q. The L_p norm of one over the denominator (or any of the derivatives of that) is bounded by a (sum of) product(s) of L_{2p} norms of two factors, each of which is finite for $p>1$; and these bounds are independent of k, by the translation invariance of Lebesgue measure. To fix our ideas, take $p=4/3$ for this part of the integrand.

Lemma 15 says that the numerator, or any of its k derivatives, belongs to L_4. Thus, by the Hölder inequality, the q integral exists and is bounded by

$$\|M(k,\cdot)\|_4\,\|[(\cdot)^2+\mu^2]^{-1}\|_{8/3}^2\,,$$

with analogous bounds for k derivatives. Lemma 15 also says that these bounds are rapidly decreasing in k, so we conclude that X is the Fourier transform of a function in $\mathcal{S}$, and hence is in $\mathcal{S}$.

The reality of X is most easily seen from Equation (75). Using the reality of K_λ^{-1}, we get $\bar{M}(k,q) = M(-k,-q)$, which does the job because the denominator obeys the same law.

To put the discussion in the frame work of the typical argument for functional derivatives and the bounded difference structure, we note a standard argument:

$$\|D^l X\|_2 \leq \|(1+|x|^4)^{-1}\|_2\,\|(1+|x|^4)D^l X\|_\infty\,. \tag{88}$$

After translating $|x|^4$ and D^λ to k-space, we can eventually bound the H_n norm of X by expressions of the sort in Lemma 15. Taking into account our earlier remarks on the bounded difference structure of M, it is not hard to see that we are in the typical situation for our methods of proof, and that all statements in Lemma 16 about $\delta\varphi_F$ follow.

The proof just given gets H_n bounds more indirectly than previous arguments. Perhaps there is another way to get to the same goal, based on an expression that we shall get later, containing $\delta\varphi_F$ in terms of a trace. Without further proof, we state

Lemma 17. $\varphi_{1F} = -3\lambda K_\lambda^{-1}[\varphi\delta\varphi_F]$ *obeys exactly the same statements as* φ_{1L} *in Lemma* 13.

That completes our discussion of the one loop correction to the solution of the functional field equation, $\hbar c\varphi_1$,

$$\varphi_1 = \varphi_{1N} + \varphi_{1L} + \varphi_{1F} + b_1\lambda K_\lambda^{-1}\Delta\varphi_0 . \tag{89}$$

We did not put the piece $\varphi_{1N} = -a_{1F}\mu^2 K_\lambda^{-1}\varphi$ in a lemma, because it trivially has all the properties stated in Lemmas 13 and 17 for φ_{1L} and φ_{1F}.

2. *The Generating Functional*

It remains to discuss the one loop correction to the generating functional. To compute it formally is a standard manipulation [1]. At first, we follow that procedure, which is a fps functional integration of polynomials in φ, to get the candidate for $L_1(f)$. Then we take the formal result as a starting point, and show that it is rigorously well-defined, and has the required functional derivatives.

So we take the one loop correction to the solution of the functional field equation, and write

$$\varphi_1 = \delta L_1/\delta f . \tag{90}$$

Assuming that L_1 obeys the conditions of Lemma 9, this follows from

$$\delta L_1/\delta\varphi = -3\lambda\varphi\delta\varphi_R + \lambda b_1\Delta\varphi . \tag{91}$$

All of the terms except

$$\delta L_{1F}/\delta\varphi = -3\lambda\varphi\delta\varphi_F \tag{92}$$

are trivial to integrate formally, and the answer is displayed in Theorem 20. This term is not so bad either, as a fps in λ. Remembering the definition of u in Equation (55), we get

$$\begin{aligned}\frac{\delta L_{1F}}{\delta\varphi} &= \sum_{m=2}^{\infty}(-1)^{m+1}3\lambda\varphi(x)\langle x|K^{-1}(uK^{-1})^m|x\rangle \\ &= \frac{1}{2}(\delta/\delta\varphi)\sum_{m=3}^{\infty}\frac{1}{m}\int\langle x|(-uK^{-1})^m|x\rangle dx \\ &= \frac{1}{2}(\delta/\delta\varphi)\sum_{m=3}^{\infty}\frac{1}{m}\mathrm{Tr}(-A)^m ,\end{aligned} \tag{93}$$

$$A \equiv 3\lambda\varphi K^{-1}\varphi . \tag{94}$$

This gives the formal expression

$$L_{1F}(f) = -\frac{1}{2}\operatorname{Tr}\log[(I+A)e^{\frac{A^2}{2}-A}], \tag{95}$$

which we now study.

We are going to find that A is a non-negative, self-adjoint, compact operator on L_2, and that A^3 has a trace. The last fact should already be expected from Lemma 16, by putting $m=2$ and replacing K_λ^{-1} by K^{-1} in Equation (73) for X. We get that more directly from the following:

Lemma 18. *Let u_1 and u_2 belong to H_3. Then $O \equiv K^{-1}u_1K^{-1}u_2K^{-1}$ is a trace class operator on L_2, and the usual trace norm obeys*

$$\|O\|_{\mathrm{Tr}} \leq C\|u_1\|_{2,3}\,\|u_2\|_{2,3}\,. \tag{96}$$

Proof. Let $A(x)$ and $B(k)$ be functions, and let A and B be the corresponding operator functions of the operators x and $-i\nabla$ on L_2. It is well known that if $A(\cdot)$ and $B(\cdot)$ are L_2 functions in their respective variables, then AB is a Hilbert-Schmidt operator with

$$\operatorname{Tr} B^*A^*AB = (2\pi)^{-n}\,\|A(\cdot)\|_2^2\,\|B(\cdot)\|_2^2\,, \tag{97}$$

and that if $A(\cdot)$ and $B(\cdot)$ are continuous functions that vanish at infinity, then AB is compact[5]. We thus know that $K^{-1}u$ is compact, for u in H_3. We also know that $K^{-(1+\varepsilon)}u$ is Hilbert-Schmidt, and that $u_1K^{-3}u_2$ is of trace class.

For the operator O, the idea is to shove enough K^{-1}'s to the middle by means of the commutation relation:

$$\begin{aligned}[K^{-1},u] &= K^{-1}[2(\nabla u)\cdot\nabla + (\nabla^2 u)]K^{-1} \\ &= 2K^{-1}\nabla\cdot(\nabla u)K^{-1} - K^{-1}(\nabla^2 u)K^{-1}\,.\end{aligned} \tag{98}$$

For each commutator, there is a net gain of one inverse power of ∇ in the middle of the expression for O, coming from the term with the ∇K^{-1} factor on the right or left, as the case may be. Thus, consider

$$\begin{aligned}K^{-1}u_1K^{-1}u_2K^{-1} &= u_1K^{-3}u_2 + u_1K^{-2}[u_2,K^{-1}] \\ &\quad + [K^{-1},u_1]K^{-2}u_2 + [K^{-1},u_1]K^{-1}[u_2,K^{-1}]\,.\end{aligned} \tag{99}$$

The first three terms on the r.h.s. have effective powers of at least $K^{-5/2}$ in the middle, and derivatives of at most second order on the u's. The last term has one piece, containing ∇u_1 and ∇u_2, where the effective power in the middle is only K^{-2}. Shoving one more K^{-1} through in that term gives $K^{-5/2}$ in the middle, and derivatives of third order on one of the u's. Any outside factors K^{-1} are bounded, so the result follows from the statements in the first paragraph of the proof, plus the Schwartz inequality for the Hilbert-Schmidt inner product. □

Lemma 19. *Let*

$$A = 3\lambda\varphi K^{-1}\varphi, \qquad B = \log[(I+A)e^{\frac{A^2}{2}-A}]\,. \tag{100}$$

[5] I want to thank I. Herbst for making these facts known to me

(i) *A is non-negative, self-adjoint, and compact on L_2; A^3 is in the trace class; and so is B. In particular*

$$0 \leqq B \leqq A^3/3. \tag{101}$$

(ii) *Let $T = \mathrm{Tr}\, B$. Then for $h \in H_3$,*

$$|\Delta_\varphi(h)T| = O(\|h\|_{2,3}); \tag{102}$$

and T is analytic at $\lambda \in [0, \infty)$.

(iii) *For any $g \in \mathscr{S}$,*

$$\delta_\varphi(g)T = 3\lambda\, \mathrm{Tr}[A^2(I+A)^{-1}(\varphi K^{-1}g + gK^{-1}\varphi)]. \tag{103}$$

Proof. (i) Positivity of A is evident from that of K^{-1}, and self-adjointness, too, since only bounded operators are involved. The trace condition for A^3 follows from Lemma 18, and the estimate on B can be taken as straightforward if A is replaced by a non-negative real number. That will be clear from the integral representation in the next step.

(ii) The integral representation

$$B = \int_0^1 A^3\eta^2/(I+\eta A)\,d\eta \tag{104}$$

converges in $B(L_2)$ norm, for positive A, and in trace norm, if A^3 also has a trace. Since $\Delta_\varphi(h)\,\mathrm{Tr}\,B = \mathrm{Tr}[\Delta_\varphi(h)B]$, we consider

$$\Delta_\varphi(h)B = \int_0^1 \{(A_h^3 - A^3)(I+\eta A)^{-1} - A^3(I+\eta A)^{-1}\eta(A_h - A)(I+\eta A)^{-1}\}\,\eta^2\,d\eta. \tag{105}$$

Taking into account the cyclic property of the trace, and factoring out certain $B(L_2)$ norms, we get

$$\|\Delta_\varphi(h)B\|_{\mathrm{Tr}} \leqq \|A_h^3 - A^3\|_{\mathrm{Tr}}/3 + 2^{-2}|A_h - A|\,\mathrm{Tr}\,A^3. \tag{106}$$

To get the result, look at

$$A_h - A = 3\lambda(hK^{-1}\varphi + \varphi K^{-1}h + hK^{-1}h). \tag{107}$$

In the first term on the r.h.s. of Equation (106), expand A_h by Equation (107), cycle the trace to bring a factor h to the outside, and write $\|hO\|_{\mathrm{Tr}} \leqq |h|\,\|O\|_{\mathrm{Tr}}$. The factor O left behind is still of trace class, by Lemma 18. For any factors h^2 remaining in the terms in O, we remove $|h|$ by cycling the trace, still leaving a trace class operator of the sort in Lemma 18. We conclude that $\|O\|_{\mathrm{Tr}}$ is uniformly bounded for small $\|h\|_{2,3}$, because $|h|$ is easily bounded by $C\|h\|_{2,3}$. The result for the second term in Equation (106) is immediate.

Analyticity in λ follows from that of φ after similar arguments.

(iii) Now put $h = \varepsilon g$ in Equation (105) for $\Delta_\varphi(h)B$, and note that, except for the factor $(I+\eta A_{\varepsilon g})^{-1}$, the rest is a polynomial in ε, of order ε for small ε, with trace class coefficients, because of Equation (107) for $A_{\varepsilon g} - A$. To handle the factor, note that

$$|\Delta_\varphi(\varepsilon g)(I+\eta A)^{-1}| = O(\varepsilon), \tag{108}$$

from the resolvent formula, uniformly in η for positive η. Thus, cycling the trace, we get

$$\delta_\varphi(g)T = \mathrm{Tr}\left\{\int_0^1 [3A^2(1+\eta A)^{-1} - A^3\eta(1+\eta A)^{-2}]\eta^2\, d\eta\, \delta_\varphi(g)A\right\}, \tag{109}$$

where

$$\delta_\varphi(g)A = 3\lambda(\varphi K^{-1}g + gK^{-1}\varphi) \tag{110}$$

can be taken as simply a notation, although in fact this formula makes sense in $B(H_n)$ for integer $n \geqq 0$. The integral may be evaluated from the functional calculus, a triviality since A is not only Hermitean but compact; and that gives the result in the Lemma. □

To make the connection with our earlier expression for $\delta\varphi_F$, we use the bounded operator identity:

$$KK_\lambda^{-1}\varphi = \varphi(I+A)^{-1}, \tag{111}$$

and its Hermitean conjugate. A straightforward manipulation gives

$$\begin{aligned}\delta_\varphi(g)T &= 6\lambda\, \mathrm{Tr}[g\varphi K^{-1}uK_\lambda^{-1}uK^{-1}] \\ &= 6\lambda \int g(x)\varphi(x)\langle x|K^{-1}uK_\lambda^{-1}uK^{-1}|x\rangle dx.\end{aligned} \tag{112}$$

One may question in the last step how the expression $\langle x|O|x\rangle$ is defined, and whether the definition agrees with our earlier definition of $\delta\varphi_F$. In general, the Hilbert-Schmidt kernel $\langle x|O|y\rangle$ of a trace class operator O has an L_1 restriction to the diagonal $x=y$ in a certain sense. In our case, we can use Lemma 15 and the argument in the proof of Lemma 16 to show that the Hilbert-Schmidt kernel is the Fourier transform of a function in $L_1(dk_1\, dk_2)$, and hence is continuous in (x, y); and the restriction to $x=y$ can be carried out directly and seen to belong to L_1.

With this sketchy proof, we conclude that

$$\delta T/\delta\varphi = 6\lambda\varphi\,\delta\varphi_F. \tag{113}$$

where $\delta\varphi_F$ is the function in $\mathrm{Re}\mathscr{S}$ defined earlier.

The smeared functional derivatives of T, with values in the real numbers, could be discussed directly in terms of the trace norm. Our results on the first, unsmeared functional derivative, part of the correction to the solution of the functional field equation, with values in $\mathrm{Re}\mathscr{S}$, are of course stronger than what that discussion would give directly.

To put it all together for the one loop correction, we have

Theorem 20. (i) *The one loop correction to the generating functional is* $\hbar c L_1(f)$, *where*

$$\begin{aligned}L_1(f) = &-\int(\tfrac{1}{2}\lambda\mu^2 a_{1F}\varphi^2 - \tfrac{1}{2}\lambda b_1\varphi\Delta\varphi + \tfrac{1}{4}\lambda^2 c_{1F}\varphi^4)dx \\ &- 9/4(2\pi)^{-2}\lambda^2 \int \varphi^2(x)\check{I}_1(x-y)\varphi^2(y)\,dx\,dy \\ &- \tfrac{1}{2}\mathrm{Tr}\log[(1+A)\exp(\tfrac{1}{2}A^2 - A)], \\ A \equiv\ & 3\lambda\varphi K^{-1}\varphi.\end{aligned} \tag{114}$$

and where φ *is the solution of the* CFE *corresponding to* f, *and* $\check{I}_1$ *is defined in Equations* (67) *and* (68). $L_1(f)$ *and its smeared functional derivatives of all orders with respect to* φ *and* f *exist, are analytic at* $\lambda \in [0, \infty)$, *and are continuous in* $f \in \mathrm{Re}\mathscr{S}$.

(ii) *The first unsmeared functional derivative* $\varphi_1 = \delta L_1/\delta f$ *belongs to* $\mathrm{Re}\mathscr{S}$, *and has smeared functional derivatives of all orders with respect to* φ *and* f, *belonging to* $\mathrm{Re}\mathscr{S}$. *The functional derivatives exist in* H_n *norm for all integers* $n \geqq 0$, *and* φ_1 *and its functional derivatives are analytic at* $\lambda \in [0, \infty)$ *and continuous in* f *in* $\mathrm{Re}\mathscr{S}$ *in those norms.*

(iii) *The unsmeared functional derivatives*

$$L_1(x_1, \ldots, x_m; f) = \delta^m L_1(f)/\delta f(x_1)\ldots\delta f(x_m) \tag{115}$$

are real, symmetric, tempered distributions.

Proof. (i) The only thing we haven't verified already is that the polynomical terms in φ have the correct properties. By now, we feel we have the right to claim that that is trivial.

(ii) This is already verified in Lemmas 13, 17, and 19, up to trivialities.

(iii) The argument that we have tempered distributions is the same as in the tree approximation, based on the nuclear theorem. The argument that these distributions are symmetric in the permutation of x's is also the same, based on the bounded difference property, which is true by Lemmas 13, 17, and 19, and Lemmas 6 and 7. □

Remarks 2, 3, 5, and 6 on the tree approximation apply equally well to the one loop correction.

Appendix I. Bounds on Multiplication Operators

Instead of the expression (20) for $|O|_n$, we use the definition of the operator norm as the supremum of absolute values of matrix elements

$$\langle \psi, K^n O \psi \rangle, \qquad \|\psi\|_{2,n} = 1\,. \tag{A.1}$$

The procedure is to integrate each $\nabla \cdot \nabla$ in K^n by parts once, putting one ∇ on the left-hand ψ. Then let the other gradients act to the right, giving a sum of terms of the form

$$|\langle D^{l_1}\psi, (D^{l_2}O)D^{l_3}\psi\rangle| \leqq C\|\psi\|_{2,n}\,\|D^{l_2}O\,D^{l_3}\psi\|_2\,, \tag{A.2}$$

where the first factor on the r.h.s. follows because $|l_1| \leqq n$. We also have $|l_2| + |l_3| \leqq n$, and we estimate the second factor by

$$\begin{aligned}\|D^{l_2}O\|_4\,\|D^{l_3}\psi\|_4 &\leqq C\|O\|_{2,|l_2|+1}\,\|\psi\|_{2,|l_3|+1}\\ &\leqq C'\|O\|_{2,n+1}\,\|\psi\|_{2,n}\,,\end{aligned} \tag{A.3}$$

for $|l_3| + 1 \leqq n$, and by $\|O\|_\infty\,\|\psi\|_{2,n}$ times a constant for $|l_3| = n$.

Appendix II. Proof of Lemma 3

The argument is by induction, after settling the cases $n=1$ and 2. For larger n, we can make effective use of the $B(H_n)$ norm.

$n=1$: This case is settled by Rauch's a priori estimate:

$$\|\varphi_1-\varphi_2\|_{2,1} \leqq \|f_1-f_2\|_{2,-1}\,. \tag{A.4}$$

Rauch's proof is very short: multiply the difference of field equations by $\varphi_1-\varphi_2$; drop the positive term

$$\lambda(\varphi_1-\varphi_2)(\varphi_1^3-\varphi_2^3) \geqq 0\,; \tag{A.5}$$

and apply the Schwartz inequality.

Before looking at $n=2$, we want to look at some general features of the case $n \geqq 2$. Applying the CFE, we get

$$\begin{aligned}(\|\varphi_1-\varphi_2\|_{2,n})^2 &= \langle \varphi_1-\varphi_2, K^{n-1}[f_1-f_2-\lambda(\varphi_1^3-\varphi_2^3)]\rangle \\ &\leqq \|\varphi_1-\varphi_2\|_{2,n}\,\|f_1-f_2\|_{2,n-2} \\ &\quad +\lambda|\langle \varphi_1-\varphi_2, K^{n-1}[(\varphi_1-\varphi_2)Q]\rangle|\,,\end{aligned} \tag{A.6}$$

where

$$Q \equiv \varphi_1^2+\varphi_1\varphi_2+\varphi_2^2\,.$$

We have to keep partial track of the two cases:

odd n: Estimate the second term on the r.h.s. by

$$\|\varphi_1-\varphi_2\|_{2,n-1}\,\|K^{[(n-1)/2]}(\varphi_1-\varphi_2)Q\|_2\,,$$

where $[a]$ means the integer part of a. The first factor is bounded by $\|\varphi_1-\varphi_2\|_{2,n}$, and we divide it out.

even n: Estimate the second term by

$$\|\varphi_1-\varphi_2\|_{2,n}\,\|K^{[(n-1)/2]}(\varphi_1-\varphi_2)Q\|_2\,.$$

Immediately divide out the first factor.

That leaves the second factor, involving Q. The argument that follows is going to work for any polynomial Q in φ_1 and φ_2, in particular for the Q that would result from replacing the interaction term φ^3 in the CFE by any other polynomial. So from now on we let Q be any polynomial.

$n=2$: The factor to be estimated is[6]

$$\begin{aligned}\|(\varphi_1-\varphi_2)Q\|_2 &\leqq \|\varphi_1-\varphi_2\|_4\,\|Q\|_4 \\ &\leqq C\|\varphi_1-\varphi_2\|_{2,1}\,\|Q\|_4\,.\end{aligned} \tag{A.7}$$

The difference in the H_1 norm gets bounded by $\|f_1-f_2\|_{2,-1} \leqq \mu^{-1}\|f_1-f_2\|_2$, from the case $n=1$. To estimate the Q factor, we note from $n=1$ and Lemma 2 that

[6] We learned this estimate from J. Rauch

$\|\varphi\|_{2,2} \leqq \|f\|_2 P_1$. Now bound the Q factor by sums of products of L_p norms of φ's, $p \leqq 2 < \infty$, which in turn are bounded by H_2 norms, then by L_2 norms of f's.

Induction Step. Suppose we have proved the result up to some $n \geqq 2$. Then, for $n+1$ we have to estimate

$$\begin{aligned} \|K^{[n/2]}[(\varphi_1 - \varphi_2)Q]\|_2 &= \|Q(\varphi_1 - \varphi_2)\|_{2,n'} \\ &\leqq |Q|_{n'} \|\varphi_1 - \varphi_2\|_{2,n'}, \end{aligned} \tag{A.8}$$

where $n' = 2[n/2]$ is n for even n and $n-1$ for odd n. Since $n' \leqq n$, $\|\varphi_1 - \varphi_2\|_{2,n'}$ is bounded by $\|f_1 - f_2\|_{2,n'-2}$ times a polynomial in $H_{n'-2}$ norms of f's, by the induction hypothesis, which is better than the H_{n-1} norm we need.

The Q factor can be bounded by a polynomial in $\|\varphi\|_{2,n'+1}$, from Equation (21), because $n' \geqq 2$. Since $n'+1 \leqq n+1$, we conclude from Lemma 2 and the induction hypothesis that this is bounded by a polynomial in H_{n-1} norms of f's. □

Appendix III. Proof of Lemma 4

(i) By Theorem 1, $\varphi^2 \in \mathrm{Re}\mathscr{S}$. Thus, multiplication by $3\lambda\varphi^2$ is a bounded, Hermitean operator on L_2. The operator K is known to be self-adjoint on the domain $H_2 \subset L_2$, which is easy to check in k-space. Thus, K_λ is self-adjoint with domain H_2.

(ii) Strict positivity is evident because $K_\lambda \geqq K \geqq \mu^2 I$. The existence and boundedness of the inverse follows from the functional calculus and the fact that the spectrum of K_λ is contained in $[\mu^2, \infty)$, so that of K_λ^{-1} is in $[0, \mu^2]$.

To see that $K_\lambda^{-1} H_n \subset H_{n+2}$, note first that $K_\lambda^{-1} L_2 = H_2$. Indeed, for any strictly positive, selfadjoint operator A, one has the range $\mathscr{R}(A) = L_2$, and $\mathscr{D}(A) = \mathscr{R}(A^{-1})$. Thus, for any $h \in H_n$, we know there is a $\psi \in H_2$ such that

$$K_\lambda \psi = h\,. \tag{A.9}$$

Suppose we know that $K_\lambda^{-1} H_s \subset H_{s+2}$ for $s < n$. Taking the gradient of the above equation, we get

$$\begin{aligned} K_\lambda \nabla\psi &= \nabla h - 6\lambda\varphi\nabla\varphi\psi \in H_{n-1} \\ &\Rightarrow \nabla\psi \in H_{n+1} \Rightarrow \psi \in H_{n+2}\,. \end{aligned} \tag{A.10}$$

(iii) That K_λ maps $\mathscr{S}$ continuously onto itself, and $\mathrm{Re}\mathscr{S}$ continuously onto itself is evident. We need only show that the maps are onto, for the inverse of a continuous, linear, one-to-one map of $\mathscr{S}$ onto itself is automatically continuous.

Let $K_\lambda \psi = h \in \mathscr{S}$. We have shown above that then $\psi \in H_n$ for all positive integers n. Now for any $n > 2$, every function in H_n is the Fourier transform of an L_1 function, and hence is continuous, uniformly bounded, and zero at infinity. Thus, ψ is infinitely differentiable, with uniformly bounded derivatives vanishing at infinity; so in particular, $\psi \in 0_M$. But then

$$K\psi = h - 3\lambda\varphi^2\psi \in \mathscr{S}\,. \tag{A.11}$$

The result for $\mathscr{S}$ follows, because K and K^{-1} map $\mathscr{S}$ onto $\mathscr{S}$, as well as $\mathrm{Re}\mathscr{S}$ onto $\mathrm{Re}\mathscr{S}$. The result for $\mathrm{Re}\mathscr{S}$ follows, because K_λ preserves reality, and if $K_\lambda \psi = h$ is real, then $K_\lambda \mathrm{Re}\psi = h \Rightarrow \psi = \mathrm{Re}\psi$.

(iv) The operator

$$A_n = K^{n/2} K K_\lambda^{-1} K^{-n/2} \tag{A.12}$$

is bounded on L_2, because $K_\lambda^{-1} K^{-n/2}$ maps L_2 into H_{n+2}, and $K^{1+n/2}$ maps H_{n+2} onto L_2; so A_n is defined on all of L_2, while A_n^* is densely defined, on $H_{n+2} \subset L_2$; so $A_n = A_n^{**}$ is a closed operator defined on all of L_2. Thus, KK_λ^{-1} is bounded on H_n.

Next,

$$K^{n/2} K_\lambda^{-1} K K^{-n/2} = A_{n-2} \tag{A.13}$$

which easily gives boundedness of $K_\lambda^{-1} K$ on H_n, except for the case $n=1$, which yields to an argument that $\mathscr{D}(K_\lambda^{1/2}) = H_1$, and that $K^{1/2} K_\lambda^{-1/2}$ is bounded on L_2.

(v) For a fixed f, consider the operators $K_{\lambda_0}^{-1}$ and $K_{\lambda_0+\varepsilon}^{-1}$, where $\lambda_0 \in [0, \infty)$, and ε is a small, complex number. The resolvent expansion gives

$$\begin{aligned} K_{\lambda_0+\varepsilon}^{-1} &= K_{\lambda_0}^{-1} \sum_{m=0}^{\infty} (-hK_{\lambda_0}^{-1})^m \\ &= \left[\sum_{m=0}^{\infty} (-hK_{\lambda_0}^{-1})^m\right] K_{\lambda_0}^{-1}, \end{aligned} \tag{A.14}$$

where $h = 3(\lambda_0 + \varepsilon)\varphi_\varepsilon^2 - 3\lambda_0 \varphi^2$, and the φ's correspond to $\lambda_0 + \varepsilon$ and λ_0, respectively. The strategy is to show that, for each H_n, there is a complex neighborhood of zero in ε where the expansion converges uniformly in norm. The same will then be true if we multiply the expansions from the right or left by K, because the factors $KK_{\lambda_0}^{-1}$ and $K_{\lambda_0}^{-1} K$ are bounded on H_n for integers $n \geqq 0$.

The first step is to show that h, considered as a multiplication operator on H_n, is small in norm. Thus,

$$|h|_n \leqq 3|\varepsilon|\, |\varphi_\varepsilon|_n^2 + 3\lambda_0 |\varphi_\varepsilon - \varphi|_n\, |\varphi_\varepsilon + \varphi|_n . \tag{A.15}$$

From Equation (21), we may replace the $|\cdot|_n$ on the r.h.s. by $H_{n'}$ norms, and the whole thing goes to zero like ε as $\varepsilon \to 0$, because φ_ε is analytic (and hence continuous and uniformly bounded) in $H_{n'}$ norm, by Theorem 1.

That is good enough for the norm convergence of the resolvent expansion in every H_n, for integer $n \geqq 0$, uniformly for ε small, because $K_{\lambda_0}^{-1} = K_{\lambda_0}^{-1} K K^{-1}$ is bounded, and $|hK_{\lambda_0}^{-1}|_n$ can be made uniformly less than one for ε small.

To complete the proof, we need only remark that h is an analytic function of ε, with values in $B(H_n)$, because φ_ε is, due to Theorem 1 and Equation (21). □

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Communicated by J. Glimm

Received February 9, 1977