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On the Type of Local Algebras in Quantum Field Theory

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Abstract. We give a simple sufficient condition for a von Neumann algebra to be Type III and apply it to some classes of algebras in QFT. For dilatation invariant local systems in particular we find that all sufficiently regular local algebras are Type III.

I. Introduction

An old problem of algebraic quantum field theory is the question of how far the axioms of Haag Araki determine the algebraic structure of local algebras. It is known by now that local algebras are of infinite type [1, 2], that certain types of nonlocal ones are Type III [3] resp. Type III_1 factors [4], and that in theories which are invariant with respect to dilatations of space time local algebras cannot be Type I in general [5].

The aim of this paper is to prove the Type III property for larger classes of algebras. In Section 2 we give a simple sufficient condition for a von Neumann algebra to be Type III which in Section 3 will be applied to various situations in QFT. For dilatation invariant theories in particular we shall get Type III for all local algebras belonging to not too irregularly shaped regions. However, since our argument involves short and long distance behaviour we cannot conclude that theories with well defined Gell Mann Low limit will have the same property without further assumptions as was argued in [5] for the case involving only the short distance behaviour.

Our arguments, however, lend support to the ancient conjecture that Type III should be a general feature of local algebras in QFT, a conjecture which up to now was apparently only based on the explicit computation of the type for the free field algebras [6].

II. The Type Theorem

Theorem. *Let $\mathscr{H}$ be a separable Hilbert space and $\mathscr{M}$ a von Neumann algebra in $\mathscr{L}(\mathscr{H})$ with a separating vector Θ; let $\mathscr{M}_1 \subset \mathscr{M}$ be a subalgebra of infinite type and*

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$\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathrm{Aut}(\mathscr{L}(\mathscr{H}))$ *a sequence of transformations such that*

a) $\alpha_n(\mathscr{M}_1) \subset \mathscr{M}_1$, $\forall n \in \mathbb{N}$.

b) $w\text{-}\lim_{n\to\infty} \alpha_n(M_1) = \omega(M_1)I$, $\forall M_1 \in \mathscr{M}_1$, *with* $\omega \in \mathscr{M}_{1,*}$, $\omega \neq 0$.

c) $s\text{-}\lim_{n\to\infty} [\alpha_n(M_1), M] = 0$, $\forall M_1 \in \mathscr{M}_1$, $\forall M \in \mathscr{M}$.

Then $\mathscr{M}$ *is Type* III.

Proof. Let $P \in \mathscr{M}$ be a nonzero finite projection such that $\mathscr{M}_P := P\mathscr{M}P$ is finite and thus has a normal faithful trace τ. Since $P\Theta$ is separating for $\mathscr{M}_P$ in $P\mathscr{H}$ τ can be implemented by a unit vector $\Psi \in P\mathscr{H}$. By b) and c) we then get for $A, B \in \mathscr{M}_1$:

$$\lim(\Psi, \alpha_n(A)P\alpha_n(B)\Psi) = \lim(\Psi, \alpha_n(AB)\Psi) = \omega(AB).$$

Ψ being a trace vector this implies $\omega(AB) = \omega(BA)$, $\forall A, B \in \mathscr{M}$, such that ω is a finite nonzero trace on $\mathscr{M}_1$, a contradiction. Thus $\mathscr{M}$ has no finite nonzero projection and thus is Type III.

III. Applications

Since algebras in QFT use to be infinite the problem in applying the previous theorem consists in satisfying b) and c). Candidates for the required transformations are translations, boosts, dilatations (if they exist in the given theory), or maybe some shrewd combinations of these. From the following simple examples the reader can easily generalize to more complicated situations. Except for example A) space time may have arbitrary dimension $n \geqq 2$. We use the standard assumptions of algebraic QFT without special reference.

A) Let $n \geqq 3$, let $\mathcal{O}_1$ be a double cone and $\mathcal{O} := \bigcup_{\lambda \geqq 0} (\mathcal{O}_1 + \lambda a)$; a a spacelike vector. Then $\mathscr{R}(\mathcal{O}) := \left(\bigcup_{\lambda \geqq 0} \mathscr{R}(\mathcal{O} + \lambda a) \right)''$ is Type III.

Proof. Take $\mathscr{M} = \mathscr{M}_1 = \mathscr{R}(\mathcal{O})$; α_k the translation by $k \cdot a$, $k \in \mathbb{N}$. Let Ω be the vacuum vector and ω_0 the corresponding state on $\mathscr{L}(\mathscr{H})$. Then a) is trivially satisfied and b) results as a standard application of clustering; since $n \geqq 3$ we even have:

$$w\text{-}\lim_{n\to\infty}(\alpha_n^{-1}(A)) = w\text{-}\lim_{n\to\infty}(\alpha_n(A)) = \omega_0(A)I$$

for all $A \in \mathscr{R}(\mathcal{O})$. For arbitrary $B \in \mathscr{R}(\mathcal{O})$ and $\varepsilon > 0$ there exists by the Reeh Schlieder property a local operator $B_\varepsilon \in \mathscr{R}(\mathcal{O})$ such that $\|(B_\varepsilon - B)\Omega\| \subseteq \varepsilon$. For $A \in \mathscr{R}(\mathcal{O})$ we then have by b) and locality:

$$\lim_{n\to\infty} \|[\alpha_n(A), B]\Omega\| \leqq \lim_{n\to\infty} \|(\alpha_n(A)B_\varepsilon - B\alpha_n(A))\Omega\| + \varepsilon\|A\|$$
$$= \lim_{n\to\infty} \|\alpha_n^{-1}(B - B_\varepsilon)A\Omega\| + \varepsilon\|A\| = \|(B - B_\varepsilon)\Omega\| \; \|A\Omega\| + \varepsilon\|A\| \leqq 2\varepsilon\|A\|.$$

By uniform boundedness of the sequence $\{[\alpha_n(A), B]\}_{n\in\mathbb{N}}$ and cyclicity of Ω for $\mathscr{R}(\mathcal{O})'$ we have c). q.e.d.

B) Let the theory be dilatation invariant, i.e. $\delta_\lambda \mathscr{R}(\mathcal{O}) \subseteqq \mathscr{R}(\lambda\mathcal{O})$ and $\omega_0 \cdot \delta_\lambda = \omega_0$ for all local open regions $\mathcal{O}$ and $\lambda \in \mathbb{R}_+$ (see [5] for details). Let $\mathcal{O}$ be a double cone with the origin in the boundary of its base (no loss of generality). Then $\mathscr{R}(\mathcal{O})$ is Type III.

Proof. Take $\mathscr{M} = \mathscr{M}_1 = \mathscr{R}(\mathcal{O})$ and $\alpha_n = \delta_{1/n}$, $n \in \mathbb{N}$. Then again a) results from covariance and the fact that $\delta_{1/n}\mathcal{O} \subset \mathcal{O}$ for all $n \in \mathbb{N}$; b) results from [5]. For $B \in \mathscr{R}(\mathcal{O})$

and $A \in \mathscr{L}(\mathscr{H})$ we further have:

$$\lim_{n\to\infty}(B\Omega, \alpha_n^{-1}(A)\Omega) = \lim_{n\to\infty}(\alpha_n(B)\Omega, A\Omega) = \omega_0(A)\omega_0(B^*)$$

so w-$\lim \alpha_n^{-1}(A) = \omega(A)I$ for each A localized in the causal complement of a neighborhood of the origin; for the $\{B_\varepsilon\}$ chosen in this manner c) is then shown as in example A) even without the Reeh Schlieder property.

Remarks. To substantiate our claim that for dilatation invariant theories the type problem for local algebras is essentially solved we observe that the previously used methods give Type III for each $\mathscr{R}(\mathscr{O})$ with $\mathscr{O}_1 \subset \mathscr{O} \subset \{x \in \mathbb{R}^n, x_1 < 0, |x_1| > |x_0|\}$, where $\mathscr{O}_1$ is a double cone with the origin in the boundary of its base. For local regions for which this can not be achieved by a Poincaré transformation (e.g. balls) one applies an algebraic version of results in [9] and then argues essentially as before.

If $\mathscr{R}(\mathscr{O})$ is the gauge invariant part of a field algebra $\mathscr{F}(\mathscr{O})$, as described in [5], by [10] and Lemma 2.6.5 in [11] we get Type III also for the field algebras.

C) Fields on a null plane $\Sigma = \{x \in \mathbb{R}^n; x_0 + x_1 = 0\}$: assume there exists a nontrivial, pseudolocal [7] net structure $\sigma \to \mathscr{R}_0(\sigma)$ on Σ and let the boosts in x_1 direction act in a natural way on this structure; assume further that this structure is imbedded in a local structure, i.e. $\sigma \subset \mathscr{O} \Rightarrow \mathscr{R}_0(\sigma) \subset \mathscr{R}(\mathscr{O})$ (for details see [7]). Let $\sigma = \{\bar{x} \in \Sigma, 0 < x_0 < a,\ x_2^2 + x_3^2 < b$; then $\mathscr{R}_0(\sigma)$ is of infinite type by [7]. In fact, $\mathscr{R}_0(\sigma)$ is Type III.

Proof. Take $\mathscr{M} = \mathscr{M}_1 = \mathscr{R}_0(\sigma)$ and α_n the boost which makes $(x_0 - x_1) \to n^{-1}(x_0 - x_1)$. Then a) is again trivially satisfied and b) results from $\bigcap_n \alpha_n(\mathscr{R}_0(\sigma)) = \{\mathbb{C} \cdot 1\}$ and $\omega_0 \cdot \alpha_n = \omega_0$ [8]. c) is shown as before.

Final Remark. In some theories as e.g. GFF and $P(\Phi)_2$ one can show that such null plane algebras exist and are contained in suitable local algebras. Composing a suitable sequence of automorphisms from elements of $U(a, \Lambda)_{a,\Lambda \in P_+^\uparrow}$ one can then show the Type III property also for a class of local algebras in the local systems corresponding to these theories. Whether our methods suffice to solve the type problem in general is an open problem.

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