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Application of Commutator Theorems to the Integration of Representations of Lie Algebras and Commutation Relations*

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Abstract. Sufficient conditions on unbounded, symmetric operators A and which imply that

$$\exp(itA)\exp(isB)\exp(-itA)$$

satisfies the well known "multiple commutator" formula are derived. Th formula is then applied to prove new necessary and sufficient conditions for tl integrability of representations of Lie algebras and canonical commutatic relations and the commutativity of the spectral projections of two commutin unbounded, self-adjoint operators. A classic theorem of Nelson's is obtained ɛ a corollary. Our results are useful in relativistic quantum field theory.

1. Introduction

In this note we discuss sufficient conditions for the multiple commutator formu

$$e^{itA}e^{isB}e^{-itA}=\exp is\left\{B+\sum_{n=1}^{\infty}\frac{(it)^n}{n!}\,\mathrm{ad}^n A(B)\right\}, \tag{1.}$$

to hold. Here A and B are unbounded operators and, formally,

$$\begin{aligned}\mathrm{ad}\,A(B)&=[A,B]\,,\\ \mathrm{ad}^n A(B)&=[A,\mathrm{ad}^{n-1}A(B)]\,.\end{aligned} \tag{1.}$$

Our results have applications in group theory and quantum field theory.

They are a direct outgrowth of recent work of Driessler and the author [2 concerning the Haag-Kastler axioms [12] in relativistic quantum field theory and subsequent alternate proof of the main result of [2] due to Glimm and Jaffe [3

The main result of [2, 3], a sufficient condition for the bounded functions of tw unbounded, symmetric operators A and B to commute, is a special case of tl results proven in the following sections.

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The basic strategy for a proof of (1.1) is to find a self-adjoint operator $N \geqq 1$ in terms of which A, B and $i^n \operatorname{ad}^n A(B)$, $n=1,2,\ldots$, can be dominated in the sense of the commutator theorem of [4]; see also [6]. (Different forms of the commutator theorem may be found in [8].)

2. The Main Results

We start with describing the general set-up and recalling the commutator theorem.

Let $\mathscr{H}$ be a separable Hilbert space, and N a *positive*, *self-adjoint* operator on $\mathscr{H}$ satisfying

$$N \geqq 1 \,. \tag{2.1}$$

We let $\mathscr{H}_n$ be (the completion of) $D(N^{n/2})$ in the norm

$$\|\psi\|_n = \|N^{n/2}\psi\|\,, \pm n = 1,2,3\ldots,$$

and $\mathscr{L}(\mathscr{H}_n, \mathscr{H}_m)$ the *bounded operators* from $\mathscr{H}_n$ to $\mathscr{H}_m$.

The domain of an operator C is denoted $D(C)$, and a subspace $\mathscr{D} \subset \mathscr{H}$ is called a *core* for C if

$$C = (C \upharpoonright \mathscr{D})^- \,.$$

Here $C \upharpoonright \mathscr{D}$ denotes the restriction of C to the subspace $\mathscr{D}$ and $(\ )^-$ the closure of $(\ )$.

We assume that A is a symmetric operator in $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_{-1})$; i.e., on some form core for N ($\equiv$ core for $N^{1/2}$),

$$\pm A \leqq K_1 N \,, \tag{2.2}$$

for some finite constant K_1 (in the quadratic form sense). Then

$$\dot{A} \equiv i[N, A] \tag{2.3}$$

is defined as an element of $\mathscr{L}(\mathscr{H}_3, \mathscr{H}_{-3})$ in the obvious way; see e.g. [8].

Theorem 0. *Let A be as above and assume, in addition, that $\dot{A} \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_{-1})$, i.e., on a form core for N,*

$$\pm \dot{A} \leqq K_1 N \,. \tag{2.4}$$

Then A determines a densely defined, symmetric operator – also denoted A – on $\mathscr{H}$ with

$$D(N) \subseteq D(A), \|A\psi\| \leqq k_1 \|N\psi\|\,, \quad \textit{with} \quad k_1 \leqq 2^{1/2} K_1 \,, \tag{2.5}$$

for all $\psi \in D(N)$, and

A is essentially self-adjoint on any core for N. (2.6)

Remark. This is the commutator theorem of [4], stated in a form due to [6].

In the applications a slightly different form of the commutator theorem is sometimes more useful.

Theorem 0′. *Let A be a symmetric operator on $\mathscr{H}$ with the properties that*

$$\begin{aligned} &D(A) \text{ contains a core } \mathscr{D} \text{ for } N\,, \\ &\|A\psi\| \leqq k_1 \|N\psi\|\,, \quad \text{and} \\ &\pm i\{(N\psi, A\psi) - (A\psi, N\psi)\} \leqq K_1 \|N^{1/2}\psi\|^2 \end{aligned} \tag{2.4′}$$

for all $\psi \in \mathscr{D}$.

Then the conclusions of Theorem 0 remain true.

For proofs of Theorems 0 and 0′, related results and references see [8], (Theorems X.36, X.36′, and X.37).

We now state our main results.

Theorem 1_M. *Let A, $\dot{A}$, B, and $\{C_n\}_{n=0}^{M}$ be operators in $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_{-1})$ satisfying the hypotheses of Theorems 0 or 0′. Assume that $C_0 = B$, and*

$$C_n = i[A, C_{n-1}]\,, \tag{2.7}$$

in $\mathscr{L}(\mathscr{H}_2, \mathscr{H}_{-2})$ (i.e., weakly on $D(N) \times D(N)$), for all $n = 1, \dots, M$. Then

$$e^{itA} e^{isB} e^{-itA} = e^{isB_t}\,,$$

with

$$B_t = \left[\left\{B + \sum_{n=1}^{M-1} \frac{t^n}{n!} C_n + \int_0^t dt_1 \dots \int_0^{t_{M-1}} dt_M e^{it_M A} C_M e^{-it_M A}\right\} \restriction D(N)\right]^- . \tag{2.8}$$

Theorem 1_∞. *Let A, $\dot{A}$, B and $\{C_n\}_{n=0}^{\infty}$ be as in the hypotheses of Theorem 1_M, (for $M = \infty$), and assume, in addition, that there is some finite constant K_2 such that, on a form core for N*

$$\left.\begin{aligned} &\pm C_n \leqq K_2^n n! N \\ &\pm \dot{C}_n \leqq K_2^n n! N\,, \end{aligned}\right\} \tag{2.9}$$

for all $n = 1, 2, 3, \dots$.

Then, for $|t| < K_2^{-1}$,

$$\operatorname*{s\text{-}lim}_{M \to \infty} \left\{B + \sum_{n=1}^{M} \frac{t^n}{n!} C_n\right\} \restriction D(N) \quad \text{exists}\,,$$

has a self-adjoint closure, denoted B_t, and

$$e^{itA} e^{isB} e^{-itA} = e^{isB_t}. \tag{2.10}$$

Remarks. 1) One may also denote C_n by $i^n \operatorname{ad}^n A(B)$; see (1.1) and (1.2).

2) A generalization of Theorems 1_M and 1_∞ which may be useful in various applications is presented in an Appendix; see Lemma A.1 and Theorem A.2.

3) Theorem 1_M contains as special cases sufficient conditions for the commutativity of the bounded functions of two unbounded, self-adjoint operators

(equivalent to the ones found in [2]) and for the integrability of the canonical commutation relations; see Sections 6 and 7. Theorem 1_∞ is useful for the integration of equations of motion in the "Heisenberg picture".

3. Preliminaries: Invariance of Operator Domains

Here we discuss some results concerning the invariance of the domain $D(N)$ of N under certain unitary groups. They represent a slight elaboration of results of Glimm and Jaffe [3] and may be of some interest in their own right. See also Lemma A.1 in the Appendix.

We let $Q(N)=D(N^{1/2})$ denote the quadratic form domain of N.

The main result of this section is

Lemma 2. *Let A and $\dot{A}$ satisfy the hypotheses of Theorem* 0 *or* 0'. *Then*

1) $e^{itA}Q(N)\subseteq Q(N)$, *and, for all* $\psi\in Q(N)$,

$$\|N^{1/2}e^{itA}\psi\|\leqq e^{(1/2)K_1|t|}\|N^{1/2}\psi\| ;$$

2) $e^{itA}D(N)\subseteq D(N)$, *and, for all* $\psi\in D(N)$,

$$\|Ne^{itA}\psi\|\leqq e^{k_1|t|}\|N\psi\| ;$$

3) $e^{itA}D(N^\alpha)=D(N^\alpha)$, *for* $\alpha=1/2,1$.

Remark. 3) is an immediate consequence of 1) and 2).

Proof. We first prove a simpler version of Lemma 2.

Definition. For $\lambda\geqq 0$ we set

$$R(\lambda)\equiv(N+\lambda)^{-1}, \quad N_\lambda\equiv\lambda^2R(\lambda)NR(\lambda) , \tag{3.1}$$

and

$$\overset{(\cdot)}{A}_\lambda\equiv\lambda^2R(\lambda)\overset{(\cdot)}{A}R(\lambda) . \tag{3.2}$$

Here $\overset{(\cdot)}{A}$ denotes A or $\dot{A}$.

Using the self-adjointness and strict positivity of N we get

$$N_\lambda^\alpha\leqq N^\alpha, \quad \text{for all} \quad \alpha\geqq 1 , \tag{3.3}$$

and since A and $\dot{A}$ satisfy (2.4), for some $K_1<\infty$

$$\|\overset{(\cdot)}{A}_\lambda\|\leqq\lambda K_1 . \tag{3.4}$$

Application of (2.5) yields

$$\|N^\alpha\overset{(\cdot)}{A}_\lambda\|\leqq\lambda^{1+\alpha}k_1, \quad \text{for} \quad \alpha\in[0,1] . \tag{3.5}$$

Lemma 2_λ. *Lemma 2 holds with A_λ replacing A and constants K_1 and k_1 independent of λ.*

Proof'. Using (3.4) and (3.5) and the power series expansion of e^{itA_λ} (convergent for all $|t|<\infty$, $\lambda<\infty$) we obtain:

$$e^{itA_\lambda}Q(N)=Q(N), \qquad e^{itA_\lambda}D(N)=D(N)\,. \tag{3.6}$$

For $\psi\in Q(N)$, set

$$F_\lambda(t)=(e^{itA_\lambda}\psi, Ne^{itA_\lambda}\psi)\,. \tag{3.7}$$

Then

$$\begin{aligned} dF_\lambda(t)/dt &= (e^{itA_\lambda}\psi, \dot{A}_\lambda e^{itA_\lambda}\psi)\\ &\leqq K_1(e^{itA_\lambda}\psi, N_\lambda e^{itA_\lambda}\psi)\\ &\leqq K_1 F_\lambda(t)\,, \end{aligned}$$

and we have used (3.2), (2.4), and (3.3). Hence

$$F_\lambda(t)\leqq e^{K_1|t|}F_\lambda(0)=e^{K_1|t|}\,\|N^{1/2}\psi\|^2 \tag{3.8}$$

which proves Lemma 2_λ, (1). Next, let $\psi\in D(N)$ and set

$$G_\lambda(t)=(Ne^{itA_\lambda}, Ne^{itA_\lambda}\psi)\,. \tag{3.9}$$

Then

$$\begin{aligned} dG_\lambda(t)/dt &= (Ne^{itA_\lambda}\psi, \dot{A}_\lambda e^{itA_\lambda}\psi)+(\dot{A}_\lambda e^{itA_\lambda}\psi, Ne^{itA_\lambda}\psi)\\ &\leqq 2\|Ne^{itA_\lambda}\psi\|\,\|\dot{A}_\lambda e^{itA_\lambda}\psi\|\\ &\leqq 2\|Ne^{itA_\lambda}\psi\|\,\|\lambda R(\lambda)\|\,\|\dot{A}\lambda R(\lambda)e^{itA_\lambda}\psi\|\\ &\leqq 2k_1\|\lambda R(\lambda)\|^2\,\|Ne^{itA_\lambda}\psi\|^2\leqq 2k_1 G_\lambda(t)\,, \end{aligned}$$

and we have used (3.2), (2.5), and (3.3). Therefore

$$\begin{aligned} G_\lambda(t)&\leqq e^{2k_1|t|}G_\lambda(0)\\ &= e^{2k_1|t|}\,\|N\psi\|^2\,, \end{aligned} \tag{3.10}$$

which proves Lemma 2_λ, (2). Lemma 2_λ, (3) is statement (3.6). Hence the proof is complete.

As a corollary to Lemma 2_λ, (2) we note that

$$\underset{\lambda\to\infty}{s\text{-lim}}\, e^{itA_\lambda}=e^{itA}\,. \tag{3.11}$$

Since $\{e^{itA_\lambda}:\lambda\geqq 0\}$ and e^{itA} are unitary operators, it suffices to prove weak convergence on a dense set. For φ and ψ in $D(N)$,

$$\begin{aligned} &(\varphi, \{e^{itA}-e^{itA_\lambda}\}\psi)\\ &\quad = i\int_0^1 ds(\{1-\lambda R(\lambda)\}e^{-isA}\varphi, (AN^{-1})\lambda R(\lambda)Ne^{i(t-s)A_\lambda}\psi)\\ &\quad\quad + i\int_0^1 ds(\{1-\lambda R(\lambda)\}e^{-isA}A\varphi, e^{i(t-s)A_\lambda}\psi)\,. \end{aligned} \tag{3.12}$$

Since $\{1-\lambda R(\lambda)\}$ tends to 0 strongly, AN^{-1} is bounded by (2.5), $\|\lambda R(\lambda)\|\leqq 1$ and $\|Ne^{i(t-s)A_\lambda}\psi\|$ is bounded uniformly in λ, the integrands tend to 0, for all s. Since the integrands are bounded uniformly in s and λ, the r.h.s. of (3.12) tends to 0 by the Lebesgue dominated convergence theorem.

We summarize:

i) For $\psi \in D(N^{1/2})$, $\|N^{1/2} e^{itA_\lambda}\psi\| \leqq e^{(1/2)K_1|t|} \|N^{1/2}\psi\|$;

ii) for $\psi \in D(N)$, $\|N e^{itA_\lambda}\psi\| \leqq e^{k_1|t|} \|N\psi\|$, uniformly in λ, and

iii) Equation (3.11).

We now complete the proof of Lemma 2.

Combination of i)—iii) with the spectral theorem applied to N immediately gives Lemma 2, 1) and 2). To prove 3), note that by 1) and 2)

$$e^{\pm itA} D(N^\alpha) \subseteqq D(N^\alpha), \quad \text{for all} \quad |t| < \infty$$

and $\alpha = 1/2, 1$. Thus

$$\begin{aligned} D(N^\alpha) &= e^{itA}\{e^{-itA} D(N^\alpha)\} \\ &\subseteqq e^{itA} D(N^\alpha) \end{aligned} \tag{3.13}$$

hence $D(N^\alpha) = e^{itA} D(N^\alpha)$, for all $|t| < \infty$. Q.E.D.

Remarks. Lemma 2, 2) and the trick of using a differential inequality for $G_\lambda(t)$ are due to [3]; (we have applied it in a slightly different form, and an extension is presented in an Appendix: Proof of sufficient conditions for

$$e^{itA} D(N^\alpha) = D(N^\alpha), \ \alpha \in (-\infty, \infty)) .$$

We note that, by Lemma 2, 3) and Theorem 0, (0′)

$$e^{-itA} D(N) \quad \text{is a } \textit{core} \text{ for} \quad B \tag{3.14}$$

if B satisfies the hypotheses of Theorem 0, (0′).

Lemma 2 may be summarized as follows: $\{e^{itA}\}$ *determines unique, exponentially bounded one parameter groups on the spaces* $\mathscr{H}_n$, for $n = -2, -1, 0, 1, 2$.

4. Proofs of Theorems 1_M and 1_∞

Let A, B, and $\{C_n\}$ satisfy the hypotheses of Theorem 1_M or 1_∞. By Theorem 0, B is essentially self-adjoint on $D(N)$. Its closure is also denoted B. Let

$$B = \int \lambda dE(\lambda)$$

be the spectral decomposition of B. We set

$$\left.\begin{aligned} B_t &= \int \lambda dE_t(\lambda), \quad \text{with} \\ E_t(\cdot) &= e^{itA} E(\cdot) e^{-itA} . \end{aligned}\right\} \tag{4.1}$$

Then we conclude from (3.14) (by the fundamental criterion) that $D(N)$ is a *core* for B_t, i.e., B_t is essentially self-adjoint on $D(N)$. By Lemma 2, 2) and (4.1)

$$B_t = e^{itA} B e^{-itA}, \quad \text{on} \quad D(N) .$$

Let ψ and θ be in $D(N)$. Then, using Lemma 2, 2), the hypotheses on B and C_1, see (2.7), and Theorem 0, (2.5), we obtain

$$\begin{aligned} d(\psi, B_t\theta)/dt &= i\{(Ae^{-itA}\psi, Be^{-itA}\theta) \\ &\quad -(Be^{-itA}\psi, Ae^{-itA}\theta)\} \\ &= (\psi, e^{itA}C_1e^{-itA}\theta)\,, \end{aligned}$$

and

$$\begin{aligned} |d(\psi, B_t\theta)/dt| &\leqq k_2 \|\psi\|\,\|Ne^{-itA}\theta\| \\ &\leqq k_2 e^{k_1|t|}\,\|\psi\|\,\|N\theta\|\,. \end{aligned}$$

Thus, for all $\theta \in D(N)$,

$$dB_t\theta/dt = e^{itA}C_1e^{-itA}\theta\,,$$

and hence

$$(A_1)\ B_t\theta = B\theta + \int_0^t ds e^{isA}C_1e^{-isA}\theta\,. \tag{4.2}$$

Since $D(N)$ is a core for B_t, we conclude:

$$B_t = \left[\left\{B + \int_0^t ds e^{isA}C_1e^{-isA}\right\} \upharpoonright D(N)\right]^-\,. \tag{4.3}$$

We now proceed by *induction:* Assume

$$\begin{aligned} (A_n)\ B_t = \Bigg[\Bigg\{B &+ \sum_{m=1}^{n-1} \frac{t^m}{m!} C_m \\ &+ \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n e^{it_nA}C_ne^{-it_nA}\Bigg\} \upharpoonright D(N)\Bigg]^-\,. \end{aligned}$$

By hypothesis on $\{C_m\}_{m=0}^{\infty}$ (see Theorems 1_{M}, 1_∞), Theorem 0 and Lemma 2, 2), $D(N)$ is contained in the domain of

$$e^{isA}C_me^{-isA}, \quad \text{for all} \quad m < \infty\,. \tag{4.4}$$

As in the proof of (A_1) we show that

$$e^{it_nA}C_ne^{-it_nA} = C_n + \int_0^{t_n} dt_{n+1} e^{it_{n+1}A}C_{n+1}e^{-it_{n+1}A}\,,$$

on $D(N)$; [just replace B by C_n, C_1 by C_{n+1}, t by t_n and use (4.4)].

Inserting this equation into (A_n) and using again that $D(N)$ is a *core* for B_t, we immediately obtain (A_{n+1}).

This completes the proof of Theorem 1_{M}.

The proof of Theorem 1_∞ is now easy:

Using inequalities (2.9) (see Theorem 1_∞), (2.5) (see Theorem 0) and Lemma 2, 2) we obtain the estimate

$$\begin{aligned}&\left\|\int_0^t dt_1 \dots \int_0^{t_{n-1}} dt_n e^{it_nA} C_n e^{-it_nA}\theta\right\| \\ &\leqq 2^{1/2} K_2^n n! \int_0^{|t|} dt_1 \dots \int_0^{t_{n-1}} dt_n \|Ne^{-it_nA}\theta\| \\ &\leqq 2^{1/2}(K_2|t|)^n e^{k_1|t|} \, \|N\theta\| \,, \end{aligned} \tag{4.5}$$

for all $\theta \in D(N)$.

For $|t| < K_2^{-1}$, the r.h.s. of (4.5) tends to 0, as $n \to \infty$, and this gives the first part of Theorem 1_∞.

The second part then follows by using once more that $D(N)$ is a core for B_t.

In the following sections we indicate some applications and in the Appendix an extension of Theorems 1_{M} and 1_∞.

5. An Application to Lie Groups

5.1. Integration of Representations of Lie Algebras

Let G be a simply connected Lie group with Lie algebra $\mathfrak{G}$; let $\{\xi_1, \dots, \xi_n\}$ be a *basis* for $\mathfrak{G}$ and $\{c_{ijk}\}$ the structure constants. We consider a representation π of $\mathfrak{G}$ on a Hilbert space $\mathscr{H}$ and define

$$X_j = i\pi(\xi_j), \quad j = 1, \dots, n \,. \tag{5.1}$$

Furthermore $N \geqq 1$ is some self-adjoint operator on $\mathscr{H}$.

The following result gives new sufficient conditions for the integrability of $\pi(\mathfrak{G})$, different from the classic ones found by Nelson in [7] (and extended e.g., in [10]).

Theorem 3. *Assume that X_j and $\dot{X}_j$ satisfy the hypotheses of Theorem 0 or 0′, for all $j = 1, \dots, n$, and that there is a core $\mathscr{D}$ for N such that*

$$[X_l, X_j] = i \sum_{k=1}^{n} c_{ljk} X_k \,, \tag{5.2}$$

weakly on $\mathscr{D} \times \mathscr{D}$.

Then $\pi(\mathfrak{G})$ is the differential of a continuous unitary representation π' of G on $\mathscr{H}$ (i.e., the representation π of $\mathfrak{G}$ on $\mathscr{H}$ can be integrated to a representation π' of G on $\mathscr{H}$).

Proof. Let $\alpha_1, \dots, \alpha_n$ be arbitrary real numbers. By Theorem 0 (or 0′)

$$D\left(\sum_{j=1}^{n} \alpha_j X_j\right) \supseteqq D(N) \,,$$

and

$$\sum_{j=1}^{n} \alpha_j X_j$$

is essentially self-adjoint on $D(N)$, for all $j=1,\dots,n$. Using (2.5) for X_k, $k=1,\dots,n$, and (5.2) we obtain

$$[X_l,X_j]=i\sum_{k=1}^{n} c_{ljk}X_k\,, \qquad \text{weakly on} \quad D(N)\times D(N)\,. \tag{5.3}$$

By Lemma 2, 3)

$$\exp\left(it\sum_{j=1}^{n}\alpha_jX_j\right)D(N)=D(N)\,, \tag{5.4}$$

for all $|t|<\infty$.

Since $\mathfrak{G}$ is the Lie algebra of G, there is some open neighborhood U of $0\in\mathfrak{G}$ which is mapped diffeomorphically onto an open neighborhood W of the identity $e\in G$ by the exponential mapping. Thus, for $g\in W$, there exists some

$$\xi=\sum_{j=1}^{n}\alpha_j\xi_j\in U\subset\mathfrak{G}$$

with

$$g=e^{\xi}\,.$$

We define

$$\pi'(g)=e^{-iX}\,,$$

where

$$X=\sum_{j=1}^{n}\alpha_jX_j\,. \tag{5.5}$$

Suppose now that $g,\hat{g}$ and $g\cdot\hat{g}$ are in W. We must show that

$$\pi'(g\hat{g})=\pi'(g)\pi'(\hat{g})\,. \tag{5.6}$$

Without loss of generality we may assume that, for all $t\in[0,1]$, $e^{t\xi}$ and $e^{t\xi}\hat{g}$ are in W. Then

$$e^{t\xi}=e^{t\sum_{j=1}^{n}\alpha_j\xi_j}$$

and

$$e^{t\xi}\hat{g}=e^{\sum_{j=1}^{n}\hat{\alpha}_j(t)\xi_j}\,, \tag{5.7}$$

where $\hat{\alpha}_j(t)$ is continuously differentiable in t in some neighborhood of $[0,1]$, for all $j=1,\dots,n$.

In order to prove (5.6) we now compare

$$F(t)=\pi'(e^{t\xi}\hat{g})\psi$$

with

$$\pi'(e^{t\xi})\pi'(\hat{g})\psi=e^{-itX}F(0)\,, \tag{5.8}$$

where ψ is an arbitrary vector in $D(N)$.

From (5.7) and (5.8) we know that

$$F(t)=e^{-iY(t)}\psi ,$$

where

$$Y(t)\equiv \sum_{j=1}^{n} \hat{\alpha}_j(t)X_j$$

is essentially self-adjoint on $D(N)$. We set

$$Y'(t)=\sum_{j=1}^{n} \frac{d}{dt}\hat{\alpha}_t(t)X_j .$$

Then $Y'(t)$ is essentially self-adjoint on $D(N)$ and by Theorem 0 (resp. 0′)

$$\|Y'(t)\varphi\| \leqq k_3 \|N\varphi\| ,$$

uniformly in $t\in[0,1]$; moreover

$$\|Ne^{-isY(t)}\varphi\| \leqq k_4 \|N\varphi\| , \tag{5.9}$$

uniformly in $s\in[0,1]$, $t\in[0,1]$, for all $\varphi\in D(N)$.

From the formula

$$e^{iA}-e^{iB}=\left(i\int_0^1 ds e^{isA}(A-B)e^{i(1-s)B}\right)^{-}$$

and the above estimates we conclude that $e^{-isY(t)}$ is continuous in t and that $F(t)$ is differentiable, with the following derivative:

$$\begin{aligned}\frac{d}{dt}F(t)&=-i\int_0^1 ds e^{-isY(t)}Y'(t)e^{-i(1-s)Y(t)}\psi\\ &=-i\int_0^1 ds e^{-isY(t)}Y'(t)e^{isY(t)}F(t) .\end{aligned} \tag{5.10}$$

The hypotheses of Theorem 3 permit us to apply the multiple commutator formula proven in Theorem 1_∞ to compute

$$Y(s,t)\equiv e^{-isY(t)}Y'(t)e^{isY(t)}$$

on $D(N)$, and then integrate over s: From Theorem 0 (resp. 0′) and (3.14) we know that $Y(s,t)$ and $\int_0^1 dsY(s,t)$ are essentially self-adjoint on $D(N)$, so that it suffices to identify

$$\int_0^1 dsY(s,t)\restriction D(N) .$$

We now claim that

$$\int_0^1 dsY(s,t)\restriction D(N)=X\restriction D(N) . \tag{5.11}$$

Using the hypotheses of Theorem 3, Theorem 0 (resp. 0′) and Lemma 2 we easily derive the required estimates which guarantee that we may apply Theorem 1_∞. This theorem then tells us that (5.11) holds if it holds formally, i.e., if

$$\int_0^1 ds e^{s\sum_{j=1}^n \hat{\alpha}_j(t)\xi_j} \left(\sum_{j=1}^n \frac{d}{dt} \hat{\alpha}_j(t)\xi_j \right) e^{-s\sum_{j=1}^n \hat{\alpha}_j(t)\xi_j} = \xi$$

holds as an equation between two elements of $\mathfrak{G}$. This, however, is obvious.

From (5.10) and (5.11) we conclude that $F(t)$ satisfies

$$dF(t)/dt = -iXF(t) .$$

Since $F(t) \in D(N)$, for all $t \in [0, 1]$, and $D(N)$ is a *core* for X, and since $F(0) = \pi'(\hat{g})\psi$, we conclude that $F(1) = e^{-iX}F(0) = \pi'(e^\xi)\pi'(\hat{g})\psi$.

The proof of (5.6) is now complete, because $D(N)$ is *dense* in $\mathscr{H}$. Q.E.D.

We note that the idea of using a differential equation and Theorem 1_∞ to prove $F(t) = e^{-itX}F(0)$, for all $t \in [0, 1]$, avoids the use of the Baker-Hausdorff-Campbell formula in the proof of (5.6) (see also [7, 10]).

From Theorem 3 we immediately obtain the following classic result of Nelson [7].

Corollary 4. *If*

$$N' = \sum_{j=1}^n X_j^2 + 1$$

is essentially self-adjoint on a dense domain $\mathscr{D} \subset \mathscr{H}$ *then*

$$N = \overline{N' \upharpoonright \mathscr{D}}$$

has the properties of the operator N of Theorem 3, and all the conclusions of Theorem 3 hold.

Remark. Corollary 4 and the results of [7] show that the converse of Theorem 3 is true, too.

5.2. A Trivial Application to the Rotation Group

Here we consider $G = \mathrm{SU}(2)$.

Let $\boldsymbol{x}$ denote the vectors in R^3, and

$$\mathscr{H} = L^2(R^3), \quad \boldsymbol{p} = -i\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right),$$

with x_i the i^{th} component of $\boldsymbol{x}$. Define

$$\boldsymbol{L} = (L_1, L_2, L_3) = \boldsymbol{x} \wedge \boldsymbol{p} ; \quad (\boldsymbol{L}, \boldsymbol{a}) = \sum_{j=1}^3 L_j a_j .$$

We set $N = \frac{1}{2}(\boldsymbol{x}^2 + \boldsymbol{p}^2 + 1)$.

It is a well known exercise to show that the hypotheses of Theorem 3 are fulfilled for this choice of N and $X_j \doteq L_j$, $j=1,2,3$; e.g.

$$\pm(\psi,(\boldsymbol{L}\,\boldsymbol{a})\psi) \leqq |\boldsymbol{a}|\,\| |\boldsymbol{x}|\psi\| \cdot \| |\boldsymbol{p}|\psi\| \leqq \frac{|\boldsymbol{a}|}{2}\{\| |\boldsymbol{x}|\psi\|^2 + \| |\boldsymbol{p}|\psi\|^2\} \leqq |\boldsymbol{a}|(\psi, N\psi)\,,$$

and

$$(\boldsymbol{L}\,\boldsymbol{a})^{\cdot} = 0, \quad \text{on} \quad \mathscr{L}(\mathscr{H}_3, \mathscr{H}_{-3})\,.$$

Thus

$$\{e^{i(\boldsymbol{L},\boldsymbol{a})} : \boldsymbol{a} \in R^3\}$$

is a continuous unitary representation of SU(2), by Theorem 3. If we know, *a priori*, that $(\boldsymbol{L}\,\boldsymbol{L})$ is essentially self-adjoint on some domain $\mathscr{D}$ we may re-define

$$N = \{\overline{(\boldsymbol{L}\,\boldsymbol{L}) + 1\} \upharpoonright \mathscr{D}}$$

and arrive at the same conclusions.

6. Commutativity of Unbounded Operators and an Application to Relativistic Quantum Field Theory

Theorem 5 [2]. *Let $A, \dot{A}$, and B satisfy the hypotheses of Theorem 0 (or 0′), and $[A,B]=0$, weakly on $D(N)\times D(N)$. Then all bounded functions of A and B commute.*

Proof. An immediate consequence of Theorem 1_1.

Remark. If there is some domain $\mathscr{D}$ dense in $\mathscr{H}$ such that

$$(A^2+B^2)\upharpoonright \mathscr{D}$$

is essentially self-adjoint then the hypotheses of Theorem 5 are true with

$$N = \overline{(A^2+B^2+1)\upharpoonright \mathscr{D}}\,.$$

[On the other hand: If all bounded functions of A and B commute then $\mathscr{D} \equiv D(A^2) \cap D(B^2)$ is obviously dense and $(A^2+B^2+1)\upharpoonright \mathscr{D}$ is self-adjoint].

Application to Quantum Field Theory [2, 3]. Let $\mathscr{H}, H, \{\varphi(f): f \in \mathscr{S}_{\text{real}}(R^d)\}$ denote the Hilbert space, the Hamiltonian, the quantum fields, respectively, of a quantum field theory satisfying all Wightman axioms [11] and, in addition,

$$\pm\varphi(f) \leqq |f|(H+1) \tag{6.1}$$

for some norm $|\cdot|$ continuous on Schwartz space. Then the bounded functions of $\{\varphi(f): f \in \mathscr{S}_{\text{real}}(R^d)\}$ generate a net of local von Neumann algebras satisfying all Haag-Kastler axioms, [12].

Proof. Set

$$N = H+1\,. \tag{6.2}$$

If f and g are two test functions with space-like separated support set

$$A=\varphi(f), \quad B=\varphi(g). \tag{6.3}$$

Then A, B, and N satisfy the hypotheses of Theorem 5; see [2]. [This is a consequence of (6.1) and Wightman's form of locality.] For details see [2]. Q.E.D.

Let $\mathscr{D}_W$ be the Wightman (polynomial) domain in $\mathscr{H}$. The remark following Theorem 5 tells us that if (6.1) is replaced by the condition that $(\varphi(f)^2+\varphi(g)^2)\restriction\mathscr{D}_W$ be essentially self-adjoint, for all test functions f and g with space-like separated supports the theory also fulfills all Haag-Kastler axioms.

It is easy to see that with N as in (6.2), $A=\varphi(f)$, $f\in\mathscr{S}_{\text{real}}(R^d)$, all hypotheses of Lemmas 2 and A.1 follow from (6.1). These lemmas then tell us that, for $\psi\in D(H^\alpha)$, (e.g., $\psi\in\mathscr{D}_W$ or $\psi=\Omega$, the physical vacuum)

$$e^{i\varphi(f)}\psi\in D(H^\alpha), \tag{6.4}$$

for all $\alpha=1,2,3,\ldots$.

7. Integration of Canonical Commutation Relations

Let $\mathscr{H}, H$, and $\{\varphi(f):f\in\mathscr{S}_{\text{real}}(R^{d-1})\}$, $\{\pi(f):f\in\mathscr{S}_{\text{real}}(R^{d-1})\}$ be the Hilbert space, the Hamiltonian, the time 0-fields and their canonically conjugate momenta, resp., of a canonical quantum field theory [1] that satisfies, in addition,

$$\varphi(f)\leqq|f|_1\cdot N,$$

with $N\equiv H+1$,

$$\pi(f)=i[N,\varphi(f)], \quad \text{on} \quad \mathscr{L}(\mathscr{H}_3,\mathscr{H}_{-3}), \tag{7.1}$$

$$\pi(f)\leqq|f|_2\cdot N$$

and

$$\pi(f)^{\cdot}\leqq|f|_3\cdot N,$$

for some Schwartz space norms $|\cdot|_1$, $|\cdot|_2$, and $|\cdot|_3$. Since we are dealing with a *canonical* field theory, we must have

$$[\varphi(f),\pi(g)]=i(f,g), \tag{7.2}$$

weakly on $D(N)\times D(N)$.

Then $\varphi(f)$ and $\pi(f)$ are essentially self-adjoint on $D(N)$, for all $f\in\mathscr{S}_{\text{real}}(R^{d-1})$, and

(1) $e^{i\varphi(f)}e^{i\pi(g)}=e^{i\pi(g)}e^{i\varphi(f)}e^{-i(f,g)}$,

(the Weyl relations; set $A=\varphi(f)$, $B=\pi(g)$ and apply Theorem 1_2).

(2) $e^{i\varphi(f)}\pi(g)e^{-i\varphi(f)}=\pi(g)-(g,f)$;

(set $A=\varphi(f)$, $B=\pi(g)$; apply Lemma 2, 2), and (7.2)*). A similar equation holds with $\varphi(f)\to\pi(f)$, $\pi(g)\to\varphi(g)$.*

(3) $e^{\pm i\varphi(f)}He^{\pm i\varphi(f)}=H\mp\pi(f)+\frac{1}{2}\|f\|_2^2$

on $D(N)$; *(set* $A=\varphi(f)$, $B=H$ *and use Lemma 2, 2)*, (7.1), *and Theorem 0). Hence*

$$\pm\pi(f)\leqq H+\tfrac{1}{2}\|f\|_2^2 \quad on \quad Q(H)\,.$$

These results are in some sense a converse to the results of Herbst, [5].

They [in particular 1)—3)], are very useful to give an easy proof of the fact that the $P(\phi)_2$ quantum field models [9] define canonical quantum field theories, [4], which requires only information on Euclidean Green's functions.

Other application of Theorems 0, 0′, 1_M, 1_∞ to canonical commutation relations (e. g., *a simple proof of von Neumann's uniqueness theorem*) can be worked out quite easily and are therefore not discussed here.

Appendix

In this appendix we generalize Lemma 2 of Section 3 and reformulate Theorems 1_M, 1_∞. First we prove

Lemma A.1. *Suppose* $A, \dot{A}, \dot{A}^{\cdot}\equiv(\dot{A})^{\cdot}\equiv\overset{(2)}{A},\ldots,\overset{(|n|)}{A}$ *all satisfy the hypotheses of Theorem 0, (resp. 0′). Then*

$$e^{itA}D(N^n)=D(N^n)\,,$$

and

$$\|N^n e^{itA}\psi\|\leqq e^{k_3|t|}\,\|N^n\psi\|\,,$$

for some finite constant k_3, *all* $|t|<\infty$; $(\pm n=1,2,3,\ldots)$.

Proof. We may assume $n>0$. As in Section 2 we first prove a Lemma $\mathrm{A.1}_\lambda$, but

$$A_\lambda=\lambda^{2n}R(\lambda)^n A R(\lambda)^n\,. \tag{A.1}$$

Then

$$A_\lambda^m e^{itA_\lambda}D(N^n)\subseteqq D(N^n) \tag{A.2}$$

is obviously true, for $m=0,1,2,\ldots$.

Let $\psi\in D(N^n)$. By (A.2) we may define

$$G_\lambda(t)=(N^n e^{-itA_\lambda}\psi, N^n e^{-itA_\lambda}\psi)\,.$$

Then

$$\begin{aligned} dG_\lambda(t)/dt &= i(e^{-itA_\lambda}\psi,[A_\lambda,N^{2n}]e^{-itA_\lambda}\psi)\\ &= i\sum_{k=0}^{2n-1}(e^{-itA_\lambda}\psi, N^k\dot{A}_\lambda N^{2n-k-1}e^{-itA_\lambda}\psi)\end{aligned}$$

is easily shown by use of (A.1), (A.2) and the hypotheses of Lemma A.1. Next one can show that

$$\begin{aligned} N^k\dot{A}_\lambda N^{2n-k-1} &= N^{k+1}\dot{A}_\lambda N^{2n-k-2}+iN^k\dot{A}^{\cdot}_\lambda N^{2n-k-2}\\ &= N^{k+1}\dot{A}_\lambda N^{2n-k-2}+iN^{k+1}\dot{A}^{\cdot}_\lambda N^{2n-k-3}\\ &\quad -N^k\dddot{A}_\lambda N^{2n-k-3}\\ &= \ldots,\ \text{on } D(N^n)\times D(N^n)\,.\end{aligned}$$

Applying this equation repeatedly, (proceeding e.g., by induction on k and on n), using the hypotheses on $A, \dot{A}, \ldots, \overset{(n)}{A}$, Theorem 0 (resp. 0′) and the trivial inequality

$$\|N^k\psi\| \leqq \|N^l\psi\|, \quad \text{for} \quad k \leqq l,$$

we obtain

$$\begin{aligned} dG_\lambda(t)/dt &\leqq 2k_3 \|N^n e^{-itA_\lambda}\psi\|^2 \\ &= 2k_3 G_\lambda(t), \end{aligned}$$

for some finite constant k_3 *independent* of λ, which after integration yields the desired Lemma A.1_λ. From this Lemma A.1 follows by essentially the same arguments that gave Lemma 2 as a corollary to Lemma 2_λ. Q.E.D.

Remarks. 1) Careful inspection of the proof of Lemma A.1 shows that the conclusions of this lemma remain true if we only assume that

$$\left\|\overset{(k)}{A}\psi\right\| \leqq k_3 \|N^k\psi\| \tag{A.3}$$

for some finite constant k_3 and all $\psi \in D(N^k)$, $k = 1, 2, \ldots, |n|$.

2) Let $\alpha \in [-|n|, |n|]$; since

$$N^{2|\alpha|} \leqq N^{2|n|}, \quad \text{on} \quad D(N^{|n|}) \times D(N^{|n|}),$$

Lemma A.1 holds for *all* $\alpha \in [-|n|, |n|]$ if $A, \dot{A}, \ldots, \overset{(|n|)}{A}$ all satisfy the hypotheses of Lemma A.1, $\left(\text{resp. } \overset{(k)}{A} \text{ satisfies (A.3)}, k = 1, \ldots, |n|.\right)$

We may now re-formulate Theorems 1_M and 1_∞.

Theorem A.2. *Let B and $A, \dot{A}, \ldots, \overset{(\alpha)}{A}$, (for some $\alpha = 1, 2, 3, \ldots$) all satisfy the hypotheses of Theorem 0.*

Let $C_0 = B$, and

$$C_n = i[A, C_{n-1}], \quad \text{weakly on} \quad D(N^\alpha) \times D(N^\alpha),$$

and

$$\|C_n\psi\| \leqq K_2^n n! \|N^\alpha\psi\|; \qquad n = 1, 2, 3, \ldots.$$

Then

$$B_t = \left[\left\{B + \sum_{m=1}^{M-1} \frac{t^m}{m!} C_m + \int_0^t dt, \ldots, \int_0^{t_{M-1}} dt_M e^{it_M A} C_M e^{-it_M A}\right\} \restriction D(N^\alpha)\right]^-$$

and, for $|t| < K_2^{-1}$,

$$B_t = \left[\underset{M\to\infty}{s\text{-lim}} \left\{B + \sum_{m=1}^{M} \frac{t^m}{m!} C_m\right\} \restriction D(N^\alpha)\right]^-$$

both satisfy

$$e^{itA} e^{isB} e^{-itA} = e^{isB_t}.$$

Proof. By Lemma A.1 $D(N^\alpha) = e^{\pm itA} D(N^\alpha)$, for all $|t| < \infty$. $D(N^\alpha)$ is a *core* for N, hence a core for B (by Theorem 0). Thus $D(N^\alpha)$ is also a *core for* B_t [defined as in (4.1)].

But on $D(N^\alpha)$

$$B_t = e^{-itA} B e^{-itA}$$

by Lemma A.1.

Applying now Lemma A.1 and the hypotheses of Theorem A.2 (concerning $\{C_n\}_{n=0}^{\infty}$), the proof of Theorem A.2 can be completed as in Section 4.

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Note Added in Proof: Ed Nelson has informed me that he has derived the conclusions of Theorem 5 under the only assumptions that A and B satisfy the hypotheses of Theorem 0, and $[A, B] = 0$, weakly on $D(N) \times D(N)$. His proof involves showing that $A + iB$ is a normal operator. Subsequently we found a proof of this result based on a straight forward extension of the methods of this paper. Moreover we proved Theorem 3 without the extra-hypothesis on $\dot{X}_j$.