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The Central Limit Theorem and the Problem of Equivalence of Ensembles

R. L. Dobrushin
Institute of Problems of Information Transmission, Ak. Nauk. U.R.S.S., Moscow, USSR

Brunello Tirozzi*
Istituto Matematico dell'Università di Camerino, I-62032 Camerino, Italy

Abstract. In this paper we show that the local limit theorem is a consequence of the integral central limit theorem in the case of a Gibbs random field ξ_t, $t \in Z^\nu$ corresponding to a finite range potential.

We apply this theorem to show that the equivalence between Gibbs and canonical ensemble is a consequence of the integral central limit theorem and of very weak conditions on decrease of correlations.

Introduction

The integral and local limit theorems for sums of random variables belonging to a random process with dependent values have been considered in a lot of papers on probability theory, see for example [1] and the literature there quoted and also some more recent papers [2, 3]. In connection with the developments of the theory of random fields and their applications to physics in the last years some papers appeared concerning the integral limit theorem [6, 14, 15], and the local limit theorem for random fields [5, 4, 7].

From the point of view of statistical physics these local theorems are interesting because they are strongly connected with the problem of the equivalence of canonical and grand canonical ensembles, if one considers, as we do in this article, this equivalence not only in the sense of equality between thermodynamical functions, but also in the sense of equality between all the correlation functions. The known proofs of the local limit theorems can be applied in particular cases and are founded on some special methods developed for studying the Gibbs random field. We use different techniques and emphasize the connection between local and integral limit theorems which up to now have never been pointed out.

The main result of this paper (§ 1.1) is that the integral limit theorem for an integer valued Markov field with non vanishing conditional probabilities (or, which is the same, [8] for a Gibbs random field with a range R finite and bounded potential) implies the local limit theorem.

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The proof of this proposition happens to be very simple.

Applying the methods of [9] one can easily reduce the problem of estimating the tail of the characteristic function, which is especially needed for the proof of the local limit theorem, to similar estimation for sums of independent random variables. And so it is possible to apply standard methods of probability theory which are presented in the pioneer work of Gnedenko [10] (see also [11, 1]).

Although these methods are well known to specialist of probability theory, they are not sufficiently popular in the usual mathematical physics literature: for this reason and to make this article self-contained, we give a detailed presentation of the methods. In § 1.2 a generalisation of the above proposition to the case of an unbounded potential and an additive functional of the random field is presented.

Therefore, the problem of the proof of the local central limit theorem is reduced to the proof of the integral central limit theorem. In 1.3 there is a discussion about what situations it is possible to prove the central integral limit theorem. In particular, it is possible to prove that the integral central limit theorem follows from the analyticity properties of the free energy as a function of the activity and so it can be derived from the well known results on this kind of analyticity.

In Section 2 the results of Section 1 are applied to the problem of the equivalence of the ensembles. The main results consists, roughly speaking, in the possibility of establishing the equivalence in all cases when the integral central limit theorem has been already proven. We note that in the recent years Thompson [12], and Georgii [13] have developed an interesting approach to the problem of equivalence of ensembles based on the concept of the Gibbs canonical state. But from the fundamental result of Georgii on the coincidence of extremal canonical and extremal usual Gibbs states it is not possible to obtain results of the type of those in § 2 of our work. In fact, as far as we know, there are no methods for checking if the limit canonical state is extremal or not which avoid the construction of the type used here.

0. Notations and Definitions

D.1. Space and Configurations. Let Z^ν be the integer lattice in the ν dimensional space. Let X be a finite set with more than one point (here and in the following $|V|$ is the cardinality of the set V), $|X|>1$. A configuration in the volume V will be denoted by $X=(x_t, t\in V)$. The set of all such configurations X^V will be called the space of configurations in the volume V.

D.2. Random Field. A random field is a system of random variables $\xi_t, t\in \boldsymbol{Z}^\nu$ taking values in X.

Let P_V be a probability measure on $(X^V, \mathfrak{B}_V)$, where $\mathfrak{B}_V$ is the σ-algebra in X^V generated by cylindrical sets such that

$$P_V(A)=P_r((\xi_t, t\in V)\in A), \qquad A\in\mathfrak{B}_V\,.$$

From Kolmogorov's theorem it follows that the probability distribution of the field $P = P_{Z^\nu}$ is uniquely defined by the set of finite dimensional distributions P_V, i.e. by distributions P_V with $|V| < \infty$.

D.3. Potential. We will consider the following situations. The energy

$$U(x_t, t \in V) = \sum_{S \subset V} \Phi(x_t, t \in S) \tag{0.1}$$

where the potential Φ is a function with values in the extended real line $(-\infty, +\infty]$ defined on the space $L_{Z^\nu} = \bigcup_{\substack{J \subset Z^\nu \\ 0 \leqq |J| < \infty}} X^J$ and such that

$$\Phi(x_t, t \in J) = 0 \quad \text{if} \quad \sup_{t, s \in J} |s - t| > R \tag{0.2}$$

i.e. Φ is finite range and

$$\Phi(\phi) = 0 \tag{0.3}$$

where ϕ is the empty set, and for every $\bar{x} \in X^{Z^\nu \setminus I}$ where $|I| < \infty$, there exists at least one $x \in X^I$ such that the interaction energy

$$\Phi_I(x|\bar{x}) = \sum_{\substack{J: J \cap I \neq \phi \\ J \subset Z^\nu}} \Phi(x'_t, t \in J) \quad \text{where} \quad x'_t = \begin{cases} x_t & t \in I \\ \bar{x}_t & t \in Z^\nu \setminus I \end{cases} \tag{0.4}$$

is finite.

Here and in the following we use the conventions:

$$a + \infty = \infty, \qquad a > -\infty, \qquad e^{-\infty} = 0.$$

Further we suppose that $\Phi(x)$ is invariant with respect to the translation group acting on L_{Z^ν} defined by $(\Theta_t x)_s = x_{s-t}$ where $x \in L_{Z^\nu}$ i.e.

$$\Phi(\Theta_t x) = \Phi(x), \qquad x \in L_{Z^\nu}. \tag{0.5}$$

D.4. Gibbs Distribution and Gibbs Random Field. The Gibbs distribution in the finite volume I, $|I| < \infty$ with boundary conditions $\bar{x} \in x^{Z^\nu \setminus I}$ is the probability distribution

$$q_I(x|\bar{x}) = \frac{\exp\{-\Phi_I(x|\bar{x})\}}{\sum_{y \in X^I} \exp\{-\Phi_I(y|\bar{x})\}}, \qquad x \in X^I. \tag{0.6}$$

A random field is a Gibbs random field if for every finite $I \subset Z^\nu$ its conditional distributions are given by

$$Pr\{\xi_t = x_t, t \in I \,|\, \xi_t = \bar{x}_t, t \in Z^\nu \setminus I\} = q_I(x|\bar{x}) \tag{0.7}$$

for all $(x_t, t \in I) \in X^I$ and almost all $\bar{x} \in X^{Z^\nu \setminus I}$. Consider a sequence of cubes $\{V_k\}$ such that

$$V_k = \{t \in Z^\nu : 0 \leqq t_i \leqq a_k, i = 1, \ldots, \nu\}$$

where $\{a_k\}$, $k\in Z^+$ is a monotone increasing sequence. We use this sequence $\{V_k\}$ for the sake of simplicity but it is easy to see that our considerations are valid also for any sequence of volumes going to infinity in the sense of van Hove [20].

Let $\{V_k\}$ be a sequence defined before. We shall say that the sequence $\{P_k\}$ of probability measures on $\{X^{V_k}, \mathfrak{B}_{V_k}\}$ is Gibbsian if it is defined in one of these two ways:

1) Given a sequence of boundary conditions $\{\bar{x}_t, t\in \hat{V}_k\}=\bar{x}^k$

$$P_k(A)=\sum_{x\in A} q(x|\bar{x}^k), \qquad A\in\mathfrak{B}_{V_k}$$

where

$$\hat{V}_k=\left\{t\,\middle|\,\min_{s\in V_k}|t-s|\leqq R,\ t\notin V_k\right\}$$

or

2) $P_k(A)=P_{V_k}(A) \qquad A\in\mathfrak{B}_{V_k}$

where $P_{V_k}(\cdot)$ is the restriction on $(X^{V_k},\mathfrak{B}_{V_k})$ of the Gibbs measure corresponding to the Gibbs random field defined above.

D.5. Integral and Local Central Limit Theorem. We say that the random variable ξ is lattice distributed with step h if it takes integer values $k_1,\dots,k_s$ such that $Pr\{\xi=k_i\}>0$ and if $k_i-k_j=l_{ij}h$ then l_{ij} are integers such that their greatest common divisor is one, see [10], Chapter 8.

Let $\{S_k\}$ be a sequence of random variables corresponding to the cubes $\{V_k\}$, set

$$ES_k=\sum_u uPr\{S_k=u\}, \qquad DS_k=\sum_u (u-ES_k)^2\, Pr\{S_k=u\} \tag{0.8}$$

and

$$\bar{S}_k=\frac{S_k-ES_k}{\sqrt{DS_k}} \quad \text{if} \quad DS_k>0\,.$$

Then we shall say that $\{S_k\}$ satisfies the integral central limit theorem if:

α) $DS_k \underset{k\to\infty}{\sim} D|V_k|$,

β) $D>0$,

γ) $Pr\{\bar{S}_k<x\}\underset{k\to\infty}{\longrightarrow}(2\pi)^{-1/2}\int_{-\infty}^{x} e^{-u^2/2}du \qquad -\infty<x<+\infty.$

Moreover suppose that each S_k takes integer values, let p by any integer and set

$$P_k(p)=Pr\{S_k=p\}$$

$$Z_p^k=\frac{p-ES_k}{\sqrt{DS_k}} \tag{0.9}$$

then we say that the sequence $\{S_k\}$ satisfies the local central limit theorem if α), β) are true and

$$\sup_p\left|\sqrt{DS_k}\,P_k(p)-(2\pi)^{-1/2}e^{-\frac{(Z_p^k)^2}{2}}\right|\underset{k\to\infty}{\longrightarrow}0\,. \tag{0.10}$$

In particular we may consider special expressions for S_k. Let ξ_t, $t\in Z^\nu$ be a Gibbs random field and set

$$S_k = \sum_{t\in V_k} \xi_t \tag{0.11}$$

and suppose that the quantities $Pr\{S_k=u\}$ are given by a Gibbsian sequence of probability measure $\{P_k\}$ as defined above.

Under these two assumptions we shall say that the Gibbs random field with some potential satisfies integral central limit theorem if α), β), γ) are verified, and that it satisfies local central limit theorem if α, β (0.10) is verified, for any Gibbsian sequence corresponding to this potential.

1. The Local Limit Theorem for a Gibbs Random Field and Its Connections with the Integral Limit Theorem

1.1.

The aim of this section is to show how the local limit theorem for a Gibbs random field can be derived from the integral central limit theorem. Since we use a finite range potential, we have some simplifications: we think that the extension to the case of long range potentials is not straightforward and requires perhaps the use of more general techniques like those in [1] combined in some way with our arguments.

Consider a Gibbs random field ξ_t, $t\in Z^\nu$ defined on the lattice Z^ν, let $\{V_k\}$ be a sequence of cubes defined as before.

Theorem 1. *If the potential Φ is bounded then the local central limit theorem for the corresponding Gibbs field follows from the integral central limit theorem.*

Proof. Using standard constructions we have the following representation for the variables defined in (0.9)

$$2\pi P_k(p) = \frac{1}{\sqrt{DS_k}} \int_{-\pi\sqrt{DS_k}}^{+\pi\sqrt{DS_k}} E(e^{i\tau\bar{S}_k}) e^{-i\tau Z_p^k} d\tau \tag{1.1}$$

$$2\pi \frac{1}{\sqrt{2\pi}} e^{-\frac{(Z_p^k)^2}{2}} = \int_{-\infty}^{+\infty} e^{-i\tau Z_p^k - \frac{\tau^2}{2}} d\tau . \tag{1.2}$$

Then we have

$$\begin{aligned}\sup_p 2\pi \left| \sqrt{DS_k} P_k(p) - (2\pi)^{-1/2} e^{-\frac{(Z_p^k)^2}{2}} \right| &\leq \int_{-A}^{+A} \left| E e^{i\tau\bar{S}_k} - e^{-\frac{\tau^2}{2}} \right| d\tau \\ &+ \int_{|\tau|\geq A} e^{-\frac{\tau^2}{2}} d\tau + \int_{A\leq|\tau|\leq\delta\sqrt{DS_k}} |E e^{i\tau\bar{S}_k}| d\tau + \int_{\pi\delta\sqrt{DS_k}\leq|\tau|\leq\pi\sqrt{DS_k}} |E e^{i\tau\bar{S}_k}| d\tau \\ &= I_1 + I_2 + I_3 + I_4 \end{aligned} \tag{1.3}$$

where I_j are equal to the corresponding terms in the middle part of the inequality (1.3) and A, δ are some positive constants ($A<\delta\sqrt{DS_k}$, $\delta<\pi$) to be choosed later in a suitable way. Now we are going to give estimates for these four integrals. Let $\varepsilon > 0$ be fixed.

I_1. For every A $I_1 \leqq \varepsilon/4$ $k \geqq k_\varepsilon$, because the integral central limit theorem is true [10, Chap. 8].

I_2. Choosing A big enough we obtain $I_2 \leqq \varepsilon/4$.

I_3. We can take for sake of simplicity and without loss of generality R to be an integer, and set

$$n_k = \left[\frac{|V_k|}{(2R+1)^\nu}\right], d = |V_k| - n_k(2R+1)^\nu .$$

Let us now consider a family $\alpha_k = \{A_j^k\}$, $j = 1, \ldots, n_k$ of disjoint cubes A_j^k on the lattice each of them containing $(2R+1)^\nu$ points; this family forms a partition of the cube with site of length $(n_k)^{1/\nu}(2R+1)$, shown on Figure 1 in the two dimensional case. We use the following notations:

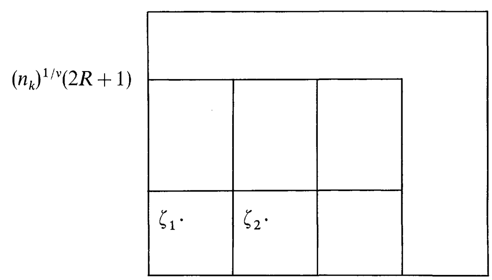

Fig. 1

t_j^k is the center of the cube A_j^k

$$\zeta_j = \xi_{t_j^k},\ \bar{V}_k = V_k \backslash (t_1^k \cup \ldots \cup t_{n_k}^k) \qquad \eta_s = \xi_s, \qquad s \in \bar{V}_k$$

γ_k is a vector random variable $\gamma_k = (\eta_s, s \in \bar{V}_k)$.

Then we define the σ-algebra generated by the following events

$$\{\eta_s = y_s, s \in \bar{V}_k\}, \qquad y_s \in X, \qquad s \in \bar{V}_k \tag{1.4}$$

and we denote it $\mathfrak{B}_{\bar{V}_k}$. In other words $\mathfrak{B}_{\bar{V}_k}$ is the σ-algebra generated by the random variables in the points of V_k except for the centers of the cubes of the family α_k.

We have the following representation of $\bar{S}_k$

$$\bar{S}_k = \varphi_1 + \varphi_2, \qquad \varphi_1 = \sum_{i=1}^{n_k} \frac{\zeta_i - E\zeta_i}{\sqrt{DS_k}}, \qquad \varphi_2 = \sum_{t \in \bar{V}_k} \frac{\eta_t - E\eta_t}{\sqrt{DS_k}}. \tag{1.5}$$

Using the fact that φ_2 is measurable with respect to the σ-algebra $\mathfrak{B}_{\bar{V}_k}$, and using the conditional expectations respect to this σ-algebra, we have:

$$Ee^{it\bar{S}_k} = E(E(e^{it\bar{S}_k}|\mathfrak{B}_{\bar{V}_k})) = E(e^{it\varphi_2}E(e^{it\varphi_1}|\mathfrak{B}_{\bar{V}_k})) . \tag{1.6}$$

From (1.6) we have

$$|Ee^{i\tau\bar{S}_k}| \leqq E(|e^{i\tau\varphi_2}|\,|E(e^{i\tau\varphi_1}|\mathfrak{B}_{\bar{V}_k})|) \leqq E(|E(e^{i\tau\varphi_1}|\mathfrak{B}_{\bar{V}_k})|). \tag{1.7}$$

Let us define some probability distributions which are needed in the following calculations. We can define the following conditional probabilities associated to the σ-algebra $\mathfrak{B}_{\bar{V}_k}$

$$Pr\{\zeta_j = p | \eta_s = y_s, s \in \bar{V}_k\}, \qquad j = 1, \ldots, n_k. \tag{1.8}$$

From the hypothesis that $\xi_t, t \in Z^\nu$ is a Gibbs random field corresponding to a finite range potential we have that for every j, the conditional probability in (1.8) depends only on the values of η_s, $s \in A_j^k$.

Furthermore the variables ζ_j are conditionally independent under conditions of this type. So introducing the new vector notations

$$\gamma^j = (\eta_s, s \in A_j^k \setminus t_j^k), \qquad \gamma^d = \left(\eta_s, s \in V_k \Big\backslash \bigcup_{j=1}^{n_k} A_j^k\right) \tag{1.9}$$

we have that the conditional distributions of every variable ζ_j, can be written as

$$Pr\{\zeta_j = u | \gamma^j = \bar{\gamma}^j\} = P_{\bar{\gamma}^j}(u) \tag{1.10}$$

where $\bar{\gamma}^j$ is a possible value of the random vector γ^j. Let also

$$w_{\bar{\gamma}^j}(\tau), \qquad E_{\bar{\gamma}^j}, \qquad D_{\bar{\gamma}^j}$$

be respectively the characteristic function, mean and variation, associated to the conditional probability distribution (1.10)

$$\begin{aligned} w_{\bar{\gamma}^j}(\tau) &= \sum_{u \in X} e^{iu\tau} P_{\bar{\gamma}^j}(u) \\ E_{\bar{\gamma}^j}\zeta_j &= \sum_{u \in X} u P_{\bar{\gamma}^j}(u) \\ D_{\bar{\gamma}^j}\zeta_j &= \sum_{u \in X} (u - E_{\bar{\gamma}^j}\zeta_j)^2 P_{\bar{\gamma}^j}(u) \end{aligned} \tag{1.11}$$

Now we are able to use these considerations for estimating the integral I_3.

By using (1.7) and the fact that the ζ_j are conditionally independent, we obtain

$$\begin{aligned} |Ee^{i\tau\bar{S}_k}| &\leqq E\left(\prod_{j=1}^{n_k} \left| E\left(e^{i\tau\frac{\zeta_j - E\zeta_j}{\sqrt{DS_k}}} \,\middle|\, \gamma^j = \bar{\gamma}^j\right)\right|\right) \\ &\leqq \max_{\bar{\gamma}^1, \ldots, \bar{\gamma}^{n_k}} \prod_{j=1}^{n_k} \left| E\left(e^{i\tau\frac{\zeta_j - E\zeta_j}{\sqrt{DS_k}}} \,\middle|\, \gamma^j = \bar{\gamma}^j\right)\right|. \end{aligned} \tag{1.12}$$

In other words we can say that, owing to the fact that the ζ_j are conditionally independent, we have reduced the estimation of the integral I_3 to the same estimation as in the case of independent variables with a smaller number of summands. Now we will estimate the product appearing in the last expression of (1.12): we will do it for every conditional distribution and then we will find an estimate independent of $\bar{\gamma}^i$. Before doing this, let us note that the mean $E\zeta_j$ appearing in (1.12) is not equal to the conditional mean $E_{\bar{\gamma}^j}\zeta_i$ defined in (1.11) but we can substitute the first with the second by using the equality

$$\begin{aligned}&\left|E\left(e^{i\frac{\tau}{\sqrt{DS_k}}(\zeta_j-E\zeta_j)}\middle|\gamma^j=\bar{\gamma}^j\right)\right|\\&\quad=\left|E\left(e^{i\frac{\tau}{\sqrt{DS_k}}(\zeta_j-E_{\bar{\gamma}^j}\zeta_j)}\middle|\gamma^j=\bar{\gamma}^j\right)\right|=\left|W_{\bar{\gamma}^j}\left(\frac{\tau}{\sqrt{DS_k}}\right)\right|. \qquad (1.13)\end{aligned}$$

Now we can expand the characteristic function (1.13) and obtain

$$E\left(e^{i\frac{\tau}{\sqrt{DS_k}}(\zeta_j-E_{\bar{\gamma}^j}\zeta_j)}\middle|\gamma^j=\bar{\gamma}^j\right)=1-\frac{\tau^2}{2DS_k}D_{\bar{\gamma}^j}\zeta_j+\frac{O(\tau^2)}{DS_k}. \qquad (1.14)$$

Let us now estimate $D_{\bar{\gamma}^j}\zeta_j$. Observe that all the probabilities $P_{\bar{\gamma}^j}(l)$ are uniformly bounded in l and in j from below by a positive constant α because of the definition of conditional distributions (D.4) and of the boundness of the potential, so we obtain

$$\begin{aligned}D_{\bar{\gamma}^j}\zeta_j&=\sum_{l\in X}P_{\bar{\gamma}^j}(l)(l-E_{\bar{\gamma}^j}\zeta_j)^2\\&\geqq\alpha\sum_{l\in X}(l-E_{\bar{\gamma}^j}\zeta_j)^2\geqq\alpha g \qquad (1.15)\end{aligned}$$

where $g>0$. From (1.15) we obtain that $D_{\bar{\gamma}^j}\zeta_j$ is uniformly bounded from below so it is possible to choose a constant $\delta>0$ which is independent of j because of translation invariance, such that for $|\tau|\leqq\pi\delta\sqrt{DS_k}$ we have

$$\left|w_{\bar{\gamma}^j}\left(\frac{\tau}{\sqrt{DS_k}}\right)\right|\leqq e^{-\frac{\tau^2}{DS_k}D_{\bar{\gamma}^j}\zeta_j}\leqq e^{-\frac{\tau^2}{DS_k}g\alpha} \qquad (1.16)$$

uniformly on j and $\bar{\gamma}^j$.

Consequently (1.12) can be written in the final form

$$|Ee^{i\tau\bar{S}_k}|\leqq e^{-\frac{\tau^2}{DS_k}n_k g\alpha} \qquad (1.17)$$

for $|\tau|\leqq\pi\delta\sqrt{DS_k}$. Now choosing V_k sufficiently large and remembering condition α), β) in definition (D.5) of integral limit theorem and the construction introduced above, we see that $\frac{n_k}{DS_k}\underset{k\to\infty}{\sim}\frac{1}{(2R+1)^\nu\cdot D}$ we obtain that there exists a $c>0$ such that

$$\int_{A\leqq|\tau|\leqq\pi\delta\sqrt{DS_k}}|Ee^{i\tau\bar{S}_k}|d\tau\leqq\int_A^\infty e^{-c\tau^2}d\tau\leqq\varepsilon/4 \qquad (1.18)$$

for δ sufficiently small, A and k sufficiently large.

I_4. For estimating this integral we use the same argument as for I_3. So we have

$$I_4 \leqq \int\limits_{\pi\delta\sqrt{DS_k} \leqq \tau \leqq \pi\sqrt{DS_k}} \max_{\bar{y}^1,\dots,\bar{y}^{n_k}} \prod_{j=1}^{n_k} \left| w_{\bar{y}^j}\left(\frac{\tau}{\sqrt{DS_k}}\right)\right| d\tau$$

$$\leqq \sqrt{DS_k} \int\limits_{\delta}^{\pi} \max_{\bar{y}^1,\dots,\bar{y}^{n_k}} \prod_{j=1}^{n_k} |w_{\bar{y}^j}(\tau)| d\tau$$

$$\leqq \sqrt{DS_k} \int\limits_{\delta}^{\pi} \max_{\bar{y}^1,\dots,\bar{y}^{n_k}} e^{\frac{1}{2}\sum_{j=1}^{n_k}(|w_{\bar{y}^j}(\tau)|^2-1)} d\tau \tag{1.19}$$

Now we use the explicit form of $w_{\bar{y}^j}(\tau)$:

$$|w_{\bar{y}^j}(\tau)|^2-1 = \sum_{l,m\in X} P_{\bar{y}^j}(l)P_{\bar{y}^j}(m)\cos t(l-m)-1$$

$$= -2\sum_{l,m\in X} P_{\bar{y}^j}(l)P_{\bar{y}^j}(m)\sin^2\frac{\tau}{2}(l-m)$$

$$\leqq -2\alpha^2 \sum_{l,m\in X} \sin^2\frac{\tau}{2}(l-m) \tag{1.20}$$

and this estimate is again uniform in $\bar{y}^j$ and j. Finally

$$I_4 \leqq \sqrt{DS_k}(\pi-\delta)e^{-\alpha^2 n_k \sum_{l,m\in X}\sin^2\frac{\delta}{2}(l-m)}$$

$$\sim \sqrt{D|V_k|}\,(\pi-\delta)e^{-\alpha^2\left[\frac{|V_k|}{(2R+1)^\nu}F(\delta)\right]} \tag{1.21}$$

where $F(\delta)$ is a positive function of δ. Choosing V_k big enough we have for any fixed $\delta<\infty$ that $I_4 \leqq \frac{\varepsilon}{4}$ and this completes the proof of the theorem.

1.2.

It is possible to generalize Theorem 1 to the case when Φ is not bounded and the expression of S_k is more general. Consider a system of functions which take integer values

$$\varphi_V(x_t, t\in V) \quad \text{where} \quad x_t\in X, \quad V\in Z^\nu, \quad |V|<\infty .$$

Suppose further that the two following conditions are satisfied:

Condition A.

$$\begin{aligned} &\varphi_V(x_t, t\in V) = \varphi_{V+n}(\tilde{x}_t, t\in V+n), \quad \tilde{x}_{t+n}=x_t \\ &\varphi_V(x_t, t\in V)=0 \quad \text{if} \quad \sup_{t,s}|t-s|>\hat{R} \end{aligned} \quad n\in Z^\nu \tag{1.22}$$

without loss of generality we can suppose that the constant $\hat{R}$ in (1.22) is equal to the constant R in (0.2). We also need some additional condition about the property of being lattice distributed for the sums of the random variables $\varphi_V(\xi_t, t\in V)$.

Consider for any finite volume $V_0 \subset Z^\nu$ the random variable

$$\theta_{V_0} = \sum_{V' \cap V_0 \neq \phi} \varphi_{V'}(\xi_t, t\in V') \tag{1.23}$$

where ξ_t is a Gibbs field with given potential Φ.

Condition B. There exists a cube V_0 such that for any boundary condition $\xi_t = \bar{x}_t$, $t \in Z^\nu \setminus V_0$ the conditional distribution of the random variable θ_{V_0} under the condition $\xi_t = \bar{x}_t$, $t\in Z^\nu\setminus V_0$ is lattice with step 1. It is interesting to note that there exist situations when the condition B is not satisfied for V_0 such that $|V_0| = 1$, but it is satisfied for enough large V_0. A simple example is the case when $X = \{0, 1\}$, the potential is pair, nearest neighbour with hard core condition $U(1,1) = \infty$ and $U(0,0) \neq \infty$ and $\varphi_V(x_t, t\in V) = x_0$ if $|V| = 1$, $\varphi_V = 0$ for other V.

Let us now introduce the following random variable

$$S_k = \sum_{V \subset V_k} \varphi_V(\xi_t, t\in V) \tag{1.24}$$

then is true the following theorem:

Theorem 2. *If the sequence of random variables $\{S_k\}$ satisfies the integral central limit theorem and conditions A and B are satisfied then $\{S_k\}$ satisfies also the local central limit theorem.*

Proof. The general scheme of the proof is the same as in Theorem 1.

But here we shall use the following geometrical construction for estimating the integrals I_3, I_4.

Let R_0 be the side of the cube V_0 of the condition B and let $R' = R_0 + R$. In analogy with the construction of 1.1 we consider the family of disjoint cubes $\alpha_k = \{A_j^k\}$ but with R' in place of R. Inside each of the cubes A_j^k we introduce the subcube $\bar{A}_j^k$; which is cocentric with A_j^k, has the sides parallel to the sides of A_j^k and is congruent to the cube V_0 (see Fig. 2). Now we can introduce the same symbols as in (1.1), with only one change.

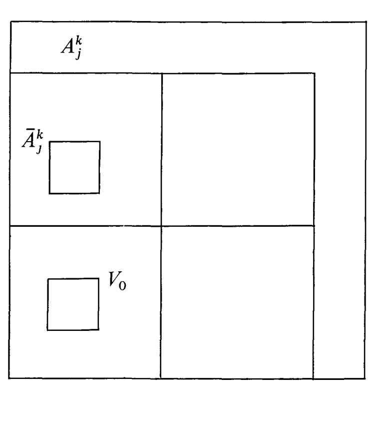

Fig. 2

Now

$$\bar{V}_k = V_k \setminus \bigcup_{j=1}^{n_k} \bar{A}_j^k, \qquad n_k = \left[\frac{|V_k|}{(2R'+1)^\nu}\right]. \tag{1.25}$$

Writing S_k in analogy with (1.5) we have the following decomposition

$$S_k = \sum_{j=1}^{n_k} \theta_{\bar{A}_j^k} + \sum_{V' \subset \bar{V}_k} \varphi_{V'}(\xi_t, t \in V'). \tag{1.26}$$

If we change the definition of γ^j in (1.9) and we set

$$\gamma^j = (\xi_t, t \in A_j^k \setminus \bar{A}_j^k) \tag{1.27}$$

then the main inequality (1.12) still holds. Now the argument needed for the evaluation of I_3 proceeds just like in the proof of the Theorem 1. The main difference between the two arguments is the bound (1.15) which must be changed by

$$D_{\bar{\gamma}^j} \theta_{\bar{A}_j^k} \geqq c \tag{1.28}$$

which follows from condition B for θ_{V_0}. The estimation of I_4 follows immediatly from the considerations of the previous sections and from the following lemma [10, Chap. 8].

Lemma 1. *If ϱ is a lattice distributed random variable then for every $\delta > 0$, it is possible to find a positive constant d_ϱ such that for every τ, $\varepsilon \leqq |\tau| \leqq \frac{2\pi}{h} - \varepsilon$ it is true the following inequality*

$$|w_\varrho(\tau)| \leqq e^{-d_\varrho}$$

where $w_\varrho(\tau)$ is the characteristics function of ϱ.

1.3. Discussion on the Hypothesis of Theorem 1

Let us now discuss in what cases the central limit theorem in the sense of (D.5) for a Gibbsian random field is verified. So we have to discuss for which Gibbs random field conditions α), β), γ) are verified.

There are many different ways to obtain the integral central limit theorem for a Gibbs random field as a corollary of well known results. Usually it is possible to check also α) with the same methods.

I. Method. The integral central limit theorem can be obtained from the fact that the Gibbs random field satisfies some condition of decreasing of correlations for large distances. Such theorems have been known for a long time [1] in the one dimensional case and recently Nakhapitan [14] and Malyshev [17] have generalized them to the case of a random field. Nakhapitan uses the Bernstein method of deletions and Malyshev uses the method of evaluating the semiinvariants. The conditions for which the decreasing of correlations in the necessary sense takes place have been found for a large class of Gibbs random fields[1] and in particular they are always true in the one dimensional case.

II. Method. It consists of making use of some equations for correlations functions which are specific for the Gibbs field [4, 5, 6, 7].

III. Method. It is possible to derive α), γ) for a Gibbsian sequence corresponding to some sequence of boundary conditions [case 1 of (D.4)] also in any case when the sequence of partition functions, corresponding to increasing volumes and these boundary conditions, converges uniformly in some open set of the complex plane to an analytic function of the chemical potential. In [18], using this note, α) and γ) are shown in the case when $\nu=1$ but it is easy to see that the same argument can be repeated for any dimension.

The same discussion can be, almost without change, extended to the case of Theorem 2. It is easily seen that $\varphi_V(\xi_t, t\in V)$ is a random field translationally invariant if ξ_t are translationally invariant and that $\varphi_V(\xi_t, t\in V)$ have the necessary properties of decreasing of correlations if the field ξ_t has the decreasing of correlations in the same sense.

β) In [19] it is shown that this condition is always true in the situation of Section 1.1 and for all non degenerate in some sense potentials in the case of Section 1.2.

So there exists a large class of explicit conditions on potential when the integral and local central limit theorems are proven. For example we can formulate the following theorem:

Theorem 3. *Let us set as usual* $U^*_{\beta,\mu}(x_t, t\in V)=\begin{cases}\beta U(x_t, t\in V), & |V|\geqq 2\\ -\mu x_t, & |V|=1\,.\end{cases}$ *Then the integral and local limit theorems for the Gibbs random field* ξ_t, $t\in Z^\nu$ *with potential* $U^*_{\beta,\mu}$ *hold when*

1) $\nu=1$ *for every* β *and* μ.

2) $\nu\geqq 1$.

If the potential is bounded, when $|\mu|\geqq\mu_0$ *or* $\beta\leqq\beta_0$ *where* μ_0, β_0 *are some constants.*

If the potential is not bounded, when

$$\mu\leqq\mu_0(\beta)$$

where $\mu_0(\beta)$: *continuous function such that it has finite limit when* $\beta\to\infty$ *and*

$$\mu_0(\beta)\sim\frac{c}{\beta},\ c>0,\ \beta\to 0.$$

3. $\nu\geqq 1$ *for* $X=\{-1,+1\}$, *for an attractive, symmetric respect the transformation* $x_t\to -x_t$ *pair potential for*

3a. $\nu\geqq 1$ *any* β *and* $\mu\neq 0$.

3b. $\nu\geqq 1$, $\mu=0$ *and* $\beta\geqq\beta_1$, *where* β_1 *some constant, for stationary Gibbs fields*

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[1] Recently in the works [15, 16] the integral central limit theorem for random fields has been demonstrated for some special conditions on the decrease of correlations but there are no methods available for verifing such conditions in the case of a Gibbs random field.

Recently integral limit theorem has been prooven by Deo [32] using conditions on the decrease of correlation which are not fulfilled for non trivial stationary Gibbs field.

which are extremal in the set of Gibbs fields, and for non stationary extremal field which was constructed in [31].

The proof of the integral central limit theorem can be obtained in the four cases of Theorem 3 from the following list of possible ways (which is far from being exaustive):

Case 1. By method I (see [1]) from the results, for example of [21], by method III from the classical results of Van Hove [20].

Case 2. By method I (see [17], or [14]) together with the results for example, in [21], or also by method II [4–7] or method III and the results of [26, 22].

Case 3a. In this case it is used method III and the result of Lee-Yang [23] (see also Ruelle [26]), or by method I [17] and see also the additional reference there.

Case 3b. It follows from an application of method I. The condition of decrease of correlations is cheked for stationary case in [24] for example, and for the non-stationary case in [31]. In the stationary case it is possible also to use Malyshev's results [17].

2. Local Limit Theorem and the Problem of Equivalence of Ensembles

In this section we apply the general mathematical results of the first section to the discussion of a problem of great interest in Statistical Mechanics: the equivalence between canonical ensemble and Gibbs ensemble.

The idea of applying local limit theorems to this problem is not new and can be found for example in the old and very interesting book of Khinchin [29]. "Mathematical Foundations of Statistical Mechanics" where he has developed the method of applying the limit theorems of probability theory to some problems of Statistical Physics and he has shown, as an example, that the local limit theorem for indipendent random variables can be applied to show the equivalence between canonical and usual description of an ideal gas.

This result is also connected with results about the equivalence of ensembles obtained by various other authors. In one of the first papers on this subject Bogoliubov, Petrina, Khazet [30] have shown in some situation the equality of correlation functions when $V \to \infty$, Halfina [7], have obtained the same results using a different approach, and Gurevich [28] has shown that for one dimensional Markov chain the multidimensional local limit theorem is a necessary and sufficient condition for the equivalence in some sense between canonical and Gibbs ensemble. In Section 2.1 we define the notion of equivalence between ensembles. The main results are contained in Section 2.2.

2.1. Definitions

D.6.1. Gibbs Ensemble. Assume that $\Phi(x_t, t \in V)$ is a potential defined like in (D.3) for $|V| \geqq 2$ and equal to zero when $|V| = 1$, we will consider it as fixed in the following considerations.

Then we introduce the family of potentials Φ_μ, $\mu\in R^1$ such that:

$$\Phi_\mu(x_t, t\in V)=\begin{cases}\Phi(x_t, t\in V) & \text{if } |V|\geqq 2\\ -\mu x_t & \text{if } V=\{t\},\end{cases} \tag{2.1}$$

where the constant $\mu\in R^1$ will be named chemical potential.

Consider the Gibbsian sequence of type 1, see (D.4), with fixed boundary conditions of probability measures $\{P_{\mu,k}\}$, [See the end of the paper for the discussion of the type 2.] corresponding to the potential Φ_μ. We shall name the Gibbs ensemble in the volume V_k the probability measure $P_{\mu,k}$. Let $P_{\mu,k}(\xi_t, t\in U)$ be the restriction $P_{\mu,k}(\cdot)$ on $(X^U, \mathfrak{B}_U)$, $U\subset V$, $E_{\mu,k}\chi$, $D_{\mu,k}\chi$, $\operatorname{cov}_{\mu,k}(x, y)$ will denote respectively the mean, variance of the random variable χ and the covariance of the random variables x, y corresponding to the probability measure $P_{\mu,k}$.

D.6.2. Canonical Ensemble. We shall say that the probability measure on $(X^{V_k}, \mathfrak{B}_{V_k})$ denoted by $q_{N,V_k}(\cdot)$ is the canonical ensemble if $q_{N,V_k}(\cdot)$ is defined by

$$q_{N,V_k}(\xi_t, t\in V_k)=P_{\mu,k}(\xi_t, t\in V_k|S_{V_k}=N) \tag{2.2}$$

where $S_{V_k}=\sum_{t\in V_k}\xi_t$ and $P_{\mu,k}(\cdot|S_{V_k}=N)$ is the conditional distribution obtained by $P_{\mu,k}$ under the condition $S_{V_k}=N$ and N is some integer such that $P_{\mu,k}(S_{V_k}=N)\neq 0$.

We emphasize that because of the condition $S_{V_k}=N$ the left part of (2.2) does not depend on μ. We shall name Gibbsian canonical sequence a sequence of canonical ensembles generated by a Gibbsian sequence and a sequence of integers N_k.

D.6.3. Equivalence between Gibbs Ensemble and Canonical Ensemble. For defining the equivalence between these two ensembles it is necessary to define (see a detailed discussion in [26, 27]) a correspondence between the thermodynamical parameters which are defined in the two different ensembles. Suppose that a certain canonical Gibbsian sequence is given such that $N_k/|V_k|\to\varrho$, where ϱ is a positive constant named density.

Choose $\mu_k=\mu_k(N_k)$ in such a way that

$$E_{\mu_k,k}S_{V_k}=N_k \tag{2.3}$$

where the mean value is made with respect to the Gibbsian probability distribution $P_{\mu_k,k}$ which belongs to the Gibbsian sequence corresponding to the considered canonical Gibbsian sequence and having chemical potential μ_k. Because of the strong convexity of the free energy in finite volume μ_k exists and is unique. We shall say that the Gibbs ensemble and canonical ensemble are equivalent, for a given canonical Gibbsian sequence if for every U and every $\xi_t\in X$, $t\in U$

$$|P_{\mu_k,k}(\xi_t, t\in U)-q_{N_k,k}(\xi_t, t\in U)|\xrightarrow[k\to\infty]{}0 \tag{2.4}$$

where the definition of $q_{N_k,k}(\xi_t, t\in U)$ is analogous to the definition of $P_{\mu_k,k}(\xi_t, t\in U)$. Suppose that there exists $\mu=\mu(\varrho)$ for some $\varrho\in R^+$ such that

$$\lim_{k\to\infty}\frac{E_{\mu(\varrho),k}S_{V_k}}{|V_k|}=\varrho \tag{2.5}$$

for any Gibbsian sequence and any sequence N_k such that $\frac{N_k}{|V_k|}\to\varrho$. From the strong convexity of the free energy as function of μ in the case of infinite volume limit, it follows that $\mu(\varrho)$ is unique if it exists (see Griffith and Ruelle [33]; Dobrushin and Nakhapitan [19]).

The function $\mu(\varrho)$ can be found explicitely by usual way by means of the Legendre transformation. If, for some fixed value of ϱ, $\mu(\varrho)$ is continuous on ϱ and the Gibbs state $P_{\mu(\varrho)}$ with $\mu=\mu(\varrho)$ is unique (this fact is realized if ϱ lies in a domain of absence of phase transition) then $\mu_k(N_k)\to\mu(\varrho)$ when $k\to\infty$ and usual compactness arguments (compare with [21]) shows that

$$|P_{\mu_k,k}(\xi_t, t\in U)-P_{\mu(\varrho)}(\xi_t, t\in U)|\xrightarrow[k]{}0 \tag{2.6}$$

for $k\to\infty$. From (2.4) and (2.6) it follows that

$$|q_{N_k,k}(\xi_t, t\in U)-P_{\mu(\varrho)}(\xi_t, t\in U)|\xrightarrow[k\to\infty]{}0 \tag{2.7}$$

and this is the most usual formulation of the equivalence of ensembles.

2.2. Equivalence of Ensembles as a Consequence of the Integral Central Limit Theorem

Let now $t_1, \ldots, t_n$, $n=0, 1, \ldots$ be n points and $Q=(t_1, \ldots, t_n)$. Let $\{P_{\mu,k}\}$ be a given Gibbsian sequence and if $Q\subset V_k$ let $P_{\mu,k}(\cdot|\bar{x}_t, t\in Q)$, $\bar{x}_t\in X$ be the conditional distribution under the conditions $\xi_t=\bar{x}_t$, $t\in Q$ calculated using the $\{P_{\mu,k}\}$, if $Q=\phi$ the quantity $P_{\mu,k}(\cdot|\bar{x}_t, t\in Q)$ coincides with $P_{\mu,k}(\cdot)$.

Theorem 4. a) *Suppose that is given a certain canonical Gibbsian sequence and a constant $d>0$*
and hypothesis b)*: suppose that for any $z\in R^1$ and for every finite Q and $\bar{x}_t\in X^Q$*

$$P_{\mu_k,k}\left(\frac{S_{V_k}-E_{\mu_k,k}S_{V_k}}{\sqrt{d|V_k|}}<z\,\middle|\,\bar{x}_t, t\in Q\right)\xrightarrow[k\to\infty]{}\int_{-\infty}^{z}(2\pi)^{-1/2}e^{-u^2/2}du$$

where $P_{\mu_k,k}$ is the Gibbsian sequence used in the definition of equivalence of ensembles and we suppose that the sequence $\{\mu_k\}$ is bounded.

Then the equivalence between Gibbs ensemble and canonical ensemble is true for this canonical Gibbsian sequence.

Proof. First we note that the local central limit theorem follows from hypothesis b) for the conditional probability distributions $P_{\mu_k,k}(\cdot|\bar{x}_t, t\in Q)$ as can be shown using the same arguments of Theorem 1 and the boundedness of μ_k. Than b) implies that:

$$\begin{aligned}&|\sqrt{d|V_k|}\,P_{\mu_k,k}(S_{V_k}=N_k)-(2\pi)^{-1/2}e^{-Z_k^2/2}|\xrightarrow[k\to\infty]{}0\\&|\sqrt{d|V_k|}\,P_{\mu_k,k}(S_{V_k}=N_k|\bar{x}_t, t\in Q)-(2\pi)^{-1/2}e^{-Z_k^2/2}|\xrightarrow[k\to\infty]{}0\end{aligned} \tag{2.8}$$

where

$$Z_k=\frac{N_k-E_{\mu_k,k}S_{V_k}}{\sqrt{d|V_k|}} \tag{2.9}$$

from (2.3) we have $Z_k=0$ and so we can compare $P_{\mu_k,k}(\cdot)$ and $P_{\mu_k,k}(\cdot|\bar{x}_t, t\in Q)$ using (2.8). Thus:

$$\lim_{k\to\infty} \frac{P_{\mu_k,k}(S_{V_k}=N_k|\bar{x}_t, t\in Q)}{P_{\mu_k,k}(S_{V_k}=N_k)}=1\,. \tag{2.10}$$

From (2.10) we obtain the thesis because of the following identity between conditional probabilities

$$\frac{P_{\mu_k,k}(\xi_t=\bar{x}_t, t\in Q|S_{V_k}=N_k)}{P_{\mu_k,k}(\xi_t=\bar{x}_t, t\in Q)}=\frac{P_{\mu_k,k}(S_{V_k}=N_k|\xi_t=\bar{x}_t, t\in Q)}{P_{\mu_k,k}(S_{V_k}=N_k)}\,. \tag{2.11}$$

The condition of the boundedness of μ_k used in Theorem 4 is not very restrictive because it is true for example if $\mu_k\to\mu(\varrho)$ for $k\to\infty$. In situations when $\mu_k\to\mu(\varrho)$ it is possible to apply without any essential change all the methods used to show the integral central limit theorem and discussed in Section 1.3. Thus it is possible to show that in the situation of Section 1.3:

$$\lim_{k\to\infty} P_{\mu_k,k}\left(\frac{S_{V_k}-E_{\mu_k,k}(S_{V_k}|\bar{x}_t, t\in Q)}{\sqrt{D_{\mu_k,k}(S_{V_k}|\bar{x}_t, t\in Q)}}<Z\,\middle|\,\bar{x}_t, t\in Q\right)\xrightarrow[k\to\infty]{}\int_{-\infty}^{Z}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}du,$$

$$D_{\mu_k,k}(S_{V_k}|\bar{x}_t, t\in Q)\sim d|V_k|\,. \tag{2.12}$$

Here we use the notation of (D.5) for $D_{\mu_k,k}(S_{V_k}|\bar{x}_t, t\in Q)$ Hypothesis b) follows from (2.12) if one can show the relations:

$$\begin{aligned} &D_{\mu_k,k}S_{V_k}\sim D_{\mu_k,k}(S_{V_k}|\bar{x}_t, t\in Q)\sim d|V_k|,\ k\to\infty \\ &V_k^{-1/2}[E_{\mu_k,k}(S_{V_k}|\bar{x}_t, t\in Q)-E_{\mu_k,k}(S_{V_k})]\xrightarrow[k\to\infty]{}0\,. \end{aligned} \tag{2.13}$$

By using the usual relations

$$\begin{aligned} E_{\mu_k,k}(S_{V_k})&=\sum_{t\in V_k} E_{\mu_k,k}\xi_t \\ E_{\mu_k,k}(S_{V_k}|\bar{x}_t, t\in Q)&=\sum_{t\in V_k} E_{\mu_k,k}(\xi_t|\bar{x}_t, t\in Q) \\ D_{\mu_k,k}(S_{V_k}|\bar{x}_t, t\in Q)&=\sum_{t,s\in V_k} \mathrm{cov}_{\mu_k,k}(\xi_t, \xi_s|\bar{x}_t, t\in Q) \\ D_{\mu_k,k}(S_{V_k})&=\sum_{t,s\in V_k} \mathrm{cov}_{\mu_k,k}(\xi_t, \xi_s) \end{aligned} \tag{2.14}$$

and comparing the corresponding terms it is possible to check the conditions (2.13) with the help of any of the types of the conditions of decrease of correlations used in the mathematical physics and probability theory literature. Besides that it is possible in enough general situations to obtain hypothesis b) from integral central limit theorem in a more usual variant.

Before we need some notations and explain some useful conditions

D.7.1. Let X be a finite space. We define on it a metric $\varrho(x, x')$ in the following may:

$$\varrho(x, \tilde{x})=\begin{cases}1 & x\neq\tilde{x}\\ 0 & x=\tilde{x}\,.\end{cases} \tag{2.15}$$

D.7.2. We define also a metric on the space of probability measures on $(X^V, \mathfrak{B}_V)$ in the following way: let P_V, Q_V be two probability measures, than:

$$R(P_V, Q_V) = \sup_{B \in \mathfrak{B}_V} |P_V(B) - Q_V(B)| . \tag{2.16}$$

D.7.3. Uniform Exponential Property. We shall say that the Gibbs random field $\xi_t, t \in Z^v$ has the uniform exponential regularity property if there exists some $c > 0$ and $C < \infty$ such that for any k, any volume $V_2 \subset Z^v$ and $\bar{V}_1 \subset Z^v \setminus V_2$ and for almost all with respect to the restriction to $(X^{V_2}, \mathfrak{B}_{V_2})$ of the measure P_{V_2} sets of variables $(x_t^1, t \in V_2)$, $(x_t^2, t \in V_2)$ the following inequality holds:

$$R(P_{\bar{V}_1 | x_t^1, t \in V_2}, P_{\bar{V}_1 | x_t^2, t \in V_2}) \leqq C \sum_{s \in \bar{V}_1} \sum_{t \in V_2} \varrho(x_t^1, x_t^2) e^{-c|s-t|} \tag{2.17}$$

where $P_{\bar{V}_1 | x_t^i, t \in V_2}$ is the conditional distribution for the set of variables $\{\xi_t, t \in \bar{V}_1\}$ under the conditions $\xi_t = x_t^i$, $t \in V_2$.

Now we formulate the following condition:

Condition C. There exists an interval $\Delta \subset R^1$ such that $\mu(\varrho) \in \Delta$ and for all $\mu \in \Delta$ the uniform exponential regularity property (2.17) is verified with c, C independent from μ.

Theorems 4, 5 of [25] imply that the condition C is true in enough general situations. Thus we can state that it is true in the cases 1, 2 of Theorem 3.

Condition D. For the Gibbsian sequence $\{P_{\mu_k, k}\}$ used in Theorem 4

$$P_{\mu_k, k}\left(\frac{S_{V_k} - E_{\mu_k, k} S_{V_k}}{\sqrt{d|V_k|}} < z\right) \xrightarrow[k \to \infty]{} \int_{-\infty}^{z} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du .$$

Proposition 1. *If $\mu_k \to \mu(\varrho)$ when $k \to \infty$ and if the Gibbs random field $\xi_t, t \in Z^v$ corresponding to the potential $\Phi_{\mu(\varrho)}$ satisfies condition C and if the condition D is true then hypothesis* b) *of Theorem 4 is true.*

Proof. Let us choose $a_k = k \in Z^+$ (see D.4) for the sake of simplicity. Suppose further that Q is contained in some cube of the sequence $\{V_k\}$ (see Fig. 3) which we shall call V_q and suppose that it has site length $2q$. Let us fix also a cube V_k with site $2k$ and a cube $D_k \in \{V_j\}_{j \in Z^+}$ with site $2[\sqrt{k}]$. Let be $q \leqq [V_k]$. Setting $W_k = V_k \backslash D_k$ we can decompose $\bar{S}_{Vk} = \dfrac{S_{V_k} - E_{\mu_k, k} S_{Vk}}{\sqrt{d|V_k|}}$ in the following way:

$$\bar{S}_{V_k} = \bar{S}_{D_k} + \bar{S}_{W_k}$$

where

$$\bar{S}_{D_k} = \frac{1}{\sqrt{d|V_k|}} \sum_{t \in D_k} (\xi_t - E_{\mu_k, k} \xi_t)$$

$$\bar{S}_{W_k} = \frac{1}{\sqrt{d|V_k|}} \sum_{t \in W_k} (\xi_t - E_{\mu_k, k} \xi_t) . \tag{2.17a}$$

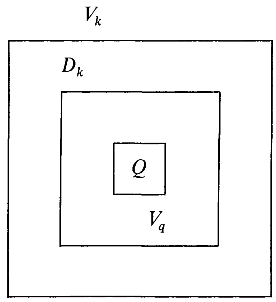

Fig. 3

Let us evaluate $D_{\mu_k,k}\tilde{S}_{D_k}$, where $\tilde{S}_{D_k}=\sqrt{d|V_k|}\bar{S}_{D_k}$

$$D_{\mu_k,k}\tilde{S}_{D_k}=\sum_{t,s\in D_k}\mathrm{cov}_{\mu_k,k}(\xi_t,\xi_s)$$
$$=\sum_{t,s\in D_k}\left(\sum_{l,m\in X}P_{\mu_k,k}(\xi_t=l,\xi_s=m)lm-\sum_{l\in X}\sum_{m\in X}lmP_{\mu_k,k}(\xi_t=l)P_{\mu_k,k}(\xi_t=m)\right) \tag{2.18}$$

where the probabilities introduced above must be intrepreted as restrictions of $P_{\mu_k,k}(\cdot)$ to the sets $\{X^V,\mathfrak{B}_V\}$ where V is respectively $\{t,s\}$, $\{t\}$, $\{s\}$. We can rewrite (2.18) in this way:

$$D_{\mu_k,k}\tilde{S}_{D_k}=\sum_{t,s\in D_k}\sum_{l\in X}lP_{\mu_k,k}(\xi_t=l)\left[\sum_{m\in X}m(P_{\mu_k,k}(\xi_s=m|\xi_t=l)\right.$$
$$\left.-P_{\mu_k,k}(\xi_s=m))\right]. \tag{2.19}$$

Thus,

$$|D_{\mu_k,k}\tilde{S}_k|\leqq\sum_{t,s\in D_k}A\sum_{m\in X}|P_{\mu_k,k}(\xi_s=m|\xi_t=l)-P_{\mu_k,k}(\xi_s=m)|$$
$$\leqq C\sum_{t,s\in D_k}Ae^{-c|t-s|}<K|D_k| \tag{2.20}$$

and C, c, K, A are some positive constants not depending on μ_k because of the hypothesis C and $k>0$. It follows immediately from (2.20) for any $k>0$

$$P_{\mu_k,k}\{\bar{S}_{D_k}>\eta\}\leqq\frac{D\tilde{S}_{D_k}}{\eta^2 d|V_k|}\leqq\frac{K}{\eta^2 d}\frac{[\sqrt{k}]^\nu}{k^\nu}\xrightarrow[k\to\infty]{}0. \tag{2.21}$$

By the same way

$$P_{\mu_k,k}\{\bar{S}_{D_k}>\eta|\bar{x}_t,t\in Q\}\xrightarrow[k\to\infty]{}0. \tag{2.22}$$

Using again condition C for any $k>0$

$$R(P_{\mu_k,k}(\bar{S}_{W_k}<z),P_{\mu_k,k}(\bar{S}_{W_k}<z|\bar{x}_t,t\in Q))$$
$$\leqq C\sum_{\substack{t\in Q\\ s\in W_k}}e^{-c|t-s|}\leqq c'\sum_{t\in Q}\sum_{n>n_0}n^{\nu-1}e^{-cn}$$
$$\leqq c'|Q|\sum_{n>n_0}n^{\nu-1}e^{-cn}, \tag{2.23}$$

where $n_0=[\sqrt{k}]-q$, $c'>0$. Since the series $\sum_n n^{\nu-1}e^{-cn}$ is convergent; the right hand side of (2.22) goes to zero when $k\to\infty$. The relations (2.17a), (2.21)–(2.23) and condition D imply hypothesis b).

We have

Theorem 5. *Equivalence between Gibbs ensemble and canonical ensemble holds in the cases 1, 2, 3.a of Theorem 3.*

Proof. In the cases 1, 2 of Theorem 3 the result can be obtained from Proposition 1. In all cases the result can be also obtained directly by checking the relations (2.13). In the case 1, 2 it is possible to use the condition C. In the case 3a it is possible to use the results of the paper [34].

Note. It is interesting also to study the problem of the equivalence of ensembles for the case when $P_{\mu,k}$ is the restriction on V_k of a Gibbs distribution in infinite volume with chemical potential μ. This case can be reduced to the considered above case when boundary condition are fixed using the following argument. It is easy to see that

$$P_{\mu,k}(x_t, t\in V_k | S_{V_k} = N_k) = \int q_{V_k, N_k}(x_t, t\in V_k | \bar{x}_k) P_{N_k}^{V_k}(d\bar{x}_k) \tag{2.24}$$

where $P_{N_k}^{V_k}$ is the measure on $X^{Z^\nu | V_k}$ defined by the joint conditional distribution of the variables $(\xi_t, t\in Z^\nu \setminus V_k)$ under the condition $S_{V_k} = N_k$. So if we show that the restriction of the distribution $q_{V_k, N_k}(\cdot | \bar{x}_k)$ on any volume U has a uniform limit with respect the boundary conditions $\bar{x}_k$ when $k \to \infty$ we obtain as a consequence that the restriction of the probability distribution (2.24) on the same volume U has the same limit. The uniform on $\bar{x}_k$ convergence follows from the convergence for any fixed sequence of boundary conditions.

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