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Analyticity Properties and Many-Particle Structure in General Quantum Field Theory

III. Two-Particle Irreducibility in a Single Channel

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Abstract. In the framework of L.S.Z. field theory in the case of a single massive scalar field, the "two-particle irreducible" parts of the n-point functions (in any single channel and for arbitrary n) are defined as the solutions of a system of integral equations suggested by the perturbative framework. These solutions enjoy the analytic and algebraic properties of general n-point functions (up to possible polar singularities of generalized C.D.D. type). Morever it is shown that the completeness of asymptotic states in the two-particle spectral region is equivalent to the analyticity of the two-particle irreducible n-point functions in the corresponding regions of complex momentum space.

1. Introduction

The previous papers in this series [1, 2] were devoted to the first steps of the off-shell non-linear program of general quantum field theory, following the line of the many-particle structure analysis of Symanzik [3].

In this program an essential role is played by the (perturbative) notion of "p-particle irreducible (p.i.) part" of a Green's function (with respect to a certain channel), which has to be rigorously incorporated in the axiomatic framework.

The present paper is devoted to the study of this problem in the case $p=2$, namely to the extraction of two-particle singularities from the n-point functions of a local field.

In other words[1], for any partition $(I, N\backslash I)$ of the set of indices $N=\{1,2,\dots,n\}$, $n\geqq 2$ arbitrary, we want to define a function $G^{I,N\backslash I}$ enjoying the following properties:

a) $G^{I,N\backslash I}$ is a general n-point function [1], i.e. it is analytic in the n-point primitive domain $D^{(n)}$ and its real boundary values satisfy Steinmann relations[2].

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1 The notations are those of [2]. For simplicity, we restrict to the case of a single mass m in the spectrum

2 For original works concerning the primitive structure of n-point functions, see [4–7]

b) The discontinuities of $G^{I,N\backslash I}$ in the channels $(I, N\backslash I)$ and $(N\backslash I, I)$ vanish in the region:

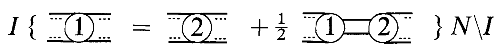
$$\mathscr{R}_I^{(2)} = \{p \in \mathbb{R}^{4(n-1)} : p_I^2 < 9m^2\}\,.$$

This property is actually our criterion for "two-particle irreducibility".

c) The functions $G^{I,N\backslash I}$ are linked with the physical n-point functions $H^{(n)}$ through a system of integral relations which is suggested by perturbation theory.

Actually a graphical definition of these two-p.i. functions can be given in the framework of perturbation theory, in the sense of formal series of Feynman amplitudes. Consider the expansion of a given n-point Green's function in terms of Feynman amplitudes and make choice of some partition $(I, N\backslash I)$ of the set of all external variables. The two-p.i. part of the considered function with respect to this partition is defined as the formal subseries of Feynman amplitudes associated with all the connected graphs which enjoy the following topological property: at least *three* internal lines must be cut in order to yield two disjoint connected subgraphs which split up the set of external lines according to the partition $(I, N\backslash I)$.

Now it turns out that the various two-p.i. parts thus obtained satisfy, as formal series, certain integral relations of the Bethe-Salpeter type usually represented under the graphic form:



which has the following algebraic meaning:

$$\begin{aligned}\tilde{\tau}_c^{(1)}(\{p_i, i\in I\}; \{p_j, j\in N\backslash I\}) &= \tilde{\tau}_c^{(2)}(\{p_i, i\in I\}; \{p_j, j\in N\backslash I\})\\ &+\tfrac{1}{2}\int \tilde{\tau}_c^{(1)}(\{p_i, i\in I\}; p_\alpha, p_\beta)\tilde{\tau}_c^{(2)}(-p_\alpha, -p_\beta; \{p_j, j\in N\backslash I\})\\ &\cdot G^{-1}(p_\alpha)G^{-1}(p_\beta)\delta(p_\alpha+p_\beta+p_I)\,dp_\alpha dp_\beta\,.\end{aligned} \tag{R}$$

Here $\tilde{\tau}_c^{(p)}(\{p_i, i\in I\}; \{p_j, j\in J\})$ denotes the p-p.i. part $(p=1,2)$ of the connected time-ordered product $\tilde{\tau}_c$, with respect to the partition (I, J) (once factored out the overall δ-function). $G(p)$ denotes the complete two-point function (with the Feynman prescription).

We shall follow the way opened by Symanzik [3], who proposed to consider integral relations of this type in order to *define* the two-particle irreducible functions in the axiomatic framework. However, for reasons explained below, we shall prefer to use *complex analogs* of these integral relations: this will be made possible by using the technique of "G-convolution" introduced in [8] and generalized in [1]. Indeed the advantages of using integral equations in complex domains are the following:

i) The new functions $G^{I,N\backslash I}$ which will be introduced as solutions of this integral system automatically appear as analytic functions whose domains can be studied by using techniques of contour deformations. Actually this approach is best suited to the general orientation of our program: here we have in mind further analytic continuation of the physical n-point functions (see some results of this type in [8]).

ii) In the course of this work, we shall take benefit of regularity properties in the complex domain which would not hold for the corresponding study in Minkowski space; these regularity properties allow a rigorous introduction of the functions $G^{I,N\backslash I}$. For example, all difficulties linked with integration at infinity will be easily overcome; moreover, equations of the type (R) in Minkowski space can be given a sense as appropriate limits of the corresponding equations in the complex domain (considering distributions as boundary values of analytic functions is a useful method for various problems involving distributions...).

iii) Finally, although some technical assumption is needed to carry out our program (the "smooth spectral condition" of [2]) and although some "pathologies" cannot be discarded (C.D.D.-type singularities [9]), these assumptions and pathologies are most clearly expressed in the complex framework.

In this approach, the construction of the two-particle irreducible four-point function (for a pseudo-scalar field) had already been presented by one of us [8].

The present work can be considered as an extension of this construction to the general case of the n-point functions of a scalar field (for arbitrary n). Moreover the proof of irreducibility[3] which is presented here is algebraically simpler (and more general) than the one given for $n=4$ in [8].

This proof relies on a detailed study of the analytic structure of the discontinuities of G-convolution integrals involving two internal lines.

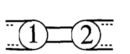
Section 2 is devoted to this mathematical study and a basic discontinuity formula is there derived (Theorem 1). In Section 3, the classical Fredholm theory is applied to the general Bethe-Salpeter equation in the complex four-point domain $D^{(4)}$ and relevant results of [8] are recalled. Section 4 is devoted to a generalization of this result for arbitrary n: a family of functions $G^{I,N\backslash I}(k_1,\ldots,k_n;\lambda)$ is constructed, each of which is meromorphic in the domain $D^{(n)}\times\mathbb{C}$.

In Section 5, it is shown that for $\lambda=1/2$, the two-particle irreducibility of the functions $G^{I,N\backslash I}$ in the relevant channel $(I, N\backslash I)$ is *equivalent* (except on a possible pathological subset corresponding to C.D.D. singularities) with the relations [10, 11, 2] stating the completeness of asymptotic states in the two-particle spectral region. Finally some technical results concerning the "permanence of smoothness" by G-convolution and Fredholm series summation are derived in two short appendices.

2. Mathematical Study: Absorptive Parts of Convolution Products

2.1. Introduction

In this section we shall consider the convolution products H^G associated in the following way with all the graphs G with two internal lines and two vertices, namely

$I\{$  $\}\, N\backslash I\,, \qquad n_1=|I|\,, \qquad n_2=|N\backslash I|\,.$

With the first (resp. second) vertex, associate a general (n_1+2)-point [resp. (n_2+2)-point] function $F^1(\hat{k}_1, k_\alpha, k_\beta)$ (resp. $F^2(-k_\alpha, -k_\beta, \hat{k}_2)$). Here the notations are the following.

Since we want to distinguish a given channel $(I, N\backslash I)$, a convenient notation in the space $\mathbb{C}^{4(n-1)}$ of the external variables $(k_1,\ldots,k_n)$ linked by $\sum_{j=1}^{n} k_j=0$

[3] In the sense of property b) described at the beginning

will be $(k_I; \hat{k}_1, \hat{k}_2)$, where $k_I = -k_{N\setminus I} = \sum_{i\in I} k_i$, $\hat{k}_1$ (resp. $\hat{k}_2$) stands for (n_1-1) [resp. (n_2-1)] independent four-vectors chosen among $\{k_i, i\in I\}$ (resp. $\{k_j, j\in N\setminus I\}$).

With the internal line α (resp. β) is associated the four-momentum k_α (resp. k_β), such that $k_\alpha + k_\beta = k_{N\setminus I} = -k_I$. $[H_0^{(2)}(k)]^{-1}$ denotes the inverse of the "bare" two-point function:

$$H_0^{(2)}(k) = -Z/(k^2-m^2)$$

with Z the "wave-function renormalization constant" of the field (as used in [2]).

Then the convolution product H^G can be written under the form [1]:

$$H^G(k_I, \hat{k}_1, \hat{k}_2) = \int_{\mathscr{C}_k} F^1(\hat{k}_1, k_\alpha, k_\beta)\, F^2(-k_\alpha, -k_\beta, \hat{k}_2)\, [H_0^{(2)}(k_\alpha) H_0^{(2)}(k_\beta)]^{-1} dk_\alpha \,. \tag{1}$$

Here $\mathscr{C}_k$ is an appropriate contour with real dimension four in the space $\mathbb{C}^4$ of the internal variable k_α, with continuous dependence on the external variables $k = (k_1, \dots, k_n)$.

More precisely ([1, 8]) $\mathscr{C}_k$ is obtained by continuous distortion of the "euclidean region" $\mathbb{R}^3 \times i\mathbb{R}$ inside the primitive domain of analyticity of the integrand, starting from the situation when the external variables are themselves euclidean. Throughout this section, the convergence of (1) at infinity on $\mathscr{C}_k$ will be assumed.

Now it has been proved [1] that H^G is a general n-point function, namely[4]:

i) H^G is analytic inside the primitive domain of analyticity, i.e. the union of the family of tubes $\{\mathscr{T}_{\mathscr{S}}, \mathscr{S}\in S(N)\}$ with appropriate complex neighbourhoods of the real, connecting these tubes together.

ii) Steinmann relations hold between the real boundary values $\{H^G_{\mathscr{S}}(p), \mathscr{S}\in S(N)\}$.

iii) coincidence relations: for any couple $(\mathscr{S}_+, \mathscr{S}_-)$ of adjacent cells separated by a partition $(J, N\setminus J)$, the corresponding boundary values $H^G_{\mathscr{S}_+}(p)$ and $H^G_{\mathscr{S}_-}(p)$ coincide on the real region

$$\mathscr{R}_J = \{p\in\mathbb{R}^{4(n-1)} : p_J^2 \neq m^2, p_J^2 < 4m^2\}$$

In each channel $(J, N\setminus J)$, the *absorptive parts* $\Delta_{\mathscr{S}_1\mathscr{S}_2} H^G$ are then defined through the extended Ruelle discontinuity formula [5]:

$$H^G_{\mathscr{S}_+}(p) - H^G_{\mathscr{S}_-}(p) = \Delta_{\mathscr{S}_1\mathscr{S}_2} H^G(p) - \Delta_{\mathscr{S}_2\mathscr{S}_1} H^G(p)$$

where $\mathscr{S}_1$ (resp. $\mathscr{S}_2$) is a well-defined cell of J (resp. $N\setminus J$) and $\Delta_{\mathscr{S}_1\mathscr{S}_2} H^G(p)$ [resp. $\Delta_{\mathscr{S}_2\mathscr{S}_1} H^G(p)$] a distribution with support in the set:

$$\Sigma_J = \{p\in\mathbb{R}^{4(n-1)} : p_J \in \bar{V}^+_{2m} \cup H^+_m\}$$

$$\text{(resp. } \Sigma_{N\setminus J} = \{p\in\mathbb{R}^{4(n-1)} : p_{N\setminus J} = -p_J \in \bar{V}^+_{2m} \cup H^+_m\})$$

$\Delta_{\mathscr{S}_1\mathscr{S}_2} H^G$ is one of the real boundary values of a certain "discontinuity function" $\Delta^J H^G$ which is the common analytic continuation inside the face $q_J = q_{N\setminus J} = 0$ of all the discontinuities $[H^G_{\mathscr{S}_+} - H^G_{\mathscr{S}_-}](p_J, \hat{k})$, with $(\mathscr{S}_+, \mathscr{S}_-)$ separated by $(J, N\setminus J)$ and $\hat{k}$ the remaining $4(n-2)$ complex variables.

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4 For a detailed review of the n-point primitive structure, see [1, 2]

More precisely $\Delta^J H^G(p_J, \hat{k})$ is a distribution in p_J (with support in $\bar{V}_{2m}^+ \cup H_m^+$), depending analytically on $\hat{k}=(\hat{k}_1, \hat{k}_2)$ inside the union of the family of "flat" tubes $\{\mathcal{T}_{\mathcal{S}_1} \times \mathcal{T}_{\mathcal{S}_2} : \mathcal{S}_1 \in S(J),\ \mathcal{S}_2 \in S(N \backslash J)\}$ with appropriate complex neighbourhoods of the real connecting all these tubes together (as a consequence of applying the edge of the wedge theorem). $\Delta_{\mathcal{S}_1 \mathcal{S}_2} H^G(p_J, \hat{k})$ denotes the branch of the "*discontinuity function*" $\Delta^J H^G$ which is analytic in the tube $\mathcal{T}_{\mathcal{S}_1} \times \mathcal{T}_{\mathcal{S}_2}$; the name "*absorptive part*" is reserved for the real boundary value $\Delta_{\mathcal{S}_1 \mathcal{S}_2} H^G(p_J, \hat{p})$ of the latter.

Then the problem we shall investigate in this section can be precisely stated as follows. Being given a convolution product H^G as above, with its convolution channel $(I, N \backslash I)$, is it possible to write for the corresponding discontinuity function $\Delta^I H^G$ a representation bringing out the contribution of the integration associated with the internal lines of G as well as those of the individual discontinuities of the vertex functions F^1, F^2.

2.2. *Preliminaries*

We first recall the following simple geometric notion. For any cell $\mathcal{S}$ of N, we define the two sets:

$$\alpha\!\uparrow\!\beta\!\uparrow\!\mathcal{S} = \{J \subset N \cup \{\alpha, \beta\} : \phi \neq J \subset \{\alpha, \beta\} \quad \text{or} \quad J \cap N \in \mathcal{S}\}$$
$$\alpha\!\downarrow\!\beta\!\downarrow\!\mathcal{S} = \{J \subset N \cup \{\alpha, \beta\} : N \subset J \neq N \cup \{\alpha, \beta\} \quad \text{or} \quad J \cap N \in \mathcal{S}\}.$$

Then these two sets are cells of $N \cup \{\alpha, \beta\}$ [4, 12].

In the space $\mathbb{C}^{4(n+1)}$ of the $(n+2)$ corresponding variables $(k_1, \ldots, k_n, k_\alpha, k_\beta)$ linked by the relation $k_\alpha + k_\beta + \sum_{i=1}^{n} k_i = 0$, let us then consider the tube $\mathcal{T}_{\alpha\uparrow\beta\uparrow\mathcal{S}}$ (resp. $\mathcal{T}_{\alpha\downarrow\beta\downarrow\mathcal{S}}$), $\mathcal{S}$ arbitrary, and the real mass-shell:

$$\sigma = (H_m^+)^2 = \{p_\alpha \in H_m^+,\ p_\beta \in H_m^+\}.$$

Being given a general $(n+2)$-point function F, we also consider its "amputated" branch:

$$F^{\mathrm{amp}}_{\alpha\uparrow\beta\uparrow\mathcal{S}}(k_\alpha, k_\beta, \hat{k}) = (k_\alpha^2 - m^2)(k_\beta^2 - m^2) F_{\alpha\uparrow\beta\uparrow\mathcal{S}}(k_\alpha, k_\beta, \hat{k})$$

(resp. $F^{\mathrm{amp}}_{\alpha\downarrow\beta\downarrow\mathcal{S}}$) analytic inside $\mathcal{T}_{\alpha\uparrow\beta\uparrow\mathcal{S}}$ (resp. $\mathcal{T}_{\alpha\downarrow\beta\downarrow\mathcal{S}}$). Here $(k_\alpha, k_\beta, \hat{k})$ is a convenient notation for the points of these tubes, with $\hat{k} \in \mathcal{T}_{\mathcal{S}}$.

Though the complexified mass-shell σ^c is not transverse to $\mathcal{T}_{\alpha\uparrow\beta\uparrow\mathcal{S}}$ (resp. $\mathcal{T}_{\alpha\downarrow\beta\downarrow\mathcal{S}}$), it has been shown in the framework of the linear program ([2], Appendix B) that $F^{\mathrm{amp}}_{\alpha\uparrow\beta\uparrow\mathcal{S}}$ (resp. $F^{\mathrm{amp}}_{\alpha\downarrow\beta\downarrow\mathcal{S}}$) can be restricted to σ^c, and yields a boundary value in the sense of distributions on σ, at all points of the open subset $\hat{\sigma}$ of non-parallel configurations (i.e. with $p_\alpha \neq p_\beta$). This restriction to σ is then denoted $\hat{F}_{\alpha\uparrow\beta\uparrow\mathcal{S}}(p_\alpha, p_\beta, \hat{k})$ [resp. $\hat{F}_{\alpha\downarrow\beta\downarrow\mathcal{S}}(p_\alpha, p_\beta, \hat{k})$]: it is a distribution on $(H_m^+)^2$ depending analytically on k inside the tube $\mathcal{T}_{\mathcal{S}}$.

Actually in the present case (when only two momenta stay on the mass-shell), $\hat{F}_{\alpha\uparrow\beta\uparrow\mathcal{S}}$ (resp. $\hat{F}_{\alpha\downarrow\beta\downarrow\mathcal{S}}$) is a distribution in $P = k_\alpha + k_\beta$, *locally analytic* in k_α inside a complex neighbourhood of σ (and analytic in $\hat{k}$ inside $\mathcal{T}_{\mathcal{S}}$). This is provided by the *two-point* analytic structure of F inside the face $q_\alpha + q_\beta = 0$.

Finally let us come back to the convolution product H^G and state a technical hypothesis (concerning the vertex functions $F^j, j=1,2$) which we shall assume throughout the rest of this section. This "smoothness assumption" can be formulated as follows. Let $(I, N\backslash I)$ the convolution channel of H^G and $(I; \{\alpha, \beta\})$ [resp. $(\{\underline{\alpha}, \underline{\beta}\}; N\backslash I)$] the corresponding channel of F^1 (resp. F^2). For any couple of adjacent cells $\mathscr{S}_{\pm}$ (resp. $\mathscr{S}'_{\pm}$) separated by $(I; \{\alpha, \beta\})$ [resp. $(\{\underline{\alpha}, \underline{\beta}\}; N\backslash I)$], we shall assume that the boundary values $F^1_{\mathscr{S}_{\pm}}(p_I, \hat{k})$ [resp. $F^2_{\mathscr{S}'_{\pm}}(-p_I, \hat{k}')$] and the corresponding discontinuities $\Delta_{\mathscr{S}_1\mathscr{S}_2}F^1(p_I, \hat{k})$ [resp. $\Delta_{\mathscr{S}'_1\mathscr{S}'_2}F^2(-p_I, \hat{k}')$], taken on the face $q_I=0$ are *continuous* functions of p_I in the region $4m^2 \leqq p_I^2 < 9m^2$.

Then as a straightforward consequence of this assumption and of the previous remarks, it is seen that the distributions $\hat{F}^j_{\alpha\uparrow\beta\uparrow\mathscr{S}}(\boldsymbol{p}_\alpha, \boldsymbol{p}_\beta, \hat{k})$ [resp. $\hat{F}^j_{\alpha\downarrow\beta\downarrow\mathscr{S}}(\boldsymbol{p}_\alpha, \boldsymbol{p}_\beta, \hat{k})$], $j=1,2, \mathscr{S}$ arbitrary can be identified with continuous functions on $(H_m^+)^2 \cap \{p_I^2 < 9m^2\}$.

In later applications (in Section 5), the physical n-point functions there considered will satisfy the above properties, which will be established on the basis of the postulate of "smooth spectral condition" previously introduced in this series [2].

2.3. A Discontinuity Formula

Now we are in a position to prove the basic

Theorem 1. *In the face $q_I = q_{N\backslash I} = 0$ associated with the convolution channel $(I, N\backslash I)$ of H^G, consider the "flat tube":*

$$\mathscr{F}^{(2)}_{\mathscr{S}_1\mathscr{S}_2} = \{(p_I, \hat{k}_1, \hat{k}_2) \in \mathbb{R}^4 \times \mathbb{C}^{4(n-2)} : p_I \in \Sigma^{(2)}, \hat{k}_1 \in \mathscr{T}_{\mathscr{S}_1}, \hat{k}_2 \in \mathscr{T}_{\mathscr{S}_2}\}$$

where $\Sigma^{(2)}$ denotes the "two-particle region":

$$\Sigma^{(2)} = \{p \in \mathbb{R}^4 : p \in V^+, 4m^2 \leqq p^2 < 9m^2\}\,.$$

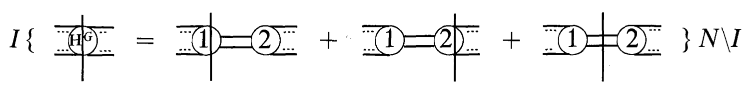
In $\mathscr{F}^{(2)}_{\mathscr{S}_1\mathscr{S}_2}$ the discontinuity function $\Delta^I H^G$ is given by the following formula, in the sense of continuous functions of p_I:

$$\begin{aligned}
&\Delta_{\mathscr{S}_1\mathscr{S}_2}H^G(p_I, \hat{k}_1, \hat{k}_2) = \\
&= \int_{\mathscr{C}_{\mathscr{S}_1}} F^1(p_I, \hat{k}_1, k_\alpha)\Delta^{N\backslash I}F^2(-p_I, -k_\alpha, \hat{k}_2)[H_0^{(2)}(k_\alpha)H_0^{(2)}(k_\alpha+p_I)]^{-1}dk_\alpha \\
&+ \int_{\mathscr{C}_{\mathscr{S}_2}} \Delta^I F^1(p_I, \hat{k}_1, k_\alpha)F^2(-p_I, -k_\alpha, \hat{k}_2)\,[H_0^{(2)}(k_\alpha)H_0^{(2)}(k_\alpha+p_I)]^{-1}dk_\alpha \\
&+ (2\pi i/Z)^2 \int \hat{F}^1_{\alpha\downarrow\beta\downarrow\mathscr{S}_1}(\hat{k}_1, \boldsymbol{p}_\alpha, \boldsymbol{p}_\beta)\hat{F}^2_{\underline{\alpha}\downarrow\underline{\beta}\downarrow\mathscr{S}_2}(\boldsymbol{p}_{\underline{\alpha}}, \boldsymbol{p}_{\underline{\beta}}, \hat{k}_2)\,\delta_\alpha^-\delta_\beta^-\delta(p_\alpha+p_\beta+p_I)dp_\alpha dp_\beta\,.
\end{aligned} \tag{2}$$

Here $p_{\underline{\alpha}} = -p_\alpha$, $\delta_\alpha^- = \theta(-p_\alpha^0)\delta(p_\alpha^2 - m^2)$ and $\mathscr{C}_{\mathscr{S}_1}$ (resp. $\mathscr{C}_{\mathscr{S}_2}$) is a contour in the space $\mathbb{C}^4$ of the internal variable k_α, which will be described in the following.

Remarks. i) The integrand of each term in the right-hand side of (2) would be an "unallowed product" of distributions in p_I if the technical "smoothness property" of the boundary values of the F^j's had not been specified.

ii) The discontinuity formula (2) brings out the contribution of each vertex function (first two terms) together with the one due to the poles carried by the internal lines. This can be graphically illustrated as follows:



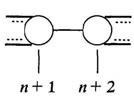
iii) The proof is rather long. The reader who is not interested in its technical details may skip the rest of this section.

2.4. Proof of the Theorem

Let us consider a couple of adjacent tubes $(\mathscr{T}_{\mathscr{S}_+}, \mathscr{T}_{\mathscr{S}_-})$ separated by the face $q_I = q_{N\setminus I} = 0$ and rewrite the expression (1) of H^G under the form:

$$\forall k \in \mathscr{T}_{\mathscr{S}_\pm} \; H^G_{\mathscr{S}_\pm}(k) = \int_{\mathscr{C}^\pm(k)} H^T(k, k_{n+1}, k_{n+2})[H_0^{(2)}(k_{n+2})]^{-1} \Big|_{k_{n+1}+k_{n+2}=0} dk_{n+1} \,. \tag{3}$$

Here H^T is the $(n+2)$-point function associated with the tree



$I\{\ \ \}N\setminus I;\ \mathscr{C}^\pm(k) = \mathbb{R}^3 \times \mathscr{L}^\pm$ is a cycle with real dimension four in the space $\mathbb{C}^4$ of k_{n+1}, with $\mathscr{L}^\pm$ some contour of the k^0_{n+1}-plane threading its way from $-i\infty$ to $+i\infty$ through the singularities of H_0^T.

The latter are "cuts" which correspond to the "vertex partitions" of the tree T [1], that is to the following channels: $\{n+1\}, \{n+2\}, I \cup \{n+1\}, J \cup \{n+1\}$ for any $J \in \mathscr{P}^*(I)$, and $L \cup \{n+2\}$ for any $L \in \mathscr{P}^*(N\setminus I)$. In the following $\{\Gamma_1, \Gamma_2, \Gamma_I, \Gamma_J, \Gamma_L\}$ will denote the corresponding singular sets and $\{\hat{\Gamma}_1, \hat{\Gamma}_2, \hat{\Gamma}_I, \hat{\Gamma}_J, \hat{\Gamma}_L\}$ their respective traces in the k^0_{n+1}-plane. In general they are note confused and the line $\mathscr{L}^\pm$ is not "pinched" (Fig. 1).

In the limit $q_I^0 = 0$, Γ_I becomes imbedded in the subspace with k^0_{n+1} real and the two subsets $\gamma_1 = \{k^2_{n+1} = m^2\}$ and $\gamma_I = \{(k_{n+1} + p_I)^2 = m^2\}$ of (respectively) Γ_1 and Γ_I have an intersection which is a sphere σ. In the k^0_{n+1}-plane, sections of σ only appear for special values of $\boldsymbol{p}_{n+1}$. At these special values of $\boldsymbol{p}_{n+1}$, the contour $\mathscr{L}^\pm$ is apparently pinched between the two coinciding poles γ_1 and γ_I. However we shall see below that the local analytic structure of the integrand in a complex neighbourhood of σ will allow us to avoid this pinching by suitable distortions of $\mathbb{R}^3 \times \mathscr{L}^\pm$ in $\mathbb{C}^4$ (provided that $p_I^2 \neq 4m^2$).

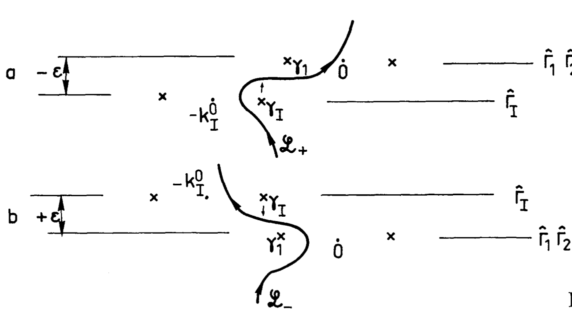


Fig. 1

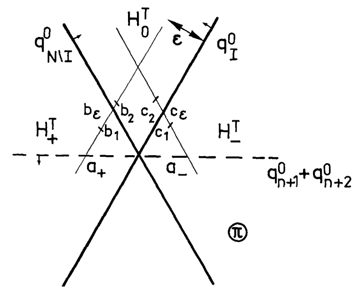


Fig. 2

2.4.1. Three Contributions

Let us choose two points k_+ (resp. k_-) in $\mathcal{T}_{\mathcal{S}_+}$ (resp. $\mathcal{T}_{\mathcal{S}_-}$) symmetrical with respect to the common face $q_I^0 = q_{N\setminus I}^0 = 0$, that is $k_\pm = (p_I^0 \pm i\varepsilon, \boldsymbol{p}_I; \hat{k})$ with $\varepsilon > 0$ and $(p_I, \hat{k})$ *fixed* in $\mathcal{F}^{(2)}_{\mathcal{S}_1\mathcal{S}_2}$.

In order to study the discontinuity

$$\Delta_{\mathcal{S}_1\mathcal{S}_2} H^G(p_I, \hat{k}) = \lim_{\varepsilon \to 0} [H^G_{\mathcal{S}_+}(k_+) - H^G_{\mathcal{S}_-}(k_-)] \tag{4}$$

it is now necessary to release the constraint $k_{n+1} + k_{n+2} = 0$ and to consider the analyticity properties of the integrand H^T in all its variables. Around the real region $\{p_I \in \Sigma^{(2)}, p_{n+1} + p_{n+2} = 0\}$, it is easily seen (Fig. 2) that there are four disconnected determinations of H^T corresponding to the four sign prescriptions $(q_I^0 \gtrless 0, q_{N\setminus I}^0 \gtrless 0)$. We denote by H_+^T (resp. H_-^T, H_0^T) the determination corresponding to the choice $(+, -)$ [resp. $(-, +)$, $(+, +)$].

Since in the following we shall be only concerned with the dependence of the contours on the variables $(q_I^0, q_{N\setminus I}^0)$, we can rewrite (3) under the form:

$$H^G_{\mathcal{S}_\pm}(k_\pm) = \int_{\mathcal{C}^\pm(a_\pm)} \tilde{H}^T_\pm(k(a_\pm), k_{n+1}) dk_{n+1} \tag{5}$$

where $a_\pm = \{\pm\varepsilon, \mp\varepsilon, 0\}$ is the projection of $k_\pm$ in the plane π of the triplet of variables $\{q_I^0, q_{N\setminus I}^0, q_{n+1}^0 + q_{n+2}^0\}$ and $\tilde{H}^T_\pm(k(a_\pm), k_{n+1})$ is a shorcut for $H^T_\pm(k(a_\pm), k_{n+1}, k_{n+2})[H_0^{(2)}(k_{n+2})]^{-1}$.

Now in view of (5) and of the above analyticity properties of H^T, the discontinuity (4) can be rewritten:

$$\begin{aligned}\Delta_{\mathcal{S}_1\mathcal{S}_2} H^G = \lim_{\varepsilon\to 0} \Bigg[& \int_{\mathcal{C}^+(b_1)} \tilde{H}^T_+(k(b_1), k_{n+1}) dk_{n+1} - \int_{\mathcal{C}^+(b_2)} \tilde{H}^T_0(k(b_2), k_{n+1}) dk_{n+1} \\ & + \int_{\mathcal{C}^+(c_2) - \mathcal{C}^-(c_2)} \tilde{H}^T_0(k(c_2), k_{n+1}) dk_{n+1} \\ & + \int_{\mathcal{C}^-(c_2)} \tilde{H}^T_0(k(c_2), k_{n+1}) dk_{n+1} - \int_{\mathcal{C}^-(c_1)} \tilde{H}^T_-(k(c_1), k_{n+1}) dk_{n+1} \Bigg].\end{aligned}$$

Here b_1, b_2, c_1, c_2 are points in the space π as shown on Figure 2. The contours $\mathcal{C}^+(b_1)$ and $\mathcal{C}^+(b_2)$ [resp. $\mathcal{C}^-(c_1)$ and $\mathcal{C}^-(c_2)$] are obtained by continuous distortion of the original $\mathcal{C}^+(a_+)$ [resp. $\mathcal{C}^-(a_-)$] inside the analycity domain of the integrand. The points $k(b_1)$ [resp. $k(c_1)$] and $k(b_2)$ [resp. $k(c_2)$] are "small perturbations" of $\hat{k}$ whose projections onto π are b_1 (resp. c_1) and b_2 (resp. c_2). We have used the homotopy of the cycles $\mathcal{C}^+(b_2)$ and $\mathcal{C}^+(c_2)$ in the analyticity domain of $\tilde{H}^T_0$.

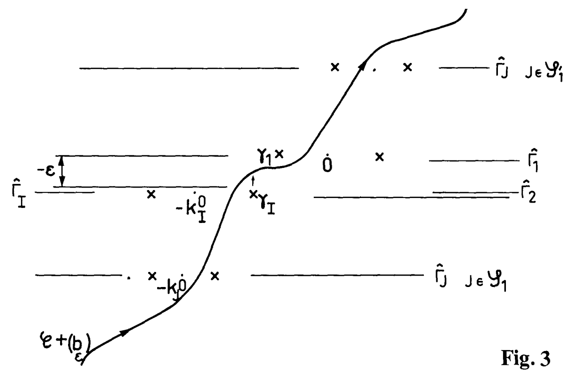


Fig. 3

Letting b_1 and b_2 (resp. c_1 and c_2) tend to b_ε (resp. c_ε), we finally obtain:

$$\begin{aligned}\Delta_{\mathcal{S}_1\mathcal{S}_2}H^G = \lim_{\varepsilon\to 0}\Big[&\int_{\mathcal{C}^+(b_\varepsilon)} [\tilde{H}_+^T - \tilde{H}_0^T](k(b_\varepsilon), k_{n+1})dk_{n+1} \\ + &\int_{\mathcal{C}^-(c_\varepsilon)} [\tilde{H}_0^T - \tilde{H}_-^T](k(c_\varepsilon), k_{n+1})dk_{n+1} \\ + &\int_{\mathcal{C}^+(c_\varepsilon)-\mathcal{C}^-(c_\varepsilon)} \tilde{H}_0^T(k(c_\varepsilon), k_{n+1})dk_{n+1}\Big] \end{aligned} \tag{6}$$

which brings out the contribution of the "pinching" of the contours (third term) and those of the discontinuities of the integrand.

2.4.2. The First Two Terms

In the first term of the right-hand side of (6), $\mathcal{C}^+(b_\varepsilon)$ is defined as the limit for $b_1\to b_\varepsilon$ of a contour $\mathcal{C}^+(b_1)=\mathbb{R}^3\times\mathcal{L}^+(b_1)$ where $\mathcal{L}^+(b_1)$ remains homotopous to $\mathcal{L}(a_+)$ in the analyticity domain of the integrand.

Figure 3 shows the situation in the limit $b_1=b_\varepsilon$, i.e. on the manifold $\{q^0_{N\setminus I}=0, p_{n+1}+p_{n+2}=0\}$: $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ lie at the distance ε, $\hat{\Gamma}_I$ and $\hat{\Gamma}_2$ are confused but, due to the presence of the factor $[H_0^{(2)}(k_{n+2})]^{-1}$ in the integrand, $\hat{\Gamma}_2$ has no pole and $\mathcal{L}^+(b_\varepsilon)$ is not pinched. The singularities $\{\hat{\Gamma}_L, L\in\mathcal{P}^*(N\setminus I)\}$ are no longer present since they correspond to partitions "in Steinmann position" with respect to the face $q^0_{N\setminus I}=0$, and that the discontinuity function $[\tilde{H}_+^T-\tilde{H}_0^T]$ does not have there any discontinuity (as a result of Steinmann relations for $\tilde{H}^T$).

Let us now investigate what happens when $\varepsilon\to 0$, starting from a fixed value ε_0: in the limit $\varepsilon=0, \mathbb{R}^3\times\mathcal{L}^+(b_\varepsilon)$ is "pinched" between the two polar manifolds γ_1 and γ_I. For $\varepsilon\leqq\varepsilon_0$, it is then necessary to modify the definition of $\mathcal{C}^+(b_\varepsilon)$ in the following way.

First notice that it is sufficient to distort $\mathcal{C}^+(b_\varepsilon)$ in its "central part", namely in the region $p_{n+1}\in\Omega$ shaded in Figure 4. Then we shall use the local analytic structure of the integrand in the neighbourhood of σ, as it has been recalled in Section 2.2, namely analyticity with respect to k_{n+1} in $V_{\hat{k}}(\Omega)\setminus(\gamma_1\cup\gamma_I)$ where $V_{\hat{k}}(\Omega)$ denotes some complex neighbourhood of Ω.

For this purpose consider the following vector field defined on Ω and depending continuously on ε for $\varepsilon_0\geqq\varepsilon\geqq 0$:

$$q_{n+1}^{(\varepsilon)}(p_{n+1})=(\boldsymbol{q}_{n+1}, q^0_{n+1})^{(\varepsilon)}=[\boldsymbol{0}, \varepsilon(1+\mu p^0_{n+1})]+(\varepsilon_0-\varepsilon)\varphi(p_{n+1})u^-(p_{n+1}) \tag{7}$$

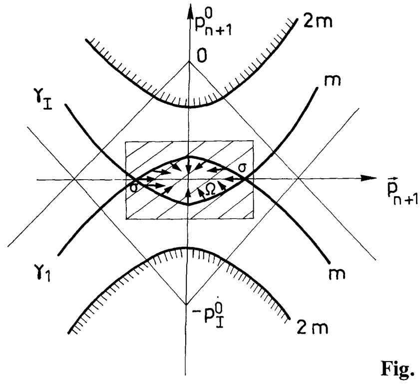


Fig. 4

with μ some positive constant, φ a continuous function with support $\bar{\Omega}$ and strictly positive in Ω and $u^-(p)=\lambda(p+p_I/2)$ a radial attractive vector field (λ negative).

For any $\varepsilon \leqq \varepsilon_0$ sufficiently small, it is then easily checked that $[p_{n+1}+iq^{(\varepsilon)}_{n+1}]$ can be kept inside $V_{\hat{k}}(\Omega)\backslash(\gamma_1 \cup \gamma_I)$, everywhere on Ω. $\mathscr{C}^+(b_\varepsilon)$ is therefore defined by the field $q^{(\varepsilon)}_{n+1}$ when $p_{n+1} \in \Omega$, and by $k^0_{n+1} \in \mathscr{L}^+(b_\varepsilon)$ outside. [$\mathscr{L}^+(b_\varepsilon)$ must be chosen to have the straight line $\varepsilon(1+\mu p^0_{n+1})$ as restriction to Ω.]

In other words, at all the points where the critical pinching situation occurs (i.e. points lying on the intersection σ of γ_1 and γ_I), the contour $\mathscr{C}^+(b_\varepsilon)$ is given a small distortion in space complex directions $\boldsymbol{q}_{n+1}$ which is kept continuous in Ω (Fig. 4).

With such a definition we can write:

$$\lim_{\varepsilon \to 0}\left[\int_{\mathscr{C}^+(b_\varepsilon)} [\tilde{H}^T_+ - \tilde{H}^T_0](k(b_\varepsilon), k_{n+1})dk_{n+1}\right] = \int_{\mathscr{C}^+(0)} [\tilde{H}^T_+ - \tilde{H}^T_0](k(0), k_{n+1})dk_{n+1} \tag{8}$$

or, in view of the tree-structure of H^T:

$$= \int_{\mathscr{C}_{\mathscr{S}_1}} F^1(p_I, \hat{k}_1, k_\alpha)\Delta^{N\backslash I}F^2(-p_I, -k_\alpha, \hat{k}_2)\,[H^{(2)}_0(k_\alpha)H^{(2)}_0(k_\alpha+p_I)]^{-1}dk_\alpha \tag{9}$$

where the notation $\mathscr{C}_{\mathscr{S}_1}$ restores the dependence of $\mathscr{C}^+(0)$ on the external variables $\hat{k}_1 \in \mathscr{T}_{\mathscr{S}_1}$. Here the assumption of continuity in p_I is required to give a sense to the product of terms in the integrand.

Similarly consider the term:

$$\int_{\mathscr{C}^-(c_\varepsilon)} [\tilde{H}^T_0 - \tilde{H}^T_-](k(c_\varepsilon), k_{n+1})dk_{n+1}. \tag{10}$$

Here the argument is slightly different from the previous one: indeed $\mathscr{C}^-(c_\varepsilon)$ is defined as the limit (for $c_1 \to c_\varepsilon$) of a contour $\mathscr{C}^-(c_1)=\mathbb{R}^3 \times \mathscr{L}^-(c_1)$, where $\mathscr{L}^-(c_1)$ remains homotopous to $\mathscr{L}(a_-)$ in the analyticity domain of the integrand. But (as it may be checked on Fig. 5) in the limit $c_1=c_\varepsilon$, $\mathscr{C}^-(c_1)$ is pinched between γ_1 and γ_I on σ.

Starting from a given value ε'_0 of $\varepsilon'=q^0_I$, $\mathscr{C}^-(c_1)$ must then be distorted above Ω as follows. Consider the vector field $q^{(\varepsilon')}_{n+1}$ defined on Ω and depending con-

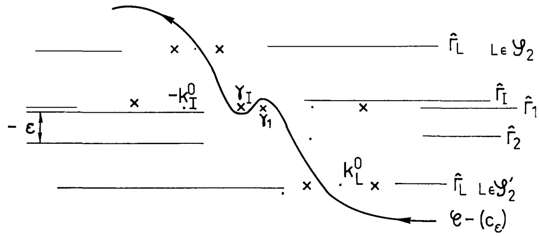


Fig. 5

tinuously on ε' for $\varepsilon_0' \geqq \varepsilon' \geqq 0$:

$$q_{n+1}^{(\varepsilon')}(p_{n+1}) = [\mathbf{0}, \varepsilon'(1-\mu p_{n+1}^0)] + (\varepsilon_0' - \varepsilon')\varphi(p_{n+1})u^+(p_{n+1})$$

with $u^+(p) = -u^-(p)$ a radial repulsive vector field. For any $\varepsilon' \leqq \varepsilon_0'$, $\mathscr{C}^-(c_1)$ is then defined by the field $q_{n+1}^{(\varepsilon')}$ for $p_{n+1} \in \Omega$, and by $k_{n+1}^0 \in \mathscr{L}^-(c_1)$ outside [$\mathscr{L}^-(c_1)$ having the straight line $\varepsilon'(1-\mu p_{n+1}^0)$ as restriction to Ω].

With these specifications, (10) is meaningful. Moreover in the limit $\varepsilon = 0$, $\mathscr{C}^-(c_\varepsilon)$ is not pinched and the second term of (6) can be written:

$$\int_{\mathscr{C}_{\mathscr{S}_2}} \Delta^I F^1(p_I, \hat{k}_1, k_\alpha) F^2(-p_I, -k_\alpha, \hat{k}_2)\, [H_0^{(2)}(k_\alpha) H_0^{(2)}(k_\alpha + p_I)]^{-1} dk_\alpha$$

where the notation $\mathscr{C}_{\mathscr{S}_2}$ restores the dependence of $\mathscr{C}^-(0)$ on the external variables $\hat{k}_2 \in \mathscr{T}_{\mathscr{S}_2}$. There again the product of terms in the integrand is meaningful on the basis of our smoothness assumption in p_I.

2.4.3. The Double Residue on the Sphere σ

Let us now consider the third term of (6), namely:

$$\int_{\mathscr{C}^+(c_\varepsilon) - \mathscr{C}^-(c_\varepsilon)} \tilde{H}_0^T(k(c_\varepsilon), k_{n+1}) dk_{n+1}\,.$$

Instead of the contours $\mathscr{C}^\pm(c_\varepsilon)$, it will be convenient to use other representatives $\underline{\mathscr{C}}^\pm(c_\varepsilon)$ in the same class of homology in the analyticity domain of $\tilde{H}_0^T$. These contours $\underline{\mathscr{C}}^\pm(c_\varepsilon)$ can be described as follows.

Since $\mathscr{C}^+(c_\varepsilon)$ [resp. $\mathscr{C}^-(c_\varepsilon)$] has been defined by continuous distortion of $\mathscr{C}^+(c_2)$ [resp. $\mathscr{C}^-(c_1)$] when the point c_2 (resp. c_1) tends to c_ε as shown on Figure 2, we shall first introduce a new representative of $\mathscr{C}^+(c_2)$ [resp. $\mathscr{C}^-(c_1)$]. We choose $\underline{\mathscr{C}}^+(c_2)$ as a "handle-shaped" domain whose sections at fixed $\boldsymbol{p}_{n+1}$ are made up of the union of the two following contours in the k_{n+1}^0-plane (Fig. 6a):

i) a complex line $\underline{\mathscr{L}}$ which is independent of $\boldsymbol{p}_{n+1}$, crosses $\hat{\Gamma}_I$ always on the left of γ_I and threads its way between the singularities $\{\hat{\Gamma}_L, L \in \mathscr{P}^*(N \backslash I)\}$ and $\{\hat{\Gamma}_J, J \in \mathscr{P}^*(I)\}$. Let $p_{n+1}^0 = c$ the intersection of $\underline{\mathscr{L}}$ with the real axis.

ii) when $\boldsymbol{p}_{n+1}$ is such that $p_{n+1}^0 = -\omega_1 = -(\boldsymbol{p}_{n+1}^2 + m^2)^{1/2}$ (i.e. the trace of γ_1) is bigger than c, we add to $\underline{\mathscr{L}}$ an anticlockwise oriented circle $\partial\gamma_1(\boldsymbol{p}_{n+1})$ around the trace of γ_1.

The way in which the "handle" $\partial\gamma_1 = \bigcup_{|\boldsymbol{p}_{n+1}| \leqq (c^2 - m^2)^{1/2}} \partial\gamma_1(\boldsymbol{p}_{n+1})$ can be attached continuously to the fixed part $\underline{\mathscr{L}}$ is obvious.

Actually $\underline{\mathscr{L}}$ can always be chosen so as to keep fixed when c_2 tends to c_ε. Then $\underline{\mathscr{C}}^+(c_\varepsilon)$ has the form $\partial\gamma_1^+ \cup (\mathbb{R}^3 \times \underline{\mathscr{L}})$, where $\partial\gamma_1^+$ is homotopous to $\partial\gamma_1$ in the

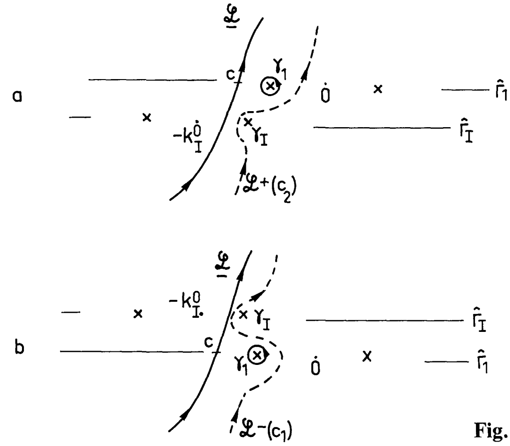


Fig. 6

domain of H_0^T. Note that $\partial\gamma_1^+$ is made up of circular sections which are transverse to γ_1 but can no longer lie in the k_{n+1}^0-plane when p_{n+1} belongs to the intersection σ of γ_1 and γ_I [i.e. when $|\boldsymbol{p}_{n+1}|=((p_I^0/2)^2-m^2)^{1/2}$].

Similarly we can introduce a new representative $\underline{\mathscr{C}}^-(c_1)$ for $\mathscr{C}^-(c_1)$ with another handle-shaped domain using the same fixed part $\mathbb{R}^3\times\underline{\mathscr{L}}$ (Fig. 6b). When c_1 tends to c_ε, the handle tends to a position $\partial\gamma_1^-$ which only differs from $\partial\gamma_1^+$ by the way it turns around the sphere σ.

Then we can write:

$$\int\limits_{\mathscr{C}^+(c_\varepsilon)-\mathscr{C}^-(c_\varepsilon)} \tilde{H}_0^T(k(c_\varepsilon), k_{n+1})dk_{n+1} = \int\limits_{\partial\gamma_1^+-\partial\gamma_1^-} \tilde{H}_0^T(k(c_\varepsilon), k_{n+1})dk_{n+1} \tag{11}$$

and in the right-hand side the integration domain can always be restricted to:

$$\bigcup_{|\boldsymbol{p}_{n+1}|\le\varrho} [\boldsymbol{p}_{n+1}, \partial\gamma_1^+(\boldsymbol{p}_{n+1})] - \bigcup_{|\boldsymbol{p}_{n+1}|\le\varrho} [\boldsymbol{p}_{n+1}, \partial\gamma_1^-(\boldsymbol{p}_{n+1})]$$

ϱ being an arbitrary number satisfying the inequalities:

$$((p_I^0/2)^2-m^2)^{1/2}\le\varrho\le(c^2-m^2)^{1/2}.$$

Now let us apply the residue theorem in its general form [13]. We get:

$$\begin{aligned}&\int\limits_{\partial\gamma_1^\pm} \tilde{H}_0^T(k(c_\varepsilon), k_{n+1})dk_{n+1}\\ &=(2i\pi)\int\limits_{\gamma_1^\pm(c_\varepsilon)} (p_{n+1}^2-m^2)\tilde{H}_0^T(k(c_\varepsilon), p_{n+1})\Big|_{p_{n+1}^2=m^2} \frac{d\boldsymbol{p}_{n+1}}{-2\omega_1(\boldsymbol{p}_{n+1})}\end{aligned} \tag{12}$$

where the "residue contour" $\gamma_1^+(c_\varepsilon)$ [resp. $\gamma_1^-(c_\varepsilon)$] has to be defined on the complex mass-shell $k_{n+1}^2=m^2$ by continuous distortion of a corresponding residue contour $\gamma_1^+(c_2)$ [resp. $\gamma_1^-(c_1)$] which we shall study now. Note that here we have used the fact that for $p_I^2 \neq 4m^2$, at any value of $c_{1,2}$ the relative situation of the manifolds γ_1 and γ_I in the neighbourhood of their intersection σ never degenerates, so that the ambient isotopy procedure [14] can be applied.

Then if we parametrize γ_1 by means of polar coordinates (r, Ω), i.e. by putting $\boldsymbol{p}_{n+1}=r\Omega$ (with $|\Omega|=1$), the residue contours $\gamma_1^+(c_2)$ and $\gamma_1^-(c_1)$ are both given

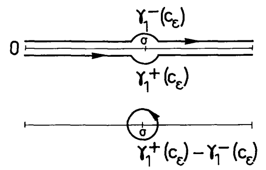


Fig. 7

by the ball $\{\Omega\in S^{(2)}\}\times\{r:0\leqq r\leqq\varrho\}$. However in the situation corresponding to c_2, the trace of γ_I in the complex r-plane (namely $r=((k_I^0/2)^2-m^2)^{1/2}$) tends to the real from above when $c_2\to c_\varepsilon$. In the situation corresponding to c_1, this trace tends to the real from below when $c_1\to c_\varepsilon$. The limiting residue contours $\gamma_1^\mp(c_\varepsilon)$ can then be pictured in the r-plane as shown on Figure 7.

In view of (11) and (12) and applying again the residue theorem in the r-plane (or equivalently in the ω-plane, $\omega=(r^2+m^2)^{1/2}$), we get:

$$\int\limits_{\partial\gamma_1^+-\partial\gamma_1^-}\tilde{H}_0^T(k(c_\varepsilon),k_{n+1})dk_{n+1}$$
$$=(2i\pi)^2\int\limits_{\sigma}(p_{n+1}^2-m^2)[(p_{n+1}+p_I)^2-m^2]\,\tilde{H}_0^T(k(c_\varepsilon),p_{n+1})\,\Big|_{\sigma}\frac{\sqrt{(p_I^0)^2-4m^2}}{8p_I^0}d\Omega$$

which can be rewritten:

$$(2i\pi)^2\int\tilde{H}_{0,\sigma}^{T,\mathrm{amp}}(k(c_\varepsilon),p_{n+1})\delta^-(p_{n+1})\delta^+(p_{n+1}+p_I)dp_{n+1}\tag{13}$$

where $\tilde{H}_{0,\sigma}^{T,\mathrm{amp}}$ stands for the restriction to the sphere σ of the "amputated" $(p_{n+1}^2-m^2)[(p_{n+1}+p_I)^2-m^2]\,\tilde{H}_0^T$. Now the limit of (13) for $\varepsilon\to 0$ is meaningful since the compact set of integration remains inside the analyticity domain of the integrand.

Finally taking into account the tree-structure of H^T and the definition of H_0^T, it is easily checked that when k_{n+1} stays in a complex neighbourhood of σ and $\hat{k}$ inside $\mathscr{T}_{\mathscr{S}_1}\times\mathscr{T}_{\mathscr{S}_2}$, the corresponding branch of $\tilde{H}_0^T$ is the following:

$$\tilde{H}_0^T=F^1_{\alpha\downarrow\beta\downarrow\mathscr{S}_1}(\hat{k}_1,k_\alpha,k_\beta)F^2_{\underline{\alpha}\downarrow\underline{\beta}\downarrow\mathscr{S}_2}(-k_\alpha,-k_\beta,\hat{k}_2)\;[H_0^{(2)}(k_\alpha)H_0^{(2)}(k_\beta)]^{-1}$$

so that:

$$\tilde{H}_{0,\sigma}^{T,\mathrm{amp}}=\frac{1}{Z^2}\hat{F}^1_{\alpha\downarrow\beta\downarrow\mathscr{S}_1}(\hat{k}_1,\boldsymbol{p}_\alpha,\boldsymbol{p}_\beta)\hat{F}^2_{\underline{\alpha}\downarrow\underline{\beta}\downarrow\mathscr{S}_2}(\boldsymbol{p}_{\underline{\alpha}},\boldsymbol{p}_{\underline{\beta}},\hat{k}_2)$$

where the notations are those of Section 2.2. Finally in the limit $\varepsilon=0$, we get

$$\lim_{\varepsilon\to 0}\left[\int\limits_{\mathscr{C}^+(c_\varepsilon)-\mathscr{C}^-(c_\varepsilon)}\tilde{H}_0^T(k(c_\varepsilon),k_{n+1})dk_{n+1}\right]$$
$$=(2\pi i/Z)^2\int\hat{F}^1_{\alpha\downarrow\beta\downarrow\mathscr{S}_1}\hat{F}^2_{\underline{\alpha}\downarrow\beta\downarrow\mathscr{S}_2}\delta_\alpha^-\delta_\beta^-\delta(p_\alpha+p_\beta+p_I)dp_\alpha dp_\beta\,.$$

This achieves the proof of Theorem 1.

2.4.4. Final Remarks

i) In the previous argument, we always supposed $p_I^2>4m^2$. When p_I^2 tends to $4m^2$ (situation where the sphere σ is degenerated), the third term vanishes as the integral of a bounded continuous function on a vanishing cycle. It is shown in [18] that it is also the case for the two other contributions.

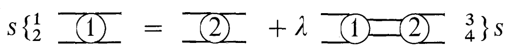
ii) An analogous discontinuity formula can be established by introducing the branch of H^T which corresponds to the $(-,-)$ choice on Figure 2. The contributions of the discontinuities of the vertex functions on the faces $q_I^0=0$ and $q_{N\setminus I}^0=0$ are of the same type as those studied in 2.4.2, with appropriate contours $\tilde{\mathscr{C}}_{\mathscr{S}_1}$ and $\tilde{\mathscr{C}}_{\mathscr{S}_2}$. The pinching contribution is the following:

$$(2\pi i/Z)^2 \int \hat{F}^1_{\alpha\uparrow\beta\uparrow\mathscr{S}_1}(\hat{k}_1, \boldsymbol{p}_\alpha, \boldsymbol{p}_\beta)\, \hat{F}^2_{\underline{\alpha}\uparrow\underline{\beta}\uparrow\mathscr{S}_2}(\boldsymbol{p}_{\underline{\alpha}}, \boldsymbol{p}_{\underline{\beta}}, \hat{k}_2)\delta(p_\alpha+p_\beta+p_I)\delta_\alpha^-\delta_\beta^-\, dp_\alpha dp_\beta\,.$$

Both expressions are equivalent: in the following, one of these will be more suitable according as we shall want to exploit the asymptotic completeness of "outgoing" or "incoming" states (see Section 5).

3. The General Bethe-Salpeter Equation

In this section we shall investigate the analyticity properties of the solution of the *general* Bethe-Salpeter equation, which is usually written under the graphic form:



(14)

and will be given a precise meaning in the *complex* four-point primitive domain $D^{(4)}$.

Let us first define our notations. We shall be dealing with general four-point functions, defined in the space $\mathbb{C}^{12}$ of four complex four-momenta $\{k_j, 1\leqq j\leqq 4\}$ linked by the relation $\sum_{j=1}^{4} k_j=0$. Since we want to distinguish a given channel, for instance $[\{1,2\};\{3,4\}]$, let us introduce the following "barycentric" independent four-vectors:

$$\begin{aligned} K&=P+iQ &&=k_1+k_2=-(k_3+k_4)\\ Z&=X+iY &&=k_1-k_2\\ Z'&=X'+iY' &&=k_3-k_4\,. \end{aligned}$$

Then in the four-point complex domain $D^{(4)}$, let us consider the following integral equation, which is the analog of (14):

$$\begin{aligned} F(K,Z,Z')&=G(K,Z,Z';\lambda)\\ &\quad+\lambda\int_\Gamma F(K,Z,Z_1)G(K,Z_1,Z';\lambda)\left[H_0^{(2)}\left(\frac{Z_1+K}{2}\right)H_0^{(2)}\left(\frac{Z_1-K}{2}\right)\right]^{-1}dZ_1\,. \end{aligned} \tag{15}$$

Here $F(K,Z,Z')$ denotes a general four-point function which is considered as given; $G(K,Z,Z';\lambda)$ is the reciprocal Fredholm kernel the analyticity properties of which are under study. $\Gamma=\mathbb{R}^3\times\mathscr{L}$ is a complex contour with real dimension four, threading its way through the singularities of the integrand, with euclidean infinite parts ([1, 8]).

Since we want to follow the perturbative theory as a heuristic guide, we choose the given function $F(K,Z,Z')$ to be the "one-particle irreducible part" $F^{\{1,2\};\{3,4\}}$ of the physical four-point function $H^{(4)}$, with respect to the considered channel

[{1, 2}; {3, 4}]. It was proved in [2] that this function enjoys the primitive structure of general four-point functions[5].

Since (from the Wightman axioms) $F^{\{1,2\};\{3,4\}}$ is expected to have a polynomial increase at infinity inside the primitive domain, it is necessary to avoid divergences on the (euclidean) infinite parts of Γ. For this purpose, F will be multiplied by the following "cut-off" factor:

$$\varrho(k_1,\ldots,k_4)=\prod_{j=1}^{4}[(m^2-\mu^2)/(k_j^2-\mu^2)]^r$$

where $\mu \gg 2m$ and r are sufficiently large positive numbers. It is clear that this regularization at infinity does not spoil the primitive analytic structure of F; moreover it does not change its restriction to the mass-shell, a point which we shall need below in Section 5.

Then the conservation of the primitive structure of four-point functions by convolution [1, 8] allows us to prove:

Proposition 1. *For each given cut-off factor* ϱ, *the unique solution* $G(K,Z,Z';\lambda)$ *of the Fredholm equation* (15) *is meromorphic in the product* $\{(K,Z,Z')\in D^{(4)}\}\times\{\lambda\in\mathbb{C}\}$.

Proof. We shall only sketch it since it has already been presented in [8]. The formulae of the classical Fredholm theory [15] indicate that (15) is identically satisfied by the following function:

$$G(K,Z,Z';\lambda)=B(K,Z,Z';\lambda)/A(K;\lambda)$$

with

$$A(K;\lambda)=\sum_{n=0}^{\infty}\lambda^n(n\,!)^{-1}A_n(K) \tag{16}$$

$$B(K,Z,Z';\lambda)=\sum_{n=0}^{\infty}\lambda^n(n\,!)^{-1}B_n(K,Z,Z') \tag{17}$$

and $A_0(K)=1$, $B_0(K,Z,Z')=F(K,Z,Z')$,

$$A_n(K)=\int_{\Gamma^n}\begin{vmatrix}F(K,Z_1,Z_1) & \ldots & F(K,Z_1,Z_n)\\ \\ F(K,Z_n,Z_1) & \ldots & F(K,Z_n,Z_n)\end{vmatrix}$$

$$\cdot\prod_{j=1}^{n}\left[H_0^{(2)}\left(\frac{Z_j+K}{2}\right)H_0^{(2)}\left(\frac{Z_j-K}{2}\right)\right]^{-1}dZ_j$$

$$B_n(K,Z,Z')=\int_{\Gamma^n}\begin{vmatrix}F(K,Z,Z') & F(K,Z,Z_1) & \ldots & F(K,Z,Z_n)\\ F(K,Z_1,Z') & F(K,Z_1,Z_1) & \ldots & F(K,Z_1,Z_n)\\ \\ F(K,Z_n,Z') & F(K,Z_n,Z_1) & \ldots & F(K,Z_n,Z_n)\end{vmatrix}$$

$$\cdot\prod_{j=1}^{n}\left[H_0^{(2)}\left(\frac{Z_j+K}{2}\right)H_0^{(2)}\left(\frac{Z_j-K}{2}\right)\right]^{-1}dZ_j\,.$$

---

[5] Up to a finite number of poles of the form $\{K^2=\alpha_\nu, 0<\alpha_\nu<m^2\}$. Such poles can in principle be produced by the zeros of the propagator [i.e. the physical two-point function $H^{(2)}(K)$], their occurence being connected with the existence of "ultraviolet" polynomial increase for $H^{(2)}$. However it is a reasonable hope that such zeros are also present (with the same order) in the physical n-point functions and consequently do not produce poles in the one-p.i. n-point functions

As recalled above, $F(K, Z, Z')$ is analytic in the domain $D^{(4)}$, minus the set of real poles $\{K^2 = \alpha_\nu, 0 < \alpha_\nu < m^2\}$. We call $\hat{D}^{(4)}$ the domain thus obtained[6] (with the hope that $D^{(4)} = \hat{D}^{(4)}$, see the footnote 5). Then in view of the conservation of analyticity by convolution proved in [1], $A_n(K)$ [resp. $B_n(K, Z, Z')$] is also analytic in $\hat{D}^{(2)}$ (resp. $\hat{D}^{(4)}$).

The independence of the definition of $G(K, Z, Z'; \lambda)$ with respect to the choice of the contour Γ in its homology class can be easily checked.

As for the (absolute) convergence of each series (16) and (17), it is established in the whole complex λ-plane by using the following bounds [8] (inspired by classical Hadamard's majorizations of determinants):

$$\begin{aligned} &|A_n(K)| \leqq C^n n^{n/2} [1 + d(K, Z, Z')^{-1}]^{nM} \\ &|B_n(K, Z, Z')| \leqq C'^n (n+1)^{\frac{n+1}{2}} [1 + d(K, Z, Z')^{-1}]^{(n+1)M} . \end{aligned} \tag{18}$$

where C and C' are positive constants, M an integer and $d(K, Z, Z')$ stands for the distance of the point (K, Z, Z') to the boundary $\partial D^{(4)}$ of $D^{(4)}$. The sums (16) and (17) are then analytic functions in the respective domains $\{K \in \hat{D}^{(2)}\} \times \{\lambda \in C\}$ and $\{(K, Z, Z') \in \hat{D}^{(4)}\} \times \{\lambda \in \mathbb{C}\}$, from which follows the meromorphy of $G(K, Z, Z'; \lambda)$ in $\{(K, Z, Z') \in D^{(4)}\} \times \{\lambda \in \mathbb{C}\}$.

Remarks. i) In addition to the possible real poles $\{K^2 = \alpha_\nu, 0 < \alpha_\nu < m^2\}$, the only singularities of $G(K, Z, Z'; \lambda)$ in $D^{(4)} \times \mathbb{C}$ are induced by the zeros of $A(K; \lambda)$ and localized on analytic manifolds of the type $f(K; \lambda) = 0$. These poles induced by the zeros of a general two-point function can be considered as generalized C.D.D. singularities [3, 9].

ii) The estimates (18) are not sufficient to imply that the analytic functions $A(K; \lambda)$ and $B(K, Z, Z'; \lambda)$ have boundary values in the sense of distributions on the boundary of their domain. Actually for each Fredholm determinant $A_n(K)$ [resp. $B_n(K, Z, Z')$], the boundary value is a distribution whose order can increase linearly with n. Therefore nothing can be said about the boundary value of the sum (16) [resp. (17)], at least in the framework of distributions.

However as a consequence of the technical postulate of "smooth spectral condition" introduced in [2], a regularity property in the convolution variable $P = \mathrm{Re}\, K$ can be established as follows.

Inside the face $\mathrm{Im}\, K = 0$, let us consider the following "flat" tubes:

$$\mathscr{F}_{\varepsilon, \varepsilon'} = \{(P, Z, Z') \in \mathbb{R}^4 \times \mathbb{C}^8 : P \in \hat{\Sigma}^{(2)}, \varepsilon\, \mathrm{Im}\, Z \in V^+, \varepsilon'\, \mathrm{Im}\, Z' \in V^+, \mathrm{Im}\, Z \neq \pm \mathrm{Im}\, Z'\}$$

with $\varepsilon, \varepsilon' = \pm 1$ and $\hat{\Sigma}^{(2)}$ the "non-C.D.D." open set [2]:

$$\hat{\Sigma}^{(2)} = \{p \in \mathbb{R}^4 : p \in V^+, 9m^2 > p^2 \geqq 4m^2, H_{\pm}^{(2)}(p) \neq 0\} .$$

Then consider the boundary values of $F(K, Z, Z')$ onto the face $\mathrm{Im}\, K = 0$, namely:

$$F_{\pm}(P, Z, Z') = \lim_{\substack{\varrho \in V^+ \\ \varrho \to 0}} F(P \pm i\varrho, Z, Z') .$$

---

6 We similarly call $\hat{D}^{(2)} = D^{(2)} \setminus \{K \in \mathbb{C}^4; K^2 = \alpha_\nu, 0 < \alpha_\nu < m^2\}$

On the basis of the results derived in [2] from the "smooth spectral condition", the following property is proved in [18]:

Proposition 2. *In each flat tube* $\mathscr{F}_{\varepsilon,\varepsilon'}$, $F_{\pm}(P, Z, Z')$ *is an analytic function of* (Z, Z') *and a (Hölder-)continuous function of* P.

Now in view of the "conservation of smoothness" in P by convolution proved in Appendix A, we can state:

Proposition 3. *In each flat tube* $\mathscr{F}_{\varepsilon,\varepsilon'}$, *the boundary values:*

$$B_n^{\pm}(P, Z, Z') = \lim_{\substack{\varrho \in V^+ \\ \varrho \to 0}} B_n(P \pm i\varrho, Z, Z')$$

$$A_n^{\pm}(P) = \lim_{\substack{\varrho \to 0 \\ \varrho \in V^+}} A_n(P \pm i\varrho)$$

are analytic in (Z, Z') *and continuous in* P.

However it is proved in Appendix B that the smoothness in P is preserved by summation of Fredholm series, in other words that:

Proposition 4. *In each flat tube* $\mathscr{F}_{\varepsilon,\varepsilon'}$, *the Fredholm series:*

$$B_{\pm}(P, Z, Z'; \lambda) = \sum_{n=0}^{\infty} \lambda^n (n!)^{-1} B_n^{\pm}(P, Z, Z')$$

$$A_{\pm}(P; \lambda) = \sum_{n=0}^{\infty} \lambda^n (n!)^{-1} A_n^{\pm}(P)$$

are analytic in (Z, Z') *and continuous in* P.

For each fixed value of λ, let us then consider the two-particle region $\Sigma^{(2)}$, minus the real C.D.D. zeros [i.e. zeros of $H_{\pm}^{(2)}(p)$] and those (generalized C.D.D. zeros) of the boundary values $A_{\pm}(p; \lambda)$ of the two-point Fredholm determinant.

Namely:

$$\hat{\Sigma}_{\lambda}^{(2)} = \{p \in \mathbb{R}^4 : p \in V^+, 4m^2 \leqq p^2 < 9m^2, H_{\pm}^{(2)}(p) \neq 0, A_{\pm}(p; \lambda) \neq 0\}.$$

It is easy to check that $\hat{\Sigma}_{\lambda}^{(2)}$ is a dense open subset of $\Sigma^{(2)}$, as a consequence of the analyticity of $H^{(2)}$ and A. Moreover from Proposition 4 it is straightforward to get:

Proposition 5. *In each flat tube:*

$$\hat{\mathscr{F}}_{\varepsilon,\varepsilon'} = \{(P, Z, Z') \in \mathbb{R}^4 \times \mathbb{C}^8 : P \in \hat{\Sigma}_{\lambda}^{(2)}, \varepsilon \operatorname{Im} Z \in V^+, \varepsilon' \operatorname{Im} Z' \in V^+, \operatorname{Im} Z \neq \pm \operatorname{Im} Z'\}$$

the boundary values:

$$G_{\pm}(P, Z, Z'; \lambda) = \lim_{\substack{\varrho \in V^+ \\ \varrho \to 0}} G(P \pm i\varrho, Z, Z'; \lambda)$$

are analytic functions of $(Z, Z'; \lambda)$ *and continuous functions of* P.

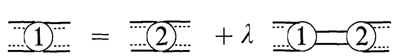
However such a regularity property cannot be obtained in the other variables. If no extra technical assumption is formulated (for instance those used in [8], p. 112), when (Z, Z') tends to the real inside $\hat{\mathscr{F}}_{\varepsilon,\varepsilon'}$, the corresponding boundary value of $G_{\pm}(P, Z, Z'; \lambda)$ is only defined as a general hyperfunction in (X, X')-space.

4. Generalization to N-Point Functions

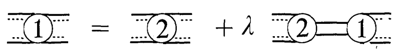
4.1. Introduction

In this section we study the analyticity properties of the solution of a *system* of integral equations which generalize the Bethe-Salpeter equation, when one is concerned with all possible channels $(I, N\backslash I)$, with $n=|N|$ and $I\subset N$ arbitrary.

Actually we shall start from the following graphical identities, which can be checked in perturbation theory (at any order):

$I\{$  $\}\, N\backslash I$ (19a)

and also:

$I\{$  $\}\, N\backslash I$. (19b)

These relations will be given a precise meaning in the *complex* n-point primitive domain $D^{(n)}$.

Let us first define our notations. We shall deal with general n-point functions defined in the space $\mathbb{C}^{4(n-1)}$ of n complex four-vectors $\{k_i, 1\leqq i\leqq n\}$ linked by the relation $\sum_{i=1}^{n} k_i=0$. Since we want to distinguish a given channel $(I, N\backslash I)$ with $n_1=|I|$ and $n_2=|N\backslash I|$, a convenient notation will be the following: $k=(k_I, \hat{k}_1, \hat{k}_2)$ where $\hat{k}_j\in\mathbb{C}^{4(n_j-1)}, j=1,2$, and $\hat{k}_1$ (resp. $\hat{k}_2$) stands for (n_1-1) [resp. (n_2-1)] independent four-vectors chosen among $\{k_i, i\in I\}$ (resp. $\{k_j, j\in N\backslash I\}$).

Then in the primitive n-point domain $D^{(n)}$, we shall write relation (19a) as follows:

$$\begin{aligned} G^{I,N\backslash I}(k_I, \hat{k}_1, \hat{k}_2;\lambda)=F^{I,N\backslash I}&(k_I, \hat{k}_1, \hat{k}_2)\\ &-\lambda\int_\Gamma F^{I;\{\alpha,\beta\}}(k_I, \hat{k}_1, k_\alpha)G^{\{\alpha,\beta\};N\backslash I}(-k_I, -k_\alpha, \hat{k}_2;\lambda)\\ &\cdot[H_0^{(2)}(k_\alpha)H_0^{(2)}(k_\alpha+p_I)]^{-1}dk_\alpha\,. \end{aligned} \tag{20}$$

Here $\Gamma=\mathbb{R}^3\times\mathscr{L}$ is a contour in the class defined in Section 3. $G^{I,N\backslash I}(k_I, \hat{k}_1, \hat{k}_2;\lambda)$ denotes the unknown kernel which is under study. $F^{I,N\backslash I}(k_I, \hat{k}_1, \hat{k}_2)$ is the general n-point function which is given. In agreement with the perturbative heuristic guide, $F^{I,N\backslash I}$ has to be the "one-p.i." part of the n-point function $H^{(n)}$ with respect to the considered channel $(I, N\backslash I)$.

The definitions and analyticity properties of the $F^{I,N\backslash I}$'s have been given and studied in [2]. It was proved there that these functions enjoy the primitive structure of general n-point functions (up to a finite number of real poles $\{k_I^2=\alpha_\nu, 0<\alpha_\nu<m^2\}$).

However, as in Section 3, in order to avoid divergences on the (euclidean) infinite parts of Γ, $F^{I,N\backslash I}$ is multiplied by the analytic cut-off factor:

$$\varrho(k_1,\dots,k_n)=\prod_{j=1}^{n}[(m^2-\mu^2)/(k_j^2-\mu^2)]^r \tag{20'}$$

which enjoys the same properties as the one already used in Section 3.

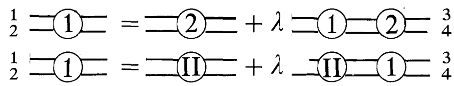
However we should take care that the perturbative relations (19a) and (19b) provide two different definitions of the function $G^{I,N\backslash I}$. In this section we shall have to prove that the corresponding integral equations in $D^{(n)}$ have the *same* solution. Actually the result we shall now prove is the following:

Starting from the integral relations (19a, b), it is possible to define *consistenly* the functions $G^{I,N\backslash I}$ for all channels $(I, N\backslash I)$. This will be done through a suitable recursion over $n_1=|I|$ and $n_2=|N\backslash I|$, which we shall describe now.

4.2. The Recursion over n_1 and n_2

i) $n_1=n_2=2$

We start from the relations:



(14)

(21)

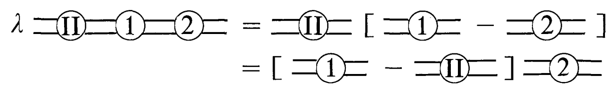
namely

$$F^{\{1,2\};\{3,4\}}=G^{\{1,2\};\{3,4\}}+\lambda F^{\{1,2\};\{\alpha,\beta\}}\odot G^{\{\underline{\alpha},\underline{\beta}\};\{3,4\}} \tag{15}$$

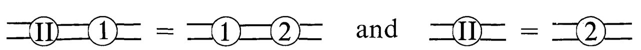
$$F^{\{1,2\};\{3,4\}}=\tilde{G}^{\{1,2\}\cdot\{3,4\}}+\lambda\tilde{G}^{\{1,2\}\cdot\{\alpha,\beta\}}\odot F^{\{\underline{\alpha},\underline{\beta}\}\cdot\{3,4\}} \tag{22}$$

where the symbol $\odot$ stands as a shortcut for the convolution integral.

The Fredholm equation (15) has been studied in Section 3 and its analyticity properties there investigated. Now we show that (15) and (22) have the same solution. Indeed it is sufficient to apply the associativity and the distributivity of convolution integrals and to write

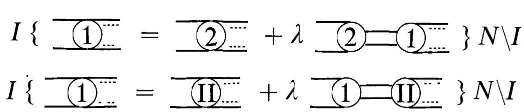


from which follows:



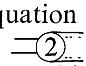
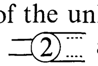
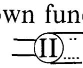
ii) $n_1=2, n_2>2$

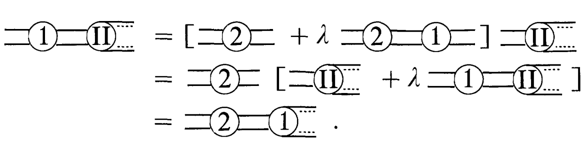
Let us consider the set of relations:



(23a)

(23b)

We notice that while (23b) is a Fredholm equation for  (23a) provides an explicit expression of the unknown function  in terms of known ones. Moreover both kernels  and  must be identical, since we can write:

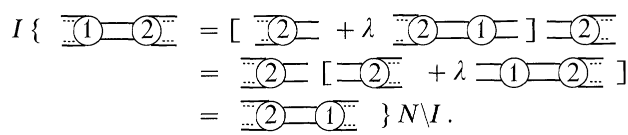


The analogue of (23a) is then chosen to provide the following *definition* of $G^{\{1,2\};N_2}$:

$$G^{\{1,2\};N_2}=F^{\{1,2\};N_2}-\lambda G^{\{1,2\};\{\alpha,\beta\}}\odot F^{\{\underline{\alpha},\underline{\beta}\};N_2} \tag{24}$$

iii) $n_1>2, n_2>2$

The relations to be used are then (19a) and (19b). We remark that, due to the previous steps of the recursion, they provide an explicit definition for $G^{I,N\backslash I}$. Notice that (19a) and (19b) are equivalent since:

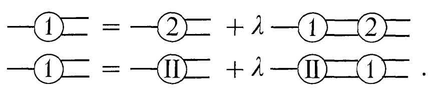


$G^{I,N\backslash I}$ is therefore given indifferently by anyone of the two following *definitions*:

$$\begin{aligned} G^{I,N\backslash I} &= F^{I,N\backslash I} - \lambda G^{I;\{\alpha,\beta\}} \odot F^{\{\underline{\alpha},\underline{\beta}\};N\backslash I} \\ &= F^{I,N\backslash I} - \lambda F^{I;\{\alpha,\beta\}} \odot G^{\{\underline{\alpha},\underline{\beta}\};N\backslash I} . \end{aligned} \tag{25}$$

iv) $n_1=1, n_2=2$

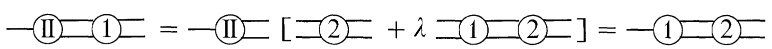
We consider the relations:



(26a)

(26b)

Both have the same solution since:

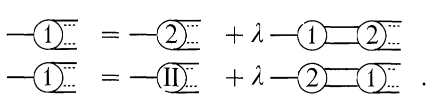


Then (26a) is chosen to give the following *definition* of $G^{\{1\};\{2,3\}}$:

$$G^{\{1\};\{2,3\}} = F^{\{1\};\{2,3\}} - \lambda F^{\{1\};\{\alpha,\beta\}} \odot G^{\{\underline{\alpha},\underline{\beta}\};\{2,3\}} . \tag{27}$$

v) $n_1=1, n_2>2$

We use the relations:

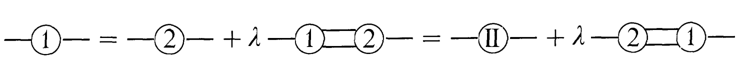


Both are equivalent, as easily checked. We choose indifferently anyone of the two as *definition* of $G^{\{1\};N_2}$:

$$\begin{aligned} G^{\{1\};N_2} &= F^{\{1\};N_2} - \lambda G^{\{1\};\{\alpha,\beta\}} \odot F^{\{\underline{\alpha},\underline{\beta}\};N_2} \\ G^{\{1\};N_2} &= F^{\{1\};N_2} - \lambda F^{\{1\};\{\alpha,\beta\}} \odot G^{\{\underline{\alpha},\underline{\beta}\};N_2} . \end{aligned} \tag{28}$$

vi) $n_1=n_2=1$

Then we have:



Both expressions are equivalent, namely:

$$\begin{aligned} G^{\{1\};\{2\}} &= F^{\{1\};\{2\}} - \lambda F^{\{1\};\{\alpha,\beta\}} \odot G^{\{\underline{\alpha},\underline{\beta}\};\{2\}} \\ &= F^{\{1\};\{2\}} - \lambda G^{\{1\};\{\alpha,\beta\}} \odot F^{\{\underline{\alpha},\underline{\beta}\};\{2\}} . \end{aligned} \tag{29}$$

To summarize, we can say that, once solved the general Bethe-Salpeter equation, it is no longer necessary to handle any Fredholm equation to get the complete set of functions $\{G^{I,N\setminus I}\}$. Indeed as shown above if the case $n_1=n_2=2$ has been solved, the rigorous counterparts of the perturbative graphical identities provide explicit definitions for the other functions.

4.3. Analyticity Properties

Now if we take into account the conservation of the primitive analytic structure by convolution [1] and the meromorphy properties of $G^{\{1,2\};\{3,4\}}$ such as given in Proposition 1, we have directly, in view of the above described recursive argument:

Proposition 6. *Each of the functions $G^{I,N\setminus I}(k;\lambda)$ introduced in Section 4.2 is meromorphic in the domain $\{k\in D^{(n)}\}\times\{\lambda\in\mathbb{C}\}$. In addition to the possible real poles $\{k_I^2=\alpha_\nu, 0<\alpha_\nu<m^2\}$ (see footnote 5), its only singularities in this domain are induced by the zeros of the two-point Fredholm determinant $A(k_I;\lambda)$. Besides, its real boundary values satisfy all the relevant linear relations of general n-point functions.*

We then turn to the smoothness properties which can be established for the $G^{I,N\setminus I}$'s on the basis of the "smooth spectral condition" [2]. First we note that the following result concerning the $F^{I,N\setminus I}$'s has been proved in [18]:

Proposition 7. *Let $\mathscr{S}_+$ and $\mathscr{S}_-$ denote two adjacent cells of N, separated by the partition $(I, N\setminus I)$ and $\mathscr{T}_{\mathscr{S}}$ denote the commun face of the two corresponding tubes $\mathscr{T}_{\mathscr{S}_+}$ and $\mathscr{T}_{\mathscr{S}_-}$ on the manifold $q_I=q_{N\setminus I}=0$. Then in the "flat" tube:*

$$\mathscr{F}_{\mathscr{S}}=\{(p_I,\hat{k})\in\mathbb{R}^4\times\mathbb{C}^{4(n-2)}:p_I\in\hat{\Sigma},\hat{k}\in\mathscr{T}_{\mathscr{S}}\}$$

*where $\hat{\Sigma}$ stands for the "non-*C.D.D.*" open set:*

$$\hat{\Sigma}=\{p\in\mathbb{R}^4:p\in V^+,p^2\geqq 4m^2,H_{\pm}^{(2)}(p)\neq 0\}$$

the boundary values $F_{\mathscr{S}_\pm}^{I,N\setminus I}(p_I,\hat{k})$ are analytic in $\hat{k}$ and (Hölder) continuous in p_I.

Now taking into account this result as well as Proposition 5 and the "conservation of smoothness" in p_I by convolution (such as given in Appendix A), we get:

Proposition 8. *In the "flat" tube:*

$$\hat{\mathscr{F}}_{\mathscr{S}}=\{(p_I,\hat{k})\in\mathbb{R}^4\times\mathbb{C}^{4(n-2)}:p_I\in\hat{\Sigma}_\lambda^{(2)},\hat{k}\in\mathscr{T}_{\mathscr{S}}\}$$

with

$$\hat{\Sigma}_\lambda^{(2)}=\{p\in\mathbb{R}^4:p\in V^+,4m^2\leqq p^2<9m^2,H_{\pm}^{(2)}(p)\neq 0,A_{\pm}(p;\lambda)\neq 0\}$$

the boundary values $G_{\mathscr{S}_\pm}^{I,N\setminus I}(p_I,\hat{k};\lambda)$ are analytic in $\hat{k}$ and continuous in p_I.

Remark. When $\hat{k}$ tends to the real inside $\mathscr{T}_{\mathscr{S}}$, the boundary value of $G_{\mathscr{S}_\pm}^{I,N\setminus I}(p_I,\hat{k};\lambda)$ is a general hyperfunction in the variables $\hat{p}$. Nothing more can be said in this general framework, but this is irrelevant for what is done in the following.

5. The Algebraic Algorithm of Irreducibility

This section is devoted to the proof that for $\lambda = 1/2$ each function $G^{I,N\setminus I}$ previously introduced is actually two-particle irreducible in the relevant channel $(I, N\setminus I)$.

In other words, we shall prove that the coincidence region of each function $G^{I,N\setminus I}(k_I, \hat{k}; 1/2)$ in the channel $(I, N\setminus I)$ has the form:

$$\mathscr{R}_I^{(2)} = \{p \in \mathbb{R}^{4(n-1)} : p_I^2 < 9m^2\} .$$

Or similarly, that each corresponding absorptive part $\Delta_{\mathscr{S}_1\mathscr{S}_2} G^{I,N\setminus I}(p_I, \hat{k}; 1/2)$, [with $\mathscr{S}_1$ (resp. $\mathscr{S}_2$) arbitrary in $S(I)$ (resp. $S(N\setminus I)$)], *vanishes* in the following "flat" tube, on the face $q_I = q_{N\setminus I} = 0$:

$$\mathscr{F}^{(2)}_{\mathscr{S}_1\mathscr{S}_2} = \{(p_I, \hat{k}) \in \mathbb{R}^4 \times \mathbb{C}^{4(n-2)} : p_I^2 < 9m^2, \hat{k} \in \mathscr{T}_{\mathscr{S}_1} \times \mathscr{T}_{\mathscr{S}_2}\} .$$

In view of the edge of the wedge theorem, both properties are indeed equivalent.

Since it can be seen from Proposition 8 that any absorptive part $\Delta_{\mathscr{S}_1\mathscr{S}_2} G^{I,N\setminus I}(p_I, \hat{k}; \lambda)$ is an analytic function of $\hat{k}$ inside $\mathscr{T}_{\mathscr{S}_1} \times \mathscr{T}_{\mathscr{S}_2}$, and a continuous function of p_I in $\hat{\Sigma}_\lambda^{(2)}$, all that follows is established at fixed p_I in the sense of continuous functions.

We first recall some basic results concerning the completeness of two-particle asymptotic states.

5.1. The Two-Particle Completeness Relations

The two-particle non-linear information of general quantum field theory is formulated in the following "completeness relations" [10, 11, 2] which express the completeness of incoming (resp. outgoing) two-particle asymptotic states.

In each two-particle region:

$$\Sigma_I^{(2)} = \{p \in \mathbb{R}^{4(n-1)} : p_I \in \Sigma^{(2)}\}$$

with

$$\Sigma^{(2)} = \{p \in \mathbb{R}^4 : p \in V^+, 4m^2 \leqq p^2 < 9m^2\} ,$$

the following relations are satisfied by any absorptive part $\Delta_{\mathscr{S}_1\mathscr{S}_2} H^{(n)}$ of the n-point functions $H^{(n)}$ in the (arbitrary) channel $(I, N\setminus I)$:

$$\Delta_{\mathscr{S}_1\mathscr{S}_2} H^{(n)}(p_I, \hat{p}) = (2\pi i/Z)^2 (2!)^{-1} [\hat{H}^{(n_1+2)}_{\alpha\uparrow\beta\uparrow\mathscr{S}_1} * \hat{H}^{(n_2+2)}_{\underline{\alpha}\uparrow\underline{\beta}\uparrow\mathscr{S}_2}](p_I, \hat{p}) \tag{30a}$$

$$= (2\pi i/Z)^2 (2!)^{-1} [\hat{H}^{(n_1+2)}_{\alpha\downarrow\beta\downarrow\mathscr{S}_1} * \hat{H}^{(n_2+2)}_{\underline{\alpha}\downarrow\underline{\beta}\downarrow\mathscr{S}_2}](p_I, \hat{p}) . \tag{30b}$$

Here both sides are distributions in $\mathbb{R}^{4(n-1)}$ and the notations are those of Section 2.2. In particular, $n_1 = |I|, n_2 = |N\setminus I|, \underline{\alpha}$ means that $p_{\underline{\alpha}} = -p_\alpha$, and $\hat{H}^{(n_j+2)}_{\alpha\uparrow\beta\uparrow\mathscr{S}_j}(p_\alpha, p_\beta, \hat{p}_j)$ denotes the restriction to the mass-shell $\{p_\alpha^2 = p_\beta^2 = m^2\}$ of the "amputated" distribution:

$$H^{\mathrm{amp}}_{\alpha\uparrow\beta\uparrow\mathscr{S}_j}(p_\alpha, p_\beta, \hat{p}_j) = (p_\alpha^2 - m^2)(p_\beta^2 - m^2) H^{(n_j+2)}_{\alpha\uparrow\beta\uparrow\mathscr{S}_j}(p_\alpha, p_\beta, \hat{p}_j) .$$

The symbol $*$ stands as a shortcut for the "mass-shell convolution" product:

$$[\hat{H}^{(n_1+2)}_{\alpha\uparrow\beta\uparrow\mathscr{S}_1} * \hat{H}^{(n_2+2)}_{\underline{\alpha}\uparrow\underline{\beta}\uparrow\mathscr{S}_2}](p_I, \hat{p}_1, \hat{p}_2)$$
$$= \int \hat{H}^{(n_1+2)}_{\alpha\uparrow\beta\uparrow\mathscr{S}_1}(p_\alpha, p_\beta, \hat{p}_1) \hat{H}^{(n_2+2)}_{\underline{\alpha}\uparrow\underline{\beta}\uparrow\mathscr{S}_2}(p_{\underline{\alpha}}, p_{\underline{\beta}}, \hat{p}_2) \delta_\alpha^- \delta_\beta^- \delta(p_\alpha + p_\beta + p_I) dp_\alpha dp_\beta . \tag{31}$$

Now (30) and (31) are in general meaningful in the sense of distributions in $\mathbb{R}^{4(n-1)}$. This is provided by the fact (shown in [10]) that, after testing in the external variables $\hat{p}_j$, each distribution $\hat{H}^{(n_j+2)}_{\alpha\uparrow\beta\uparrow\mathscr{S}_j}$ (resp. $\hat{H}^{(n_j+2)}_{\alpha\downarrow\beta\downarrow\mathscr{S}_j}$) can be identified with a square-integrable function on the mass shell $\sigma=\{p_\alpha\in H_m^-, p_\beta\in H_m^-\}$ with respect to the measure $(d\boldsymbol{p}_\alpha/2\omega_\alpha)(d\boldsymbol{p}_\beta/2\omega_\beta)$.

However if the technical property of "smooth spectral condition" is postulated [2], (30) and (31) also make sense as distributions in $\hat{p}=(\hat{p}_1,\hat{p}_2)$, having a *continuous* dependence on p_I. Indeed it can be proved ([2], Proposition 3, and the remarks made above in Section 2.2) that each $\hat{H}^{(n_j+2)}_{\alpha\uparrow\beta\uparrow\mathscr{S}_j}$ (resp. $\hat{H}^{(n_j+2)}_{\alpha\downarrow\beta\downarrow\mathscr{S}_j}$) is a continuous function on σ.

Now let us consider the "non-C.D.D." two-particle regions:

$$\hat{\Sigma}_I^{(2)}=\{p\in\mathbb{R}^{4(n-1)}: p_I\in\hat{\Sigma}^{(2)}\}$$

with

$$\hat{\Sigma}^{(2)}=\{p\in\mathbb{R}^4: p\in V^+, 4m^2\leq p^2<9m^2, H_{\pm}^{(2)}(p)\neq 0\}\,.$$

It is easily seen that $\hat{\Sigma}^{(2)}$ (resp. any $\hat{\Sigma}_I^{(2)}$) is a dense open subset of $\Sigma^{(2)}$ (resp. $\Sigma_I^{(2)}$) and we have the following

Theorem. [2] *The system of non-linear relations:*

$$\Delta_{\mathscr{S}_1\mathscr{S}_2}F^{I,N\setminus I}(p_I,\hat{p})=(2\pi i/Z)^2(2!)^{-1}[\hat{F}^{I;\{\alpha,\beta\}}_{\alpha\uparrow\beta\uparrow\mathscr{S}_1}*\hat{F}^{\{\underline{\alpha},\underline{\beta}\};N\setminus I}_{\underline{\alpha}\uparrow\underline{\beta}\uparrow\mathscr{S}_2}](p_I,\hat{p}) \tag{32a}$$

$$=(2\pi i/Z)^2(2!)^{-1}[\hat{F}^{I;\{\alpha,\beta\}}_{\alpha\downarrow\beta\downarrow\mathscr{S}_1}*\hat{F}^{\{\underline{\alpha},\underline{\beta}\};N\setminus I}_{\underline{\alpha}\downarrow\underline{\beta}\downarrow\mathscr{S}_2}](p_I,\hat{p}) \tag{32b}$$

satisfied on each corresponding $\hat{\Sigma}_I^{(2)}$ by the various one-p.i. functions $F^{I,N\setminus I}$, is equivalent with the original system of completeness relations (30a, b) *expressed in the same regions.*

In other words, the two-particle non-linear information can be expressed in terms of the one-p.i. functions, except on the "pathological set" corresponding to possible C.D.D. singularities.

Relations (32) are valid as distributions in $\hat{p}$ and continuous functions in p_I. Moreover they can be easily extended, as analytic functions of the complexified variables $\hat{k}$, in certain "flat" tubes as we shall describe now.

5.2. Extension to Flat Tubes

The left-hand side of (32a) is the boundary value of the discontinuity function $\Delta^I F^{I,N\setminus I}$ in the "flat" tube:

$$\{(p_I,\hat{k}_1,\hat{k}_2)\in\mathbb{R}^4\times\mathbb{C}^{4(n-2)}: p_I\in\hat{\Sigma}^{(2)}, \hat{k}_1\in\mathscr{T}_{\mathscr{S}_1}, \hat{k}_2\in\mathscr{T}_{\mathscr{S}_2}\}\,.$$

From the linear program (see Section 2.1) it is known that $\Delta^I F^{I,N\setminus I}$ is analytic in $\hat{k}=(\hat{k}_1,\hat{k}_2)$ inside a domain D_I which is the union of all flat tubes $\{\mathscr{T}_{\mathscr{S}_1}\times\mathscr{T}_{\mathscr{S}_2}, \mathscr{S}_1\in S(I), \mathscr{S}_2\in S(N\setminus I)\}$ on the manifold $q_I=q_{N\setminus I}=0$, with appropriate complex neighbourhoods of real regions connecting these flat tubes together.

Let us now consider the following mass-shell convolution product:

$$\begin{aligned}H^\Gamma(p_I,\hat{k}_1,\hat{k}_2)=(2\pi i/Z)^2(2!)^{-1}\int \hat{F}^{I;\{\alpha,\beta\}}_{\alpha\uparrow\beta\uparrow\mathscr{S}_1}(\boldsymbol{p}_\alpha,\boldsymbol{p}_\beta,\hat{k}_1)\hat{F}^{\{\underline{\alpha},\underline{\beta}\};N\setminus I}_{\underline{\alpha}\uparrow\underline{\beta}\uparrow\mathscr{S}_2}(\boldsymbol{p}_{\underline{\alpha}},\boldsymbol{p}_{\underline{\beta}},\hat{k}_2)\\ \cdot\delta_\alpha^-\delta_\beta^-\delta(p_\alpha+p_\beta+p_I)\,dp_\alpha dp_\beta\end{aligned} \tag{33}$$

where the argument $(p_I,\hat{k})$ is chosen to lie in the product $\hat{\Sigma}^{(2)}\times D_I$.

The integration variable lies on the compact mass-shell σ and, as recalled in Section 2.2, it is a result of the linear program that each factor $\hat{F}^{I;\{\alpha,\beta\}}_{\alpha\uparrow\beta\uparrow\mathscr{S}_1}$ (resp. $\hat{F}^{\{\underline{\alpha},\underline{\beta}\};N\setminus I}_{\underline{\alpha}\uparrow\underline{\beta}\uparrow\mathscr{S}_2}$) has an analytic dependence on $\hat{k}_1$ (resp. $\hat{k}_2$) in D_I.

This is sufficient to conclude that the right-hand side of (32a) is the boundary value in $\mathscr{T}_{\mathscr{S}_1}\times\mathscr{T}_{\mathscr{S}_2}$ of a function $H^\Gamma(p_I,\hat{k})$ analytic in D_I.

As a consequence of the edge of the wedge theorem, (32a) then implies the coincidence of the corresponding analytic functions:

$$\Delta^I F^{I,N\setminus I}(p_I,\hat{k})=H^\Gamma(p_I,\hat{k})$$

throughout the product $\{p_I\in\hat{\Sigma}^{(2)}\}\times\{\hat{k}\in D_I\}$.

A similar extension could be obtained for (30a, b) and (32b). Moreover the two non-linear systems thus obtained, satisfied on the relevant products $\hat{\Sigma}^{(2)}\times D_I$ are still *equivalent*.

Finally we shall also need the following result:

Proposition 9. *In the flat tube:*

$$\hat{\mathscr{F}}_{\mathscr{S}_1\mathscr{S}_2}=\{(p_I,\hat{k})\in\mathbb{R}^4\times\mathbb{C}^{4(n-2)}:p_I\in\hat{\Sigma}^{(2)}_\lambda,\ \hat{k}\in\mathscr{T}_{\mathscr{S}_1}\times\mathscr{T}_{\mathscr{S}_2}\}$$

the absorptive part $\Delta_{\mathscr{S}_1\mathscr{S}_2}G^{I,N\setminus I}(p_I,\hat{k};\lambda)$ is given by the following integral relation (in short):

$$\begin{aligned}\Delta_{\mathscr{S}_1\mathscr{S}_2}G^{I,N\setminus I}=\Delta_{\mathscr{S}_1\mathscr{S}_2}F^{I,N\setminus I}&-\lambda G^{I;\{\alpha,\beta\}}\odot\Delta F^{\{\underline{\alpha},\underline{\beta}\};N\setminus I}\\&-\lambda\Delta G^{I;\{\alpha,\beta\}}\odot F^{\{\underline{\alpha},\underline{\beta}\};N\setminus I}\\&-\lambda(2\pi i/Z)^2\hat{G}^{I;\{\alpha,\beta\}}_{\alpha\downarrow\beta\downarrow\mathscr{S}_1}*\hat{F}^{\{\underline{\alpha},\underline{\beta}\};N\setminus I}_{\underline{\alpha}\downarrow\underline{\beta}\downarrow\mathscr{S}_2}\end{aligned}\tag{34}$$

where $\hat{G}^{I;\{\alpha,\beta\}}_{\alpha\downarrow\beta\downarrow\mathscr{S}_1}$ denotes the (continuous) restriction to the mass-shell σ of the "amputated" function $(p_\alpha^2-m^2)(p_\beta^2-m^2)G^{I;\{\alpha,\beta\}}_{\alpha\downarrow\beta\downarrow\mathscr{S}_1}$.

Proof. Starting from the definition (25) of $G^{I,N\setminus I}$ given in Section 4:

$$G^{I,N\setminus I}=F^{I,N\setminus I}-\lambda G^{I;\{\alpha,\beta\}}\odot F^{\{\underline{\alpha},\underline{\beta}\};N\setminus I}$$

we apply the basic discontinuity formula of Theorem 1 to the convolution product of the right-hand side. Propositions 7 and 8 establish the necessary smoothness properties in p_I. (34) is then a (shortened) form of (2) with an appropriate specialization of the notations.

5.3. Proof of Irreducibility

We now intend to prove that each absorptive part $\Delta_{\mathscr{S}_1\mathscr{S}_2}G^{I,N\setminus I}(p_I,\hat{k};1/2)$ *vanishes* in the flat tube:

$$\mathscr{F}_{\mathscr{S}_1\mathscr{S}_2}=\{(p_I,\hat{k})\in\mathbb{R}^4\times\mathbb{C}^{4(n-2)}:p_I^2<9m^2,\ \hat{k}\in\mathscr{T}_{\mathscr{S}_1}\times\mathscr{T}_{\mathscr{S}_2}\}\,.$$

Or equivalently, in view of the one-particle irreducibility of the functions $F^{I,N\setminus I}$, in the flat tube:

$$\mathscr{F}^{(2)}_{\mathscr{S}_1\mathscr{S}_2}=\{(p_I,\hat{k})\in\mathbb{R}^4\times\mathbb{C}^{4(n-2)}:p_I\in\Sigma^{(2)},\ \hat{k}\in\mathscr{T}_{\mathscr{S}_1}\times\mathscr{T}_{\mathscr{S}_2}\}\,.$$

Discarding the exceptional manifolds corresponding to C.D.D. (or generalized C.D.D.) type, we can prove:

Theorem 2. *For $\lambda = 1/2$, the two-particle irreducibility property of each function $G^{I,N\setminus I}$:*

$$\Delta_{\mathscr{S}_1\mathscr{S}_2} G^{I,N\setminus I}(p_I, \hat{k}; 1/2) = 0$$

($\mathscr{S}_1, \mathscr{S}_2$ arbitrary) is satisfied in the corresponding flat tube:

$$\hat{\mathscr{F}}^{(2)}_{\mathscr{S}_1\mathscr{S}_2} = \{(p_I, \hat{k}) \in \mathbb{R}^4 \times \mathbb{C}^{4(n-2)} : p_I \in \hat{\Sigma}^{(2)}_{1/2}, \hat{k} \in \mathscr{T}_{\mathscr{S}_1} \times \mathscr{T}_{\mathscr{S}_2}\}.$$

Moreover the complete set of these relations (for any channel $(I, N\setminus I)$ and any couple of cells $(\mathscr{S}_1, \mathscr{S}_2)$) is equivalent with the non-linear system (32)*:*

$$\Delta_{\mathscr{S}_1\mathscr{S}_2} F^{I,N\setminus I}(p_I, \hat{k}) = (2\pi i/Z)^2 (2!)^{-1} [\hat{F}^{I;\{\alpha,\beta\}}_{\alpha\downarrow\beta\downarrow\mathscr{S}_1} * \hat{F}^{\{\underline{\alpha},\underline{\beta}\};N\setminus I}_{\underline{\alpha}\downarrow\underline{\beta}\downarrow\mathscr{S}_2}](p_I, \hat{k})$$

which is itself equivalent with the original two-particle completeness relations (30)*:*

$$\Delta_{\mathscr{S}_1\mathscr{S}_2} H^{(n)}(p_I, \hat{k}) = (2\pi i/Z)^2 (2!)^{-1} [\hat{H}^{(n_1+2)}_{\alpha\downarrow\beta\downarrow\mathscr{S}_1} * \hat{H}^{(n_2+2)}_{\underline{\alpha}\downarrow\underline{\beta}\downarrow\mathscr{S}_2}](p_I, \hat{k})$$

expressed in the same flat tubes $\hat{\mathscr{F}}^{(2)}_{\mathscr{S}_1\mathscr{S}_2}$.

Remarks. i) Here we have only formulated the equivalence of two-particle irreducibility with completeness relations involving two-particle outgoing states, i.e. with "negative" arrows $\downarrow$. The proof would go similarly for incoming states (see the second remark in Section 2.4.4 above).

ii) Theorem 2 establishes that the coincidence region of each function $G^{I,N\setminus I}(k_I, \hat{k}; 1/2)$ in the relevant channel $(I, N\setminus I)$ has the form:

$$\hat{\mathscr{R}}^{(2)}_I = \{p \in \mathbb{R}^{4(n-1)} : p_I^2 < 9m^2, H^{(2)}_{\pm}(p_I) \neq 0, A_{\pm}(p_I; 1/2) \neq 0\}.$$

$\hat{\mathscr{R}}^{(2)}_I$ is a dense open subset of $\mathscr{R}^{(2)}_I$. The restrictions correspond to the possible C.D.D. (or generalized C.D.D.) singularities. They can be deleted by using methods of the *on-shell* non linear program of general quantum field theory [16] (study of the unitarity relations: see also [17]).

iii) The possibility for the functions $G^{I,N\setminus I}$ to have a fixed pole at $\lambda = 1/2$, induced by a fixed zero of $A_{\pm}(p_I; 1/2)$ can also be discarded [19].

Proof. First we define for all channels $(I, N\setminus I)$:

$$\Theta_{\mathscr{S}_1\mathscr{S}_2} F^{I,N\setminus I} = \Delta_{\mathscr{S}_1\mathscr{S}_2} F^{I,N\setminus I} - (2\pi i/Z)^2 (2!)^{-1} \hat{F}^{I;\{\alpha,\beta\}}_{\alpha\downarrow\beta\downarrow\mathscr{S}_1} * \hat{F}^{\{\underline{\alpha},\underline{\beta}\};N\setminus I}_{\underline{\alpha}\downarrow\underline{\beta}\downarrow\mathscr{S}_2}$$

which is the "two-particle completeness" kernel of the function $F^{I,N\setminus I}$. Then the proof goes in three steps and starts from (34).

i) First, consider the convolution product $G^{I,\{\alpha,\beta\}} \odot \Delta F^{\{\underline{\alpha},\underline{\beta}\};N\setminus I}$ in the right-hand side. Considerations of local analyticity in the neighbourhood of the mass-shell σ have shown in Section 2.4.2 that the contour $\mathscr{C}_{\mathscr{S}_1}$ which occurs in this convolution belongs to the domain of the analytic continuation (denoted $\Delta_{\underline{\alpha}\updownarrow\underline{\beta},\mathscr{S}_2} F^{\{\underline{\alpha},\underline{\beta}\};N\setminus I}$) of the two absorptive parts $\Delta_{\underline{\alpha}\uparrow\underline{\beta},\mathscr{S}_2} F^{\{\underline{\alpha},\underline{\beta}\};N\setminus I}$ and $\Delta_{\underline{\alpha}\downarrow\underline{\beta},\mathscr{S}_2} F^{\{\underline{\alpha},\underline{\beta}\};N\setminus I}$.

We note $\Theta_{\underline{\alpha}\updownarrow\underline{\beta},\mathscr{S}_2}F^{\{\underline{\alpha},\underline{\beta}\};N\backslash I}$ the analytic continuation of the two corresponding completeness kernels. Then we can write (34) under the form:

$$\begin{aligned}\Delta_{\mathscr{S}_1\mathscr{S}_2}G^{I,N\backslash I}+\lambda\Delta G^{I;\{\alpha,\beta\}}\odot F^{\{\underline{\alpha},\underline{\beta}\};N\backslash I}&\\ =\Delta_{\mathscr{S}_1\mathscr{S}_2}F^{I,N\backslash I}-\lambda G^{I;\{\alpha,\beta\}}\odot\Theta_{\underline{\alpha}\updownarrow\underline{\beta},\mathscr{S}_2}F^{\{\underline{\alpha},\underline{\beta}\};N\backslash I}&\\ -(2\pi i/Z)^2\lambda[\hat{G}^{I;\{\lambda,\mu\}}_{\lambda\downarrow\mu\downarrow\mathscr{S}_1}+(2\,!)^{-1}G^{I;\{\alpha,\beta\}}\odot\hat{F}^{\{\underline{\alpha},\underline{\beta}\};\{\lambda,\mu\}}_{\lambda\downarrow\underline{\mu}\downarrow\underline{\alpha}\updownarrow\underline{\beta}}]*\hat{F}^{\{\lambda,\mu\};N\backslash I}_{\lambda\downarrow\mu\downarrow\mathscr{S}_1}\,.&\end{aligned}\tag{35}$$

But the last bracket can be rewritten by using the following expression:

$$\lambda G^{I;\{\alpha,\beta\}}_{\alpha\updownarrow\beta\downarrow\mathscr{S}_1}\odot\hat{F}^{\{\underline{\alpha},\underline{\beta}\};\{\lambda,\mu\}}_{\lambda\downarrow\underline{\mu}\downarrow\underline{\alpha}\updownarrow\underline{\beta}}=\hat{F}^{I;\{\lambda,\mu\}}_{\lambda\downarrow\mu\downarrow\mathscr{S}_1}-\hat{G}^{I;\{\lambda,\mu\}}_{\lambda\downarrow\mu\downarrow\mathscr{S}_1}\tag{36}$$

which is one of the definitions introduced in Section 4.2, after amputation in $\{\lambda,\mu\}$ and restriction to the mass-shell $\sigma=\{p_\lambda\in H_m^-, p_\mu\in H_m^-\}$. Here we have used the fact that the contour $\mathscr{C}_{\mathscr{S}_1}$ occuring in the left-hand side convolution is not pinched when the external variables (k_λ, k_μ) are restricted to σ.

Taking (36) into account, we get:

$$\begin{aligned}\lambda\,[\hat{G}^{I;\{\lambda,\mu\}}_{\lambda\downarrow\mu\downarrow\mathscr{S}_1}+(2\,!)^{-1}G^{I;\{\alpha,\beta\}}\odot\hat{F}^{\{\underline{\alpha},\underline{\beta}\};\{\lambda,\mu\}}_{\lambda\downarrow\underline{\mu}\downarrow\underline{\alpha}\updownarrow\underline{\beta}}]&\\ =(2\,!)^{-1}\hat{F}^{I;\{\lambda,\mu\}}_{\lambda\downarrow\mu\downarrow\mathscr{S}_1}+(\lambda-(2\,!)^{-1})\,\hat{G}^{I;\{\lambda,\mu\}}_{\lambda\downarrow\mu\downarrow\mathscr{S}_1}&\end{aligned}$$

and this allows to write (35) under the form:

$$\begin{aligned}\Delta_{\mathscr{S}_1\mathscr{S}_2}G^{I,N\backslash I}+\lambda\Delta G^{I;\{\alpha,\beta\}}\odot F^{\{\underline{\alpha},\underline{\beta}\};N\backslash I}&\\ =\Theta_{\mathscr{S}_1\mathscr{S}_2}F^{I,N\backslash I}-\lambda G^{I;\{\alpha,\beta\}}\odot\Theta_{\underline{\alpha}\updownarrow\underline{\beta},\mathscr{S}_2}F^{\{\underline{\alpha},\underline{\beta}\};N\backslash I}&\\ -(\lambda-(2\,!)^{-1})(2\pi i/Z)^2\,\hat{G}^{I;\{\lambda,\mu\}}_{\lambda\downarrow\mu\downarrow\mathscr{S}_1}*\hat{F}^{\{\underline{\lambda},\underline{\mu}\};N\backslash I}_{\underline{\lambda}\downarrow\underline{\mu}\downarrow\mathscr{S}_2}\,.&\end{aligned}\tag{37}$$

Finally for $\lambda=1/2$, we get the following algorithm which we shall apply now:

$$\begin{aligned}\Delta_{\mathscr{S}_1\mathscr{S}_2}G^{I,N\backslash I}+\tfrac{1}{2}\,\Delta G^{I;\{\alpha,\beta\}}\odot F^{\{\underline{\alpha},\underline{\beta}\};N\backslash I}&\\ =\Theta_{\mathscr{S}_1\mathscr{S}_2}F^{I,N\backslash I}-\tfrac{1}{2}\,G^{I;\{\alpha,\beta\}}\odot\Theta_{\underline{\alpha}\updownarrow\underline{\beta},\mathscr{S}_2}F^{\{\underline{\alpha},\underline{\beta}\};N\backslash I}\,.&\end{aligned}\tag{38}$$

ii) Applying relations (32), namely in any $\hat{\mathscr{F}}^{(2)}_{\mathscr{S}_1\mathscr{S}_2}$:

$$\Theta_{\mathscr{S}_1\mathscr{S}_2}F^{I,N\backslash I}=0$$

and inserting them in (38), we obtain:

$$\Delta_{\mathscr{S}_1\mathscr{S}_2}G^{I,N\backslash I}+\tfrac{1}{2}\int\limits_{\mathscr{C}_{\mathscr{S}_2}}\Delta G^{I;\{\alpha,\beta\}}F^{\{\underline{\alpha},\underline{\beta}\};N\backslash I}[H_0^{(2)}(k_\alpha)H_0^{(2)}(k_\beta)]^{-1}dk_\alpha=0\tag{39}$$

where a more detailed notation has been used.

Like in Section 4.2, the proof then goes by recursion over $n_1=|I|$ and $n_2=|N\backslash I|$. We start from the case $n_1=n_2=2$, where (39) is an homogeneous Fredholm equation with contour $\mathscr{C}_{\mathscr{S}_2}$. If $\mathbb{1}$ is the identity for $\odot$, we get:

$$\Delta G^{\{1,2\};\{\alpha,\beta\}}\odot[\mathbb{1}+\tfrac{1}{2}F^{\{\underline{\alpha},\underline{\beta}\};\{3,4\}}]=0\,.$$

But here the right kernel admits a right inverse, since:

$$[\mathbb{1}+\tfrac{1}{2}F^{\{1,2\};\{\lambda,\mu\}}]\odot[\mathbb{1}-\tfrac{1}{2}G^{\{\underline{\lambda},\underline{\mu}\};\{3,4\}}]=\mathbb{1}\tag{40}$$

which is nothing but the definition (15) of $G^{\{1,2\};\{3,4\}}$. Then we get:

$$\Delta G^{\{1,2\};\{3,4\}} = 0 \tag{41}$$

in the union of the relevant flat tubes $\hat{\mathscr{F}}^{(2)}_{1\updownarrow 2,3\updownarrow 4}$. By recursion over n_1 and n_2, inserting (41) in (39), we then achieve the proof of the first part of Theorem 2.

iii) The proof of the converse result goes in the same way. We start from the two-particle irreducibility relations in any $\hat{\mathscr{F}}^{(2)}_{\mathscr{S}_1\mathscr{S}_2}$:

$$\Delta_{\mathscr{S}_1\mathscr{S}_2} G^{I,N\setminus I} = 0$$

which we insert in (38). We get:

$$\Theta_{\mathscr{S}_1\mathscr{S}_2} F^{I,N\setminus I} - \tfrac{1}{2} G^{I;\{\alpha,\beta\}} \odot \Theta_{\underline{\alpha}\updownarrow\underline{\beta},\mathscr{S}_2} F^{\{\underline{\alpha},\underline{\beta}\};N\setminus I} = 0 \,. \tag{42}$$

For $n_1 = n_2 = 2$, this is an homogeneous Fredholm equation, with contour $\mathscr{C}_{\mathscr{S}_1}$:

$$[\mathbb{1} - \tfrac{1}{2} G^{\{1,2\};\{\alpha,\beta\}}] \odot \Theta_{\underline{\alpha}\updownarrow\underline{\beta},3\updownarrow 4} F^{\{\underline{\alpha},\underline{\beta}\};\{3,4\}} = 0$$

and the left kernel admits a left inverse in view of (40). Then we obtain:

$$\Theta_{1\updownarrow 2,3\updownarrow 4} F^{\{1,2\};\{3,4\}} = 0$$

in the union of flat tubes $\hat{\mathscr{F}}^{(2)}_{1\updownarrow 2,3\updownarrow 4}$. We achieve the converse proof by recursion over n_1 and n_2, after inserting this result in (42).

This ends the proof of Theorem 2.

6. Conclusion and Outlook

In this paper we have shown that the two-particle non-linear information of general quantum field theory (originally known through the two-particle completeness relations) can be alternatively and (up to the technical problem of C.D.D. singularities) *equivalently* formulated in terms of the two-particle irreducibility in a single channel of a given set of n-point functions.

In other words we are now provided with an *analytic formulation* of the two-particle structure of the n-point functions, which is more convenient as far as one is concerned with analytic extension properties.

Here we emphasize that this equivalence has been obtained on the only basis of two-particle irreducibility *in a single channel.* A very natural complement to our study should be the introduction of functions *simultaneously* two-p.i. in several channels. If this is not necessary as far as the above statement of equivalence is concerned, it plays a basic role when trying to carry through further steps in the non-linear program ([20, 21]).

Actually, at the present stage, one can expect progress along two directions. On the one hand, from the better analyticity properties of the two-p.i. functions, together with the algebra of the various convolution relations linking them together, it may be expected to improve the local analyticity properties of the n-point functions, and possibly isolate pieces of some Landau surfaces.

On the other hand, the mechanism which has been illustrated here for $p=2$ seems able to be reproduced at higher orders. In particular a better knowledge of the analytic structure of absorptive parts for convolution products with two vertices and $p>2$ internal lines (i.e. a generalization of Theorem 1) should certainly allow one to construct a simple algorithm [such as (38)] linking the non-linear *algebra* of p-particle completeness relations with the *analytic* formulation of p-particle irreducibility properties.

A similar question would be the derivation of $(p+k)$-particle completeness relations for the p-p.i. irreducible functions, when $p \geqq 2$, such as it was done in [2] (Theorem 3) for the one-p.i. functions.

These problems are connected with residue calculus in several complex variables [13, 14] and "pinching"-type techniques similar to those used above in Section 2. They are at present under study.

Appendix A: Conservation of Smoothness by Convolution

In this appendix, being given a general n-point function $F^{(n)}$ and a given channel $(I, N\backslash I)$, we shall say that $F^{(n)}$ is "smooth in this channel" if the following property is satisfied by its boundary values:

Let $(\mathscr{S}_+, \mathscr{S}_-)$ denote any couple of adjacent cells separated by $(I, N\backslash I)$ and $\mathscr{T}_{\mathscr{S}}$ be the common face on the manifold $q_I = q_{N\backslash I} = 0$ of the corresponding tubes $\mathscr{T}_{\mathscr{S}_+}$ and $\mathscr{T}_{\mathscr{S}_-}$. Then in the following "flat" tube:

$$\mathscr{F}_{\mathscr{S}}^{(2)} = \{(p_I, \hat{k}) \in \mathbb{R}^4 \times \mathbb{C}^{4(n-2)} : p_I \in \Sigma^{(2)}, \hat{k} \in \mathscr{T}_{\mathscr{S}}\}$$

the boundary values $F_{\mathscr{S}_\pm}^{(n)}(p_I, \hat{k})$ are analytic in $\hat{k}$ and continuous in p_I.

Here we shall prove that this property is preserved by convolution, in the following sense: starting from a vertex function F^1 (resp. F^2) smooth in the channel $[I; \{\alpha, \beta\}]$ (resp. $[\{\underline{\alpha}, \underline{\beta}\}; N\backslash I]$), the convolution product obtained is smooth in the channel $(I, N\backslash I)$.

The proof goes as follows. Consider two adjacent cells $\mathscr{S}_\pm$ separated by the convolution channel $(I, N\backslash I)$ and the corresponding boundary values $H_{\mathscr{S}_\pm}^G(p_I, \hat{k})$ of the convolution product. By using methods very similar to those of Section 2.4, it is seen that the following representation holds for these boundary values:

$$H_{\mathscr{S}_\pm}^G(p_I, \hat{k}_1, \hat{k}_2) = \int_{\mathscr{C}_{\mathscr{S}_\pm}} F^1(p_I, \hat{k}_1, k_\alpha) F^2(-p_I, -k_\alpha, \hat{k}_2) \, [H_0^{(2)}(k_\alpha) H_0^{(2)}(p_I + k_\alpha)]^{-1} dk_\alpha$$

where the contour $\mathscr{C}_{\mathscr{S}_\pm}$ can be represented by a "handle-shaped" domain of the type described in Section 2.4.4, namely the union of a fixed part $\mathbb{R}^3 \times \underline{\mathscr{L}}$ (with euclidean infinite parts) with a handle $\partial \gamma_1^\pm$.

Then the continuity in p_I of the integral on the fixed part $\mathbb{R}^3 \times \underline{\mathscr{L}}$ is easily checked, provided that one assumes that the integrand is (uniformly) bounded in p_I at infinity by an integrable function of k_α. This assumption is satisfied in Sections 3 to 5.

As for the integral on each handle $\partial\gamma_1^{\pm}$, using like in Section 2.4.4 the residue theorem in its general form [13], it can be rewritten as the integral on a compact set of a continuous function in p_I, which achieves the proof.

Appendix B: Conservation of Smoothness by Summation of Fredholm Series

This appendix is devoted to the proof of Proposition 4.

From the conservation of smoothness in P by convolution, it is first seen that each boundary value:

$$B_n^{\pm}(P,Z,Z') = \lim_{\substack{\varrho\to 0\\ \varrho\in V^+}} B_n(P\pm i\varrho, Z, Z')$$

$$A_n^{\pm}(P) = \lim_{\substack{\varrho\to 0\\ \varrho\in V^+}} A_n(P\pm i\varrho)$$

is analytic in (Z, Z') and continuous in P.

Moreover considerations of local analyticity very similar to those given in Section 2.4 display for these boundary values a representation in terms of Fredholm determinants $[F_{\pm}(P, Z_i, Z_j)]$ integrated on a contour $(\mathscr{C}_{\pm})^n$, with $\mathscr{C}_{\pm}$ of the "handle-type" described in Appendix A.

Taking into account the (uniform) bounds of each $F_{\pm}(P, Z_i, Z_j)$ and applying the Hadamard's trick [15], the (absolute) convergence of each series:

$$B_{\pm}(P,Z,Z';\lambda) = \sum_{n=0}^{\infty} \lambda^n (n!)^{-1} B_n^{\pm}(P,Z,Z')$$

$$A_{\pm}(P;\lambda) = \sum_{n=0}^{\infty} \lambda^n (n!)^{-1} A_n^{\pm}(P)$$

is then established in the whole complex λ-plane.

Considered as series of functions continuous in P, they are therefore uniformly convergent, which establishes the continuity in P of their sums. q.e.d.

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Note Added in Proof

In Section 4 an extra technical postulate has been implicitly assumed, which asserts the uniformity with respect to n of the bounds at infinity of all the n-point functions $\{F^{I,N\setminus I}\}$. This has been expressed by using analytic cut-offs (20′) with the same exponent r. If such uniform polynomial bounds do not hold, one may use more general analytic cut-offs

$$\varrho(k_1,\ldots,k_n)=\prod_{i=1}^{n}\lambda(k_i^2)$$

where $\lambda(k^2)$ is a two-point function with exponential decrease at infinity and $\lambda(m^2)=1$. Then the arguments used in Sections 4–5 remain valid, provided the same function λ is used to regularize all the $\{F^{I,N\setminus I}\}$.