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The Critical Behavior of ϕ_1^4

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Abstract. The eigenvalues, eigenfunctions, and Schwinger functions of the ordinary differential operator

$$H(\lambda, m)=\tfrac{1}{2}\{p^2+\lambda q^4+(m^2-\lambda m^{-1})q^2\}$$

are studied as $\lambda\to\infty$. It is shown that the scaling limit of the Schwinger functions equals the scaling limit of a one dimensional Ising model. Critical exponents of $H(\lambda, m)$ are shown to equal critical exponents of the Ising model, while critical exponents of the renormalized theory are shown to agree with those of a harmonic oscillator.

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1. Introduction

The purpose of this paper is to explain the behavior of the eigenvalues, eigenfunctions, and Schwinger (or correlation) functions of the ordinary differential operator

$$\tfrac{1}{2}\{p^2+\lambda q^4+(m^2-\lambda m^{-1})q^2\} \tag{1.1}$$

in the limit as λ becomes infinite. This problem is a simple, but characteristic, member of a large class of singular perturbation problems which arise naturally in mathematical physics (see e.g. [1—3]). The operator (1.1) represents the quantum mechanical Hamiltonian for a particle subject to the force $-4\lambda q^3 - 2(m^2 - \lambda m^{-1})q$.

The Schwinger functions of (1.1) describe the equilibrium statistical mechanics of an elastic string subject to the same non-linear restoring force. The limit as λ tends to infinity is studied because, as will be explained in Sections II and III, $\lambda = \infty$ is a critical point for each value of $m > 0$.

The most interesting reason for studying the critical behavior ($\lambda \to \infty$) of (1.1) is that it provides a clear illustration of two forms of the principle of universality. The first form, which says that critical exponents are independent of the detailed nature of the interaction (see Kadanoff [4]), will be illustrated in Section 7 where the exponents ν, ζ, γ, and η of (1.1) are calculated and shown to agree with the exponents of a one dimensional Ising model. The renormalized Schwinger functions associated with (1.1) are defined in Section 5, and the critical exponents of the renormalized theory are shown to agree with those of a free theory (i.e., a harmonic oscillator). The second form of universality maintains that the scaling limit of the renormalized Schwinger functions for ϕ_d^4 equals the scaling limit of the correlation functions of the Ising model in d dimensions ($d \leqq 4$) (see Glimm and Jaffe [3, 5, 6]). This conjecture will be explained and proved for ϕ_1^4 in Section 5.

For a discussion of the scaling limit conjecture for d dimensional euclidean quantum fields ($d \leqq 3$) and its relation to the construction of four dimensional quantum fields see Glimm and Jaffe [3, 5, 6].

2. Definitions

Let $\mathscr{H}$ be the Hilbert space $L_2(R^1)$, let q be multiplication by x and p be the operator $-i\partial/\partial x$.

The ϕ_1^4 theory is the study of the operator

$$H^1(\lambda, m) \equiv \tfrac{1}{2}\{p^2 + \lambda q^4 + (m^2 - \lambda m^{-1})q^2\}. \tag{2.1}$$

In this paper λ and m will always stand for real numbers for which $\lambda > 0$ and $m \neq 0$.

The following are well known properties of $H^1(\lambda, m)$ (see Jaffe [7] and Simon [2]):

p_1) $H^1(\lambda, m)$ is essentially self-adjoint on $C_0^\infty(R^1)$ and is self-adjoint on $\mathscr{D} \equiv \mathscr{D}(p^2) \cap \mathscr{D}(q^4)$.

p_2) $H^1(\lambda, m)$ has a compact resolvent.

p_3) The eigenvalues of $H^1(\lambda, m)$ are non-degenerate.

p_4) The eigenfunctions alternate parity, and the one that corresponds to the smallest eigenvalue is even.

Let $\varepsilon_0^1(\lambda, m)$ be the smallest eigenvalue of $H^1(\lambda, m)$. Define the renormalized operator $H = H(\lambda, m)$ by

$$H(\lambda, m) \equiv H^1(\lambda, m) - \varepsilon_0^1(\lambda, m). \tag{2.2}$$

Let the eigenvalues and eigenfunctions of $H(\lambda, m)$ be denoted by $\varepsilon_j = \varepsilon_j(\lambda, m)$ and $\Omega^j = \Omega^j(\lambda, m)$ for $j = 0, 1, 2, \ldots$, where $0 = \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \ldots$. Ω^0 is called the ground or vacuum state, and $\varepsilon_1(\lambda, m)$ is called the physical mass or mass gap. $\varepsilon_1(\lambda, m)$ will also be denoted by $m_p(\lambda, m)$.

Since H is self-adjoint and $H \geqq 0$, it generates a contraction semi-group $e^{-\tau H}$ for $\tau \geqq 0$. The eigenfunctions Ω^j are contained in the Schwartz space $\mathscr{S}$; $\mathscr{S}$ is invariant under p, q and $e^{-\tau H}$ (see Jaffe [7]).

These properties allow us to define the Schwinger functions $S^{(n)}$ for $n = 1, 2, \ldots$ and $t_1 \leqq t_2 \leqq \ldots \leqq t_n$ by

$$S^{(n)}(t_1, \ldots, t_n; \lambda, m) = \langle \Omega^0, q e^{-|t_2 - t_1|H} q e^{-|t_3 - t_2|H} q \ldots q e^{-|t_n - t_{n-1}|H} q \Omega^0 \rangle . \tag{2.3}$$

A consequence of the Feynman-Kac formula (see for example [8]) is that the Schwinger functions are the moments of a measure $d\mu_{\lambda,m}(q)$ on $\mathscr{S}'(R^1)$; i.e.,

$$S^{(n)}(t_1, \ldots, t_n; \lambda, m) = \int_{\mathscr{S}'(R^1)} q(t_1) \ldots q(t_n) d\mu_{\lambda,m}(q) . \tag{2.4}$$

The measure $d\mu_{\lambda,m}(q)$ is given formally by

$$d\mu_{\lambda,m}(q) = e^{-V(q)} \prod_t dq(t) \Big/ \int e^{-V(q)} \prod_t dq(t) , \tag{2.5}$$

where

$$V(q) = \tfrac{1}{2} \int_{-\infty}^{\infty} [(dq/dt)^2 + \lambda q^4 + (m^2 - \lambda m^{-1}) q^2] dt . \tag{2.6}$$

We see that (2.5) is the Gibbs measure (with the kinetic energy integrated out) of an elastic string subject to a nonlinear restoring force. Thus the Schwinger functions are correlation functions of a string in thermodynamic equilibrium. The correlation length of the string, $\xi(\lambda, m)$, defined by

$$\xi^{-1}(\lambda, m) \equiv - \lim_{|t_2 - t_1| \to \infty} \ln S^{(2)}(t_1, t_2; \lambda, m)/|t_2 - t_1| \tag{2.7}$$

is equal to $m_p(\lambda, m)^{-1}$. Values of λ and m which give rise to an infinite correlation length (i.e. zero mass) are called critical points, and will be denoted by λ_c and m_c.

At first glance the ϕ_1^4 theory appears to be a poor choice in which to look for a critical point because for $m \neq 0$ and $\lambda < \infty$, Ω^0 is non-degenerate implying that $m_p(\lambda, m) \neq 0$. However a theorem of Kac and Thompson [9] implies that $\lambda = \infty$ is a critical point for each value of $m > 0$. In fact their result implies the following theorem:

Theorem 2.1. *If $m > 0$ and $g \equiv \lambda/m^3$, then for g sufficiently large there exist positive constants A and B (independent of g) for which*

$$m_p(\lambda, m) \leqq A(g-1)^{1/2} m e^{-Bg^{1/2}} . \tag{2.8}$$

Proof. Combine Kac and Thompson [9], p. 259, Equation (2.8) with the scaling (3.21) of Section 3.

Corollary 2.2. *For each $m > 0$, $\lambda_c = \infty$ is a critical point for the ϕ_1^4 theory.*

Proof. From (2.8) it follows that

$$\lim_{\lambda \to \infty} m_p(\lambda, m) = 0 . \tag{2.9}$$

The rest of this paper is devoted to studying the critical (i.e. $\lambda \to \infty$) behavior of all the eigenfunctions, eigenvalues, and Schwinger functions of ϕ_1^4.

3. The Scaling Group

We introduce two scale transformations of the Hamiltonian (2.2). These scalings will be called the Ising model and harmonic oscillator scalings. The Ising scaling will be used to establish the approach of the Schwinger functions to the Ising model's correlation functions as the critical point is approached, while the harmonic oscillator scaling will be used to establish the approach of the spectrum of H to a (doubled) harmonic oscillator's spectrum.

For each real number $\alpha > 0$, the scaling transformation U_α is the unitary operator

$$U_\alpha \psi(x) \equiv \alpha^{1/2} \psi(\alpha x) \tag{3.1}$$

for all $\psi(x) \in \mathscr{H}$. It follows that

$$\begin{aligned} &U_\alpha^* \psi(x) = \alpha^{-1/2} \psi(\alpha^{-1} x), \; U_\alpha U_\beta = U_{\alpha\beta}, \\ &U_1 = I, \; U_{\alpha^{-1}} = U_\alpha^* = U_\alpha^{-1} \end{aligned} \tag{3.2}$$

so that the U_α form a 1-parameter group, called the scaling group.

The following are direct consequences of (3.1) and (3.2):

$$U_\alpha q U_\alpha^* = \alpha q , \tag{3.3}$$

$$U_\alpha p U_\alpha^* = \alpha^{-1} p , \tag{3.4}$$

$$U_\alpha H(\lambda, m) U_\alpha^* = \alpha^{-2} H(\alpha^6 \lambda, \alpha^2 m), \tag{3.5}$$

$$U_\alpha \Omega^j(\lambda, m) = \Omega^j(\alpha^6 \lambda, \alpha^2 m) \quad \text{for} \quad j = 0, 1, 2, \ldots \tag{3.6}$$

$$\varepsilon_j(\lambda, m) = \alpha^{-2} \varepsilon_j(\alpha^6 \lambda, \alpha^2 m) \quad \text{for} \quad j = 0, 1, 2, \ldots, \tag{3.7}$$

$$S^{(n)}(t_1, \ldots, t_n; \lambda, m) = \alpha^n S^{(n)}(\alpha^{-2} t_1, \ldots, \alpha^{-2} t_n; \alpha^6 \lambda, \alpha^2 m). \tag{3.8}$$

It is seen from (3.5) that the parameter $g = \lambda/m^3$, called "charge", is dimensionless. Scaling with the choice $\alpha = m^{-1/2}$ yields

$$U_\alpha \Omega^0(\lambda, m) = \Omega^0(g, 1) \equiv \Omega^0(g), \tag{3.9}$$

$$m_p(\lambda, m) = m m_p(g, 1) \equiv m m_p(g), \tag{3.10}$$

$$S^{(n)}(t_1, \ldots, t_n; \lambda, m) = m^{-n/2} S^{(n)}(m t_1, \ldots, m t_n; g, 1), \tag{3.11}$$

where $\Omega^0(g)$ and $m_p(g)$ are, respectively, the vacuum and mass of the Hamiltonian

$$H(g) \equiv H(g, 1) = 1/2[p^2 + g q^4 + (1 - g) q^2] - \varepsilon_0^1(g, 1). \tag{3.12}$$

The "critical charge" g_c is now defined by

$$m_p(g_c) = 0 \tag{3.13}$$

which, from Theorem 2.1, implies $g_c = \infty$.

This scaling shows that all theories with the same charge are equivalent. Thus, in order to understand the approach to the critical point, it is sufficient to study the behavior of $H(g)$ as $g \to \infty$.

In order to study the spectrum of $H(g)$, we rescale (3.12) using

$$\alpha = (g-1)^{-1/4}. \tag{3.14}$$

Let $\nu \equiv g(g-1)^{-3/2}$; then $\nu \to 0$ as $g \to g_c = \infty$. The scaling (3.14) reduces the study of $H(g)$ as $g \to \infty$ to the study of the "anharmonic oscillator" Hamiltonian

$$\begin{aligned} H_\nu &\equiv \tfrac{1}{2}[p^2 + \nu q^4 - q^2 + (4\nu)^{-1}] \\ &= \tfrac{1}{2}[p^2 + \nu(q^2 - (2\nu)^{-1})^2] \end{aligned} \tag{3.15}$$

as $\nu \to 0$. We note that H_ν also possesses properties P_1)—P_4). Its eigenvalues and corresponding eigenfunctions are denoted by $E_\nu^0 < E_\nu^1 < E_\nu^2 < \ldots$ and $\Omega_\nu^0, \Omega_\nu^1, \Omega_\nu^2, \ldots$, respectively, and

$$U_\alpha \Omega^j(g, 1) = \Omega_\nu^j, \tag{3.16}$$

$$\varepsilon_j(g, 1) = (g-1)^{1/2}(E_\nu^j - E_\nu^0), \tag{3.17}$$

and

$$\begin{aligned} &S^{(n)}(t_1, \ldots, t_n; \lambda, m) \\ &= m^{-n/2}(g-1)^{-n/4}\langle \Omega_\nu^0, q e^{-|t_2 - t_1|\tilde{H}} q \ldots q e^{-|t_n - t_{n-1}|\tilde{H}} q \Omega_\nu^0 \rangle, \end{aligned} \tag{3.18}$$

where

$$\tilde{H} \equiv m(g-1)^{1/2}(H_\nu - E_\nu^0). \tag{3.19}$$

The operator $\tilde{H}$ will be called the "Ising model scaled" Hamiltonian. In order to express $\tilde{H}$ in a more useful form, introduce

$$M_\nu \equiv E_\nu^1 - E_\nu^0 \tag{3.20}$$

then

$$m_p(\lambda, m) = m m_p(g) = m(g-1)^{1/2} M_\nu \tag{3.21}$$

and

$$\tilde{H} = m_p(\lambda, m) M_\nu^{-1}(H_\nu - E_\nu^0). \tag{3.22}$$

In what follows the anharmonic oscillator H_ν will be used to study the critical behavior of the eigenvalues and eigenfunctions of $H(g)$, and the Ising model scaled Hamiltonian $\tilde{H}$ will be used to study the critical behavior of the Schwinger functions.

4. The Anharmonic Oscillator

The behavior of the eigenvalues and eigenfunctions of the anharmonic oscillator H_ν as $\nu \to 0$ is easy to understand because the function $\nu(q^2 - 1/2\nu)^2$ approaches a potential for two independent harmonic oscillators (see Fig. 1). More precisely, expanding H_ν in a Taylor series about its minima $\pm(2\nu)^{-1/2}$,

$$H_\nu = \tfrac{1}{2}[p^2 + 2(|q| - (2\nu)^{-1/2})^2 + \delta V], \tag{4.1}$$

where for any α such that $0 \leqq \alpha \leqq 1/6$,

$$\sup_{||q|-(2\nu)^{-1/2}|<\nu^{\alpha-1/6}} |\delta V| = 0(\nu^{3\alpha}) \quad \text{as} \quad \nu \to 0. \tag{4.2}$$

Because the identical wells become widely separated by a high barrier, one expects "asymptotic eigenvalue degeneracy"; i.e.,

$$\lim_{\nu \to 0} |E_\nu^{2j} - E_\nu^{2j+1}| = 0 \quad \text{for} \quad j = 0, 1, 2, \ldots. \tag{4.3}$$

Because the wells become quadratic ($\sim 2q^2$) at their minima, one expects the eigenvalues of H_ν to approach those of $\frac{1}{2}[p^2 + 2q^2]$; i.e.,

$$\lim_{\nu \to 0} E_\nu^{2j} = 2^{-1/2}(2j+1) \quad \text{for} \quad j = 0, 1, 2, \ldots. \tag{4.4}$$

The way that the behavior of E_ν^j and Ω_ν^j, in the limit $\nu \to 0$, is rigorously established is to approximate the operator H_ν by operators $H_\nu^{(1)}$ and $H_\nu^{(2)}$ whose eigenvalues and eigenfunctions are known. Here $H_\nu^{(\alpha)}$ is said to approximate H_ν as $\nu \to 0$ if for some z, independent of ν, $\lim_{\nu \to 0} \|(H_\nu + z)^{-1} - (H_\nu^{(\alpha)} + z)^{-1}\| = 0$.

An obvious choice of an operator with which to approximate H_ν is

$$H_\nu^{(1)} \equiv \tfrac{1}{2}[p^2 + 2(|x| - (2\nu)^{-1/2})^2]. \tag{4.5}$$

The eigenfunctions and eigenvalues of (4.5) are given in terms of parabolic cylinder functions and their zeros (see Merzbacher [17]). However, the analysis of the parabolic cylinder functions can be avoided by first approximating H_ν by $H_\nu^{(1)}$ and then approximating $H_\nu^{(1)}$ by $H_\nu^{(2)}$ where

$$H_\nu^{(2)} \equiv \tfrac{1}{2}[p^2 + \tfrac{1}{2}(x - (2\nu x)^{-1})^2]^1. \tag{4.6}$$

$H_\nu^{(2)}$ has doubly degenerate eigenvalues

$$\lambda_\nu^j = 2^{-1/2}(2j+1) + (4\nu)^{-1}((2\nu)^2 + 1)^{1/2} - 1). \tag{4.7}$$

The corresponding eigenfunctions are denoted $\psi_\nu^{j,e}$ and $\psi_\nu^{j,0}$, so that

$$H_\nu^{(2)} \psi_\nu^{j,e} = \lambda_\nu^j \psi_\nu^{j,e}, H_\nu^{(2)} \psi_\nu^{j,0} = \lambda_\nu^j \psi_\nu^{j,0}. \tag{4.8}$$

$\psi_\nu^{j,e}$ is even, $\psi_\nu^{j,0}$ is odd, and both can be expressed simply in terms of generalized Laguerre polynomials times exponentials [11—13, 18, 19].

The approximation of H_ν by $H_\nu^{(2)}$ will yield the following theorem whose proof is given in [19]. Recall that E_ν^j and Ω_ν^j are the eigenvalues and eigenfunctions of H_ν.

Theorem 4.1. *In the above notation,*

(a) $\lim_{\nu \to 0} E_\nu^{2j} = \lim_{\nu \to 0} E_\nu^{2j+1} = \lim_{\nu \to 0} \lambda_\nu^j = 2^{-1/2}(2j+1)$,

(b) $\lim_{\nu \to 0} \left\| \Omega_\nu^{2j} - \psi_\nu^{j,e} \right\| = \lim_{\nu \to 0} \left\| \Omega_\nu^{2j+1} - \psi_\nu^{j,0} \right\| = 0$.

A synopsis of the proof of Theorem 4.1 is given in Appendix 1.

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[1] We thank Francesco Zirilli for helpful discussions and, in particular, for suggesting the choice of $H_\nu^{(2)}$

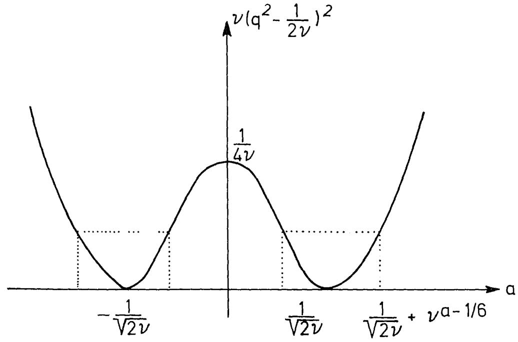

Fig. 1. The function $\nu(q^2-1/2\nu)^2$ approaches a potential for two independent harmonic oscillators

5. The Scaling Limit

A formal interchange of the limits $g\to\infty$, and $\varepsilon(=$lattice spacing$)\to 0$ in the lattice approximation to $S^{(n)}$ suggests the following "scaling limit" conjecture (see Glimm and Jaffe [3, 5, 6]):

$$\lim_{\substack{g\to g_c\\ m_p=1}} S_{\text{ren}}^{(n)} = \lim_{\substack{\varepsilon\to 0\\ \xi=1}} I^{(n)}. \tag{5.1}$$

The left side of the equation is the infinite scaling limit of the ϕ_d^4 theory, and the right side is the infinite scaling limit of the n-point correlation functions of the Ising model in d dimensions. The conjecture states that for $d\leqq 4$ the two limits exist and are equal. Here,

$$S_{\text{ren}}^{(n)} \equiv Z_3^{-n/2} S^{(n)}, \tag{5.2}$$

where Z_3 is the "field strength renormalization" which is defined to be the strength of the one particle pole in the Fourier transform of $S^{(2)}$.

To calculate Z_3 observe that

$$\begin{aligned} S^{(2)}(p;\lambda,m) &= \int_{-\infty}^{+\infty} e^{-i\tau p}\langle \Omega_{\lambda,m}^0, q e^{-|\tau| H(\lambda,m)} q\Omega_{\lambda,m}^0\rangle d\tau \\ &= \left\langle \Omega_{\lambda,m}^0, q\left[\frac{1}{H+ip}+\frac{1}{H-ip}\right] q\Omega_{\lambda,m}^0\right\rangle \\ &= \left\langle \Omega_{\lambda,m}^0, q\frac{2H}{H^2+p^2} q\Omega_{\lambda,m}^0\right\rangle. \end{aligned} \tag{5.3}$$

Applying the spectral theorem; (5.3) equals

$$\sum_{j=1}^{\infty} 2\varepsilon_j(\lambda,m)/(p^2+\varepsilon_j(\lambda,m)^2)|\langle \Omega_{\lambda,m}^0, q\Omega_{\lambda,m}^j\rangle|^2. \tag{5.4}$$

[We remark that (5.4) is the Lehmann spectral representation for the 2-point function in one dimension.] Thus,

$$Z_3 = \lim_{p^2\to -m_p^2} (m_p^2+p^2)\sum_{j=1}^{\infty} 2\varepsilon_j/(p^2+\varepsilon_j^2)|\langle \Omega_{\lambda,m}^0, q\Omega_{\lambda,m}^j\rangle|^2 \tag{5.5}$$

$$= 2m_p(\lambda,m)|\langle \Omega_{\lambda,m}^0, q\Omega_{\lambda,m}^1\rangle|^2. \tag{5.6}$$

Scaling (5.6) with $\alpha = m^{-1/2}(g-1)^{-1/4}$ yields

$$Z_3 = 2m_p(\lambda, m)m^{-1}(g-1)^{-1/2}|\langle\Omega_v^0, q\Omega_v^1\rangle|^2. \tag{5.7}$$

Combine (5.7) and (3.18)—(3.22); it follows that

$$S_{\text{ren}}^{(n)}(t_1, \ldots, t_n; \lambda, m) = 2^{-n/2} m_p(\lambda, m)^{-n/2} |\langle\Omega_v^0, q\Omega_v^1\rangle|^{-n} \langle\Omega_v^0, qe^{-|t_2-t_1|\tilde{H}} q \ldots qe^{-|t_n-t_{n-1}|\tilde{H}} q\Omega_v^0\rangle. \tag{5.8}$$

Since $m_p(\lambda, m) = mm_p(g, 1)$ the physical mass may be kept fixed at one as $g \to \infty$ by taking

$$m = m_p(g, 1)^{-1}. \tag{5.9}$$

From (3.22) and (5.9)

$$\tilde{H} = M_v^{-1}(H_v - E_v^0) \equiv \hat{H}_v. \tag{5.10}$$

Here we introduce a new symbol, $\hat{H} = \hat{H}_v$, to indicate this normalization of $\tilde{H}$. $\hat{H}$ has the eigenvalues $\hat{E}_j$ and eigenvectors Ω_v^j, where

$$\hat{E}_0 = 0 < \hat{E}_1 = 1 < \hat{E}_2 = (E_v^2 - E_v^0)M_v^{-1} < \hat{E}_3 = (E_v^3 - E_v^0)M_v^{-1} < \ldots. \tag{5.11}$$

The $\hat{H}$ normalization will be called the Ising normalization because as $v \to 0$, $\hat{E}_j \to \infty$ for $j \geqq 2$, which implies that the transfer matrix $e^{-|\tau|\hat{H}}$ converges to the transfer matrix for the Ising model.

The field strength renormalization Z_3 is necessary even in one dimension because the scaling limit and critical point are infrared divergent. For $d = 1$, $g_c = \infty$, and the right hand side of (5.1) is calculated in Appendix 2 yielding, for an Ising model whose spins are $\pm 2^{-1/2}$,

$$\lim_{\substack{\varepsilon \to 0 \\ \xi = 1}} I^{(n)} = \begin{cases} 0, & \text{for } n \text{ odd} \\ 2^{-n/2} \prod_{j=1}^{n/2} e^{-|t_{2j} - t_{2j-1}|}, & \text{for } n \text{ even}. \end{cases} \tag{5.12}$$

The infinite scaling limit (5.1) has now been transformed into the following theorem:

Theorem 5.1

$$\lim_{v \to 0} |\langle q\,\Omega_v^0, \Omega_v^1\rangle|^{-n} \langle\Omega_v^0, qe^{-|t_2-t_1|\hat{H}} q \ldots qe^{-|t_n-t_{n-1}|\hat{H}} q\Omega_v^0\rangle$$

$$= \begin{cases} 0 & \text{for } n \text{ odd} \\ \prod_{j=1}^{n/2} e^{-|t_{2j} - t_{2j-1}|} & \text{for } n \text{ even}. \end{cases}$$

Proof. For n odd, $S_{\text{ren}}^{(n)} = 0$ because of the invariance of $S^{(n)}$ under the $q \to -q$ symmetry. For n even the proof of Theorem 5.1 follows from Theorem 4.1 and the following lemma:

Lemma 5.1. *For* $k = 1, 2, \ldots$

$$\lim_{v \to 0} (2v)^k \langle\Omega_v^0, q^{2k}\Omega_v^0\rangle = \lim_{v \to 0} (2v)^{1/2} \langle\Omega_v^0, q\Omega_0^1\rangle = 1, \tag{5.13}$$

$$\langle\Omega_v^0, qe^{-|t_2-t_1|\hat{H}} q \ldots qe^{-|t_n-t_{n-1}|\hat{H}} q\Omega_v^0\rangle = 0(v^{-n/2}). \tag{5.14}$$

Assuming Lemma 5.1 and Theorem 4.1, a proof of Theorem 5.1 may be given by induction as follows. For $n=2$ what must be shown is that

$$\lim_{v\to 0} |\langle \Omega_v^0, q\Omega_v^1\rangle|^{-2}\langle \Omega_v^0, qe^{-|\tau|\hat{H}}q\Omega_v^0\rangle = e^{-|\tau|}. \tag{5.15}$$

If $\tau=0$, this follows from (5.13) with $k=1$. When $\tau \neq 0$ the spectral theorem implies

$$\langle \Omega_v^0, qe^{-|\tau|\hat{H}}q\Omega_v^0\rangle = \sum_{j=0}^{\infty} |\langle \Omega_v^0, q\Omega_v^j\rangle|^2 e^{-|\tau|\hat{E}_j}. \tag{5.16}$$

From (5.16) and the fact that the Ω^j's alternate parity

$$|\langle \Omega_v^0, q\Omega_v^1\rangle|^{-2}\langle \Omega_v^0, qe^{-|\tau|\hat{H}}q\Omega_v^0\rangle = e^{-|\tau|} + R_v, \tag{5.17}$$

where

$$R_v = |\langle \Omega_v^0, q\Omega_v^1\rangle|^{-2} \sum_{\substack{j>2 \\ j\,\mathrm{odd}}}^{\infty} e^{-|\tau|\hat{E}_j}|\langle \Omega_v^0, q\Omega_v^j\rangle|^2. \tag{5.18}$$

Because the $\hat{E}_j$ increase as j increases

$$\begin{aligned} R_v &\leqq |\langle \Omega_v^0, q\Omega_v^1\rangle|^{-2} e^{-|\tau|\hat{E}_3} \sum_{\substack{j\geqq 3 \\ j\,\mathrm{odd}}}^{\infty} |\langle \Omega_v^0, q\Omega_v^j\rangle|^2 \\ &\leqq |\langle \Omega_v^0, q\Omega_v^1\rangle|^{-2} e^{-|\tau|\hat{E}_3} \sum_{j=0}^{\infty} |\langle \Omega_v^0, q\Omega_v^j\rangle|^2 \\ &\leqq |\langle \Omega_v^0, q\Omega_v^1\rangle|^{-2} e^{-|\tau|\hat{E}_3} \langle \Omega_v^0, q^2\Omega_v^0\rangle. \end{aligned} \tag{5.19}$$

By the definition of $\hat{E}_3$ and Theorem 4.1,

$$\lim_{v\to 0} \hat{E}_3 = \lim_{v\to 0} (E_v^3 - E_v^0)/(E_v^1 - E_v^0) = +\infty. \tag{5.20}$$

By Lemma 5.1,

$$\lim_{v\to 0} (\langle \Omega_v^0, q^2\Omega_v^0\rangle)\, |\langle \Omega_v^0, q\Omega_v^1\rangle|^{-2} = 1. \tag{5.21}$$

Since $\tau \neq 0$ in this case, $\lim_{v\to 0} R_v = 0$, and Theorem 5.1 has now been established for $n=2$. The next step in the proof is to assume Theorem 5.1 true for all k less than n (where k and n are even) and show that this assumption implies the theorem is true for n. If $t_{i+1}=t_i$ for $i=1,2,\ldots,n-1$ then Theorem 5.1 reduces to Lemma 5.1, and so only the case for which there is an i such that $t_{i+1} \neq t_i$ need be considered. For simplicity i will be assumed to be even. The case when i is odd can be established by a similar argument. By the spectral theorem and the fact that $\hat{H}\Omega_v^0 = 0$

$$\begin{aligned} &\langle \Omega_v^0, qe^{-|t_2-t_1|\hat{H}}q \ldots qe^{-|t_{i+1}-t_i|\hat{H}}q \ldots qe^{-|t_n-t_{n-1}|\hat{H}}q\Omega_v^0\rangle \\ &= \sum_{j=0}^{\infty} \langle \Omega_v^0, qe^{-|t_2-t_1|\hat{H}}q \ldots qe^{-|t_i-t_{i-1}|\hat{H}}q\Omega_v^j\rangle \\ &\quad \cdot \langle \Omega_v^j, e^{-|t_{i+1}-t_i|\hat{H}}q \ldots qe^{-|t_n-t_{n-1}|\hat{H}}q\Omega_v^0\rangle \\ &= \langle \Omega_v^0, qe^{-|t_2-t_1|\hat{H}}q \ldots qe^{-|t_i-t_{i-1}|\hat{H}}q\Omega_v^0\rangle \\ &\quad \cdot \langle \Omega_v^0, qe^{-|t_{i+2}-t_{i+1}|\hat{H}}q \ldots qe^{-|t_n-t_{n-1}|\hat{H}}q\Omega_v^0\rangle + R_v. \end{aligned} \tag{5.22}$$

Let $A = qe^{-|t_2 - t_1|\hat{H}} q \ldots qe^{-|t_i - t_{i-1}|\hat{H}} q$, and

$B = qe^{-|t_{i+2} - t_{i+1}|\hat{H}} q \ldots qe^{-|t_n - t_{n-1}|\hat{H}} q$.

In this notation

$$R_\nu = \sum_{\substack{j \geqq 2 \\ j \text{ even}}}^{\infty} e^{-|t_{i+1} - t_i|\hat{E}_j} \langle \Omega_\nu^0, A\Omega_\nu^j \rangle \langle \Omega_\nu^j, B\Omega_\nu^0 \rangle$$

$$\leqq e^{-|t_{i+1} - t_i|\hat{E}_2} \sum_{j \geqq 0}^{\infty} |\langle \Omega_\nu^0, A\Omega_\nu^j \rangle| \, |\langle \Omega_\nu^j, B\Omega_\nu^0 \rangle|$$

$$\leqq e^{-|t_{i+1} - t_i|\hat{E}_2} (\langle \Omega_\nu^0, AA^* \Omega_\nu^0 \rangle)^{1/2} (\langle \Omega_\nu^0, B^* B\Omega_\nu^0 \rangle)^{1/2} . \tag{5.23}$$

Applying (5.14) to (5.23) yields

$$R_\nu \leqq e^{-|t_{i+1} - t_i|\hat{E}_2} 0(\nu^{-i/2}) 0(\nu^{-(n-i)/2})$$

$$= e^{-|t_{i+1} - t_i|\hat{E}_2} 0(\nu^{-n/2}) . \tag{5.24}$$

Theorem 5.1 becomes

$$\lim_{\nu \to 0} |\langle \Omega_\nu^0, q\Omega_\nu^1 \rangle|^{-n} \langle \Omega_\nu^0, Ae^{-|t_{i+1} - t_i|\hat{H}} B\Omega_\nu^0 \rangle$$

$$= \lim_{\nu \to 0} |\langle \Omega_\nu^0, q\Omega_\nu^1 \rangle|^{-i} \langle \Omega_\nu^0, A\Omega_\nu^0 \rangle |\langle \Omega_\nu^0, q\Omega_\nu^1 \rangle|^{-(n-i)} \langle \Omega_\nu^0, B\Omega_\nu^0 \rangle$$

$$+ \lim_{\nu \to 0} |\langle \Omega_\nu^0, q\Omega_\nu^1 \rangle|^{-n} R_\nu . \tag{5.25}$$

By the induction hypothesis the first term approaches

$$\prod_{j=1}^{i/2} e^{-|t_{2j} - t_{2j-1}|} \prod_{j=(i+2)/2}^{n/2} e^{-|t_{2j} - t_{2j-1}|}$$

$$= \prod_{j=1}^{n/2} e^{-|t_{2j} - t_{2j-1}|} . \tag{5.26}$$

By Theorem 4.1, Lemma 5.1, and (5.24) the second term approaches zero. Hence, the scaling limit is a consequence of Theorem 4.1, and Lemma 5.1.

6. The Proof of Lemma 5.1

In this section it will first be shown that as $\nu \to 0$:

$$\langle \Omega_\nu^0, q\Omega_\nu^1 \rangle \sim (2\nu)^{-1/2} , \tag{6.1}$$

$$\langle \Omega_\nu^0, q^{2k} \Omega_\nu^0 \rangle \sim (2\nu)^{-k} . \tag{6.2}$$

Here, $f(\nu) \sim g(\nu)$ as $\nu \to 0$ means that $\lim_{\nu \to 0} f(\nu)/g(\nu) = 1$. The reason (6.1) and (6.2) are true, despite the fact that Ω_ν^0 and Ω_ν^1 are converging weakly to zero, is that most of the mass of $(\Omega_\nu^0)^2$ and $(\Omega_\nu^1)^2$ is near the minima, $q = \pm (2\nu)^{-1/2}$, of the potential $(q^2 - (2\nu)^{-1})^2$. (6.1) and (6.2) will be used in finding critical exponents as well as in proving Lemma 5.1.

A computation in [18] shows that:

$$\langle \psi_\nu^{0,e}, q\psi_\nu^{0,0}\rangle \sim (2\nu)^{-1/2}, \tag{6.3}$$

$$\langle \psi_\nu^{0,e}, q^{2k}\psi_\nu^{0,e}\rangle \sim (2\nu)^{-k}. \tag{6.4}$$

To prove (6.1) and (6.2) use (6.3) and (6.4) with the following lemma.

Lemma 6.1

(a) $\lim_{\nu\to 0} \langle \Omega_\nu^0, q\Omega_\nu^1\rangle / \langle \psi_\nu^{0,e}, q\psi_\nu^{0,0}\rangle = 1$,

(b) $\lim_{\nu\to 0} \langle \Omega_\nu^0, q^{2k}\Omega_\nu^0\rangle / \langle \psi_\nu^{0,e}, q^{2k}\psi_\nu^{0,e}\rangle = 1$.

Proof

$$\begin{aligned}
&|(\langle \Omega_\nu^0, q\Omega_\nu^1\rangle / \langle \psi_\nu^{0,e}, q\psi_\nu^{0,0}\rangle) - 1| \\
&= |\langle \Omega_\nu^0, q\Omega_\nu^1\rangle - \langle \psi_\nu^{0,e}, q\psi_\nu^{0,0}\rangle| / \langle \psi_\nu^{0,e}, q\psi_\nu^{0,0}\rangle \\
&= |\langle (\Omega_\nu^0 - \psi_\nu^{0,e}), q\Omega_\nu^1\rangle + \langle \psi_\nu^{0,e}, q(\Omega_\nu^1 - \psi_\nu^{0,0})\rangle| / \langle \psi_\nu^{0,e}, q\psi_\nu^{0,0}\rangle \\
&\leq [\|\Omega_\nu^0 - \psi_\nu^{0,e}\| \, \|q\Omega_\nu^1\| + \|q\psi_\nu^{0,e}\| \, \|\Omega_\nu^1 - \psi_\nu^{0,0}\|] / \langle \psi_\nu^{0,e}, q\psi_\nu^{0,0}\rangle .
\end{aligned} \tag{6.5}$$

In order to show that $\|q\Omega_\nu^1\| = 0(\nu^{-1/2})$ observe that

$$q^2 < p^2 + \nu(q^2 - 1/2\nu)^2 + \nu^{-1} = 2H_\nu + \nu^{-1}. \tag{6.6}$$

From (6.6) and Theorem 4.1 we see that

$$\|q\Omega_\nu^1\|^2 = \langle \Omega_\nu^1, q^2\Omega_\nu^1\rangle \leq 2E_\nu^1 + \nu^{-1} = 0(\nu^{-1}). \tag{6.7}$$

Applying (6.3), (6.4), (6.7), and Theorem 4.1 to (6.5) yields Lemma 6.1a). The proof of (b) is similar. Start by observing that

$$\begin{aligned}
&|(\langle \Omega_\nu^0, q^{2k}\Omega^0\rangle / \langle \psi_\nu^{0,e}, q^{2k}\psi_\nu^{0,e}\rangle) - 1| \\
&\leq [\|\Omega_\nu^0 - \psi_\nu^{0,e}\| \, \|q^{2k}\Omega_\nu^0\| + \|q^{2k}\psi_\nu^{0,e}\| \, \|\Omega_\nu^0 - \psi_\nu^{0,e}\|] / \langle \psi_\nu^{0,e}, q^{2k}\psi_\nu^{0,e}\rangle .
\end{aligned} \tag{6.8}$$

Part (b) will follow by applying (6.4) and Theorem 4.1 to (6.8), once it has been shown that $\|q^{2k}\Omega_\nu^0\| = 0(\nu^{-k})$. This result will be the next lemma.

Lemma 6.2. *For* $n = 0, 1, 2, \ldots$

$$\langle \Omega^0, q^{2n+2}\Omega_\nu^0\rangle = 0(\nu^{-(n+1)}). \tag{6.9}$$

Proof. Induction will be used. When $n=0$, (6.6) shows that $\langle \Omega_\nu^0, q^2\Omega_\nu^0\rangle = 0(\nu^{-1})$. For $n>0$ assume that (6.9) is true for all $k<n$, and observe that

$$\begin{aligned}
&\langle \Omega_\nu^0, q^{2n+2}\Omega_\nu^0\rangle = \langle q^n\Omega_\nu^0, q^2 q^n\Omega_\nu^0\rangle \\
&\leq \langle q^n\Omega_\nu^0, (2H_\nu + \nu^{-1}) q^n\Omega_\nu^0\rangle \\
&= \langle q^n\Omega_\nu^0 \{(q^n(2H_\nu + \nu^{-1}) + [2H_\nu + \nu^{-1}, q^n]\}\Omega_\nu^0\rangle \\
&= (2E_\nu^0 + \nu^{-1})\langle \Omega_\nu^0, q^{2n}\Omega_\nu^0\rangle + \langle q^n\Omega_\nu^0, [p^2, q^n]\Omega_\nu^0\rangle .
\end{aligned} \tag{6.10}$$

However,

$$[p^2, q^n] = p[p, q^n] + [p, q^n]p = -in\{pq^{n-1} + q^{n-1}p\}$$
$$= -in\{2pq^{n-1} + [q^{n-1}, p]\}$$
$$= -in2pq^{n-1} + n(n-1)q^{n-2}. \tag{6.11}$$

When $n=1$, the second term will be zero. Substituting (6.11) in (6.10) yields

$$\langle \Omega_\nu^0, q^{2n+2}\Omega_\nu^0\rangle \leqq (2E^0 + \nu^{-1})\langle \Omega_\nu^0, q^{2n}\Omega_\nu^0\rangle$$
$$+ n(n-1)\langle \Omega_\nu^0, q^{2n-2}\Omega_\nu^0\rangle - 2ni\langle \Omega_\nu^0, q^{n-1}(qp)q^{n-1}\Omega_\nu^0\rangle. \tag{6.12}$$

However,

$$i2qp = i(qp + qp) = i(qp + pq) - 1. \tag{6.13}$$

Hence, (6.12) equals

$$(2E_\nu^0 + \nu^{-1})\langle \Omega_\nu^0, q^{2n}\Omega_\nu^0\rangle + n^2\langle \Omega_\nu^0, q^{2n-2}\Omega_\nu^0\rangle$$
$$- ni\langle q^{n-1}\Omega_\nu^0, (qp + pq)q^{n-1}\Omega_\nu^0\rangle. \tag{6.14}$$

The last term in (6.14) is zero because $q^{n-1}\Omega_\nu^0$ is a real valued function, and if ψ is any real valued function $\langle \psi, (qp + pq)\psi\rangle = 0$. To complete this proof just apply the induction hypothesis to the surviving terms in (6.14) in order to conclude from (6.10) that $\langle \Omega_\nu^0, q^{2n+2}\Omega_\nu^0\rangle = 0(\nu^{-(n+1)})$. The proof of Lemma 5.1 will be completed in the next Lemma.

Lemma 6.3. *As* $\nu \to 0$

$$\langle \Omega_\nu^0, qe^{-|\tau_1|\hat{H}}q \dots qe^{-|\tau_{2n-1}|\hat{H}}q\Omega_\nu^0\rangle = 0(\nu^{-n}). \tag{6.15}$$

Proof. When $n=1$

$$\langle \Omega_\nu^0, qe^{-|\tau_1|\hat{H}}q\Omega_\nu^0\rangle \leqq \|q\Omega_\nu^0\| \, \|e^{-|\tau_1|\hat{H}}q\Omega_\nu^0\|$$
$$\leqq \|q\Omega_\nu^0\|^2 = \langle \Omega_\nu^0, q^2\Omega_\nu^0\rangle = 0(\nu^{-1}). \tag{6.16}$$

When $n>1$, consider the case for which $\tau_j \neq 0$ for $j = 1, 2, \dots, 2n-1$. Define

$$A_j \equiv qe^{-|\tau_1|\hat{H}}q \dots qe^{-|\tau_{j-1}|\hat{H}}q, \tag{6.17}$$

$$B_j \equiv qe^{-|\tau_{j+1}|\hat{H}}q \dots qe^{-|\tau_{2n-1}|\hat{H}}q, \tag{6.18}$$

$$\psi_j \equiv A_j^*\Omega_\nu^0, \qquad \phi_j \equiv B_j\Omega_\nu^0. \tag{6.19}$$

In this notation (6.15) may be estimated by

$$\langle \Omega_\nu^0, A_n e^{-|\tau_n|\hat{H}}B_n\Omega_\nu^0\rangle \leqq \|\psi_n\| \, \|e^{-|\tau_n|\hat{H}}\phi_n\|$$
$$\leqq \|\psi_n\| \, \|\phi_n\|, \quad . \tag{6.20}$$

$$\|\psi_n\|^2 = \langle \psi_{n-1}, e^{-|\tau_{n-1}|\hat{H}}q^2e^{-|\tau_{n-1}|\hat{H}}\psi_{n-1}\rangle \tag{6.21}$$

which using (6.6) can be dominated by

$$\nu^{-1}\langle \psi_{n-1}, e^{-2|\tau_{n-1}|\hat{H}}\psi_{n-1}\rangle$$
$$+ 2\langle \psi_{n-1}, e^{-|\tau_{n-1}|\hat{H}}H_\nu e^{-|\tau_{n-1}|\hat{H}}\psi_{n-1}\rangle. \tag{6.22}$$

To bound (6.22) observe that the first term is less than

$$v^{-1}\|\psi_{n-1}\|^2. \tag{6.23}$$

We bound the second term by;

Proposition 6.1. *If* $\tau > 0$, *and* $\varepsilon > 0$, *then for* v *sufficiently small*

$$\langle \psi, e^{-\tau\hat{H}} H_v e^{-\tau\hat{H}} \psi \rangle \leqq (2^{-1/2} + \varepsilon)\,\|\psi\|^2. \tag{6.24}$$

Proof.

$$\langle \psi, e^{-\tau\hat{H}} H_v e^{-\tau\hat{H}} \psi \rangle = \sum_{j=0}^{\infty} e^{-2\tau\hat{E}_j} E_v^j |\langle \psi, \Omega_v^j \rangle|^2. \tag{6.25}$$

However

$$e^{-2\tau\hat{E}_j} E_v^j = e^{-2\tau(E_v^j - E_v^0)M_v^{-1}}(E_v^j - E_v^0) + E_v^0 e^{-2\tau(E_v^j - E_v^0)M_v^{-1}} \tag{6.26}$$

$$\leqq \sup_{x \geqq 0} e^{-2\tau M_v^{-1}x} x + E_v^0 \leqq M_v(2\tau e)^{-1} + E_v^0. \tag{6.27}$$

From Theorem 4.1, (6.27) may be made less than $2^{-1/2} + \varepsilon$ by taking v sufficiently small. Hence (6.24) holds.

Applying Proposition 6.1, and the bound (6.23) to (6.22) yields

$$\|\psi_n\|^2 \leqq (2^{-1/2} + \varepsilon + v^{-1})\,\|\psi_{n-1}\|^2 = 0(v^{-1})\,\|\psi_{n-1}\|^2. \tag{6.28}$$

Since n is arbitrary in (6.28), (6.15) follows from (6.28) and (6.16). The same argument shows $\|\phi_n\| = 0(v^{-n/2})$.

We remark that a proof of (6.15) (which is special to a q^4 interaction), that does not require all the $\tau_j \neq 0$, can be given using Lebowitz's inequality. In "A remark on the existence of ϕ_4^4" by Glimm and Jaffe [16] it is shown that as a consequence of Lebowitz's inequality

$$0 \leqq S^{n+2}(x_1, \ldots, x_{n+2}) \leqq S^{(n)}(x_1, \ldots, x_n) S^{(2)}(x_{n+1}, x_{n+2})$$
$$+ \sum_X S(X, x_{n+1}) S(X^\sim, x_{n+2}). \tag{6.29}$$

Where X denotes a subset of $x_1, \ldots, x_n$. Recursive use of (6.29) and the bound (6.16) yield (6.15).

7. Critical Behavior

In order to calculate the critical exponents v, ζ, γ, and η observe that as $g \to \infty$ the mass $m_p(\lambda, m)$ decays faster than any power of g. Thus ϕ_1^4, as in the Ising model, has $v = \infty$. This causes the exponents ζ and γ to be infinite. We overcome this by introducing the exponents $\bar{\zeta} = \zeta/v$ and $\bar{\gamma} = \gamma/v$. These parameters $\bar{\zeta}$ and $\bar{\gamma}$ then give the behavior of Z_3 and $\tilde{S}^{(2)}(0; \lambda, m)$ with respect to the mass $m_p(\lambda, m)$.

Theorem 7.1. *As* $\lambda \to \infty$ *with* m *fixed (i.e., as* $m_p \to 0$*)*

$$Z_3 \sim m_p^{\bar{\zeta}} m^{-1}, \tag{7.1}$$

$$\tilde{S}^{(2)}(0; \lambda, m) \sim m_p^{-\bar{\gamma}} m^{-1}, \tag{7.2}$$

$$S^{(2)}(\tau; \infty, m) \sim |\tau|^{1-\eta}(2m)^{-1}, \tag{7.3}$$

where $\bar{\zeta} = \bar{\gamma} = \eta = 1$.

Proof. Recall that

$$Z_3 = 2m_p(\lambda, m)\,|\langle \Omega^0(\lambda, m), q\Omega^1(\lambda, m)\rangle|^2 . \tag{7.4}$$

Thus

$$\lim_{\lambda\to\infty} m_p(\lambda, m)^{-1} Z_3 = \lim_{\lambda\to\infty} 2|\langle \Omega^0(\lambda, m), q\Omega^1(\lambda, m)\rangle|^2 \tag{7.5}$$

however scaling and (6.1) imply that

$$\lim_{\lambda\to\infty} 2|\langle \Omega^0(\lambda, m), q\Omega^1(\lambda, m)\rangle|^2 = 2\lim_{\lambda\to\infty} m^{-1}(g-1)^{-1/2}(2v)^{-1}$$
$$= 2\lim_{\lambda\to\infty} (2m)^{-1}(g-1)^{-1/2}(g-1)^{3/2}g^{-1} = m^{-1} \tag{7.6}$$

which proves (7.1). To prove (7.2) start with

$$\begin{aligned}\tilde{S}^{(2)}(0;\lambda, m) &= 2\langle \Omega^0(\lambda, m), qH^{-1}q\Omega^0(\lambda, m)\rangle \\ &= 2m_p(\lambda, m)^{-1}|\langle \Omega^0(\lambda, m), q\Omega^1(\lambda, m)\rangle|^2 \\ &\quad + 2\sum_{\substack{j\geqq 3\\ j\,\mathrm{odd}}} \varepsilon_j(\lambda, m)^{-1}|\langle \Omega^0(\lambda, m), q\Omega^j(\lambda, m)\rangle|^2 .\end{aligned} \tag{7.7}$$

Therefore

$$m_p(\lambda, m)\tilde{S}^{(2)}(0;\lambda, m) = 2|\langle \Omega^0(\lambda, m), q\Omega^1(\lambda, m)\rangle|^2 + m_p(\lambda, m)R , \tag{7.8}$$

where

$$\begin{aligned}R &= 2\sum_{\substack{j\geqq 3\\ j\,\mathrm{odd}}} \varepsilon_j(\lambda, m)^{-1}|\langle \Omega^0(\lambda, m), q\Omega^j(\lambda, m)\rangle|^2 \\ &\leqq \varepsilon_3(\lambda, m)^{-1}\langle \Omega^0(\lambda, m), q^2\Omega^0(\lambda, m)\rangle .\end{aligned} \tag{7.9}$$

The first term in (7.8) approaches m^{-1} by (7.6). The second term goes to zero by scaling Theorem (4.1) and (6.2) which imply that

$$\lim_{\lambda\to\infty} m_p(\lambda, m)\varepsilon_3(\lambda, m)^{-1} = 0 \tag{7.10}$$

and

$$\lim_{\lambda\to\infty} \langle \Omega^0(\lambda, m), q^2\Omega^0(\lambda, m)\rangle = (2m)^{-1} . \tag{7.11}$$

To prove (7.3) use

$$\begin{aligned}S^{(2)}(\tau;\lambda, m) &= \langle \Omega^0(\lambda, m), qe^{-|\tau|H}q\Omega^0(\lambda, m)\rangle \\ &= e^{-|\tau|m_p(\lambda, m)}|\langle \Omega^0(\lambda, m), q\Omega^1(\lambda, m)\rangle|^2 \\ &\quad + \sum_{\substack{j\geqq 3\\ j\,\mathrm{odd}}} e^{-|\tau|\varepsilon_j(\lambda, m)}|\langle \Omega^0(\lambda, m), q\Omega^j(\lambda, m)\rangle|^2 .\end{aligned} \tag{7.12}$$

Since as $\lambda\to\infty$, $m_p(\lambda, m)\to 0$ and $|\langle \Omega^0(\lambda, m), q\Omega^1(\lambda, m)\rangle|^2 \to (2m)^{-1}$ the first term in (7.12) approaches $(2m)^{-1}$. The second term is bounded by

$$e^{-|\tau|\varepsilon_3(\lambda, m)}\langle \Omega^0(\lambda, m), q^2\Omega^0(\lambda, m)\rangle . \tag{7.13}$$

From (7.11) and the fact that

$$\lim_{\lambda\to\infty} \varepsilon_3(\lambda, m) = \lim_{\lambda\to\infty} m(g-1)^{1/2}(E_v^3 - E_v^0) = +\infty \tag{7.14}$$

we conclude that

$$S^{(2)}(\tau; \infty, m) = (2m)^{-1}. \tag{7.15}$$

We remark that the exponents agree with those of the one dimensional Ising model and they satisfy the scaling relation $\bar{\gamma} + \bar{\zeta} = 2$. Similarly, it can be shown that the renormalized 2-point function (5.2) has the exponents $\nu = 1, \zeta = 0, \gamma = 2$, and $\eta = 0$ (which agree with those of the harmonic oscillator or "free" theory).

The next theorem contains a convenient summary of the critical behavior of ϕ_1^4.

Theorem 7.2. *If the parameter m is held fixed, then*

$$\lim_{\lambda\to\infty} S^{(2)}(\tau) = (2m)^{-1}, \tag{7.16}$$

$$\lim_{\lambda\to\infty} \frac{d}{d\tau} S^{(2)}(0^{\pm}) = \mp 2^{-1}, \tag{7.17}$$

$$\lim_{\lambda\to\infty} \tilde{S}^{(2)}(k) = \begin{cases} 0 & \text{for} \quad k \neq 0 \\ \infty & \text{for} \quad k = 0, \end{cases} \tag{7.18}$$

$$\lim_{\lambda\to\infty} Z_3 = 0, \tag{7.19}$$

$$\lim_{\lambda\to\infty} S_{\text{ren}}^{(2)}(\tau) = \infty, \tag{7.20}$$

$$\lim_{\lambda\to\infty} \frac{d}{d\tau} S_{\text{ren}}^{(2)}(0^{\pm}) = \mp\infty, \tag{7.21}$$

$$\lim_{\lambda\to\infty} \tilde{S}_{\text{ren}}^{(2)}(k) = k^2. \tag{7.22}$$

In the scaling limit, when $g\to\infty$ and $m\to\infty$ in such a way that $m_p(\lambda, m) = 1$ (denoted by "scale $\lim_{g\to\infty}$") we have

$$\text{scale} \lim_{g\to\infty} S^{(2)}(\tau) = 0, \tag{7.23}$$

$$\text{scale} \lim_{g\to\infty} \frac{d}{d\tau} S^{(2)}(0^{\pm}) = \mp 2^{-1}, \tag{7.24}$$

$$\text{scale} \lim_{g\to\infty} \tilde{S}^{(2)}(k) = 0 \quad (\text{for all } k), \tag{7.25}$$

$$\text{scale} \lim_{g\to\infty} Z_3 = 0, \tag{7.26}$$

$$\text{scale} \lim_{g\to\infty} S_{\text{ren}}^{(2)}(\tau) = 2^{-1} e^{-|\tau|}, \tag{7.27}$$

$$\text{scale} \lim_{g\to\infty} \frac{d}{d\tau} S_{\text{ren}}^{(2)}(0^{\pm}) = \mp\infty, \tag{7.28}$$

$$\text{scale} \lim_{g\to\infty} \tilde{S}_{\text{ren}}^{(2)}(k) = (k^2+1)^{-1}. \tag{7.29}$$

The proofs of (7.16)—(7.29) are similar to the proof of Theorem (7.1) see [18].

Appendix 1. The Spectral Properties of H_ν

This appendix contains a synopsis of the proof of Theorem 4.1.

Theorem 4.1 is proven as follows; first, we show that there exist positive constants A, z_0, α_0' and ν_0 such that for all $z \geqq z_0$ and $\nu \leqq \nu_0$

$$\|(H_\nu + z)^{-1} - (H_\nu^{(1)} + z)^{-1}\| \leqq A\nu^{\alpha_0} \tag{A1.1}$$

$$\|(H_\nu + z)^{-1} - (H_\nu^{(2)} + z)^{-1}\| \leqq A\nu^{\alpha_0}. \tag{A1.2}$$

In order to establish (A1.1) let

$$V \equiv H_\nu - (1/2)p^2 + c, \qquad V_1 \equiv H_\nu^{(1)} - (1/2)p^2 + c, \tag{A1.3}$$

$$R \equiv (z + c + H_\nu)^{-1}, \qquad R_1 \equiv (z + c + H_\nu^{(1)})^{-1}, \tag{A1.4}$$

where c and z are any positive numbers.

Estimates on the resolvents are reduced to estimates on the potentials V and V_1 by showing that there exist positive constants z_0 and ν_0 such that for $z \geqq z_0$ and $\nu \leqq \nu_0$

$$\|(H_\nu + c + z)^{-1}\psi\| \leqq \|V^{-1}\psi\|, \tag{A1.5}$$

$$\|(H_\nu^{(1)} + c + z)^{-1}\psi\| \leqq 2\|V_1^{-1}\psi\|. \tag{A1.6}$$

We prove (A1.5) by showing that on a core for H_ν (see Kato [10], p. 330, Theorem 2.21)

$$(H_\nu + c + z)^2 \geqq V^2. \tag{A1.7}$$

This follows from

$$\begin{aligned}(H_\nu + c + z)^2 &= (1/2p^2 + V + z)^2 \\ &= (1/4)p^4 + (V + z)^2 + p(V + z)p - (1/2)V'' \\ &\geqq V^2 + 2zV + z^2 - (1/2)V''\end{aligned}$$

which implies that

$$(H_\nu + c + z)^2 - V^2 \geqq 2zV + z^2 - (1/2)V''. \tag{A1.9}$$

The right hand side of (A1.9) is fourth degree polynomial whose minima can be estimated, and shown to be positive for all $\nu \leqq \nu_0$ and $z \geqq z_0$ (see [19]). (A1.6) is proven similarly.

We finish the proof of (A1.1) by first letting P be the projection onto the functions in $L_2(R_1)$ with support in $E_\nu \equiv \{x \,|\, |x| - (2\nu)^{-1/2}| > \nu^{(\beta - 1/2)/3}\}$. Here β is any number which satisfies $0 < \beta < 2^{-1}$. We then observe that

$$\begin{aligned}\|(R_1 - R)\psi\| &= \|R(V - V_1)R_1\psi\| \\ &\leqq \|R(V - V_1)(1 - P)R_1\psi\| + \|R(V - V_1)PR_1\psi\| \\ &\leqq (c + z)^{-2}\|(V - V_1)(1 - P)\| \, \|\psi\| \\ &\quad + \|V^{-1}(V - V_1)P\| \, \|PR_1\psi\|.\end{aligned} \tag{A1.10}$$

Using the fact that

$$\|PR_1\psi\|^2 = \langle \psi, R_1 P R_1 \psi \rangle \leqq 2(c + z)^{-1}\|V_1^{-1}P\| \, \|\psi\|^2. \tag{A1.11}$$

We may dominate (A1.10) by

$$C(\|(V-V_1)(1-P)\|+\|V^{-1}(V-V_1)P\|\,\|V_1^{-1}P\|^{1/2})\|\psi\|, \tag{A1.12}$$

where C is a positive constant independant of v. It is straightforward (see [19]) to show that for $\beta=1/8$, (A1.12) is $O(v^{1/8})$. The first term is small because V_1 approximates V uniformly in the complement of E_v (i.e., in an increasing region which contains V's minima).

The second term is small because V_1 becomes large in the region E_v as $v\to 0$. This concludes the proof of (A1.1). The proof of (A1.2) is similar (see [19]).

Choose any r satisfying $0<r<2^{1/2}$, and define

$$D_k\equiv\{z|\,|z-\lambda_v^k|\leqq r\}, \tag{A1.13}$$

$$S_n\equiv\{x+iy|0\leqq x\leqq 2^{1/2}(2n+2), \quad -1\leqq y\leqq +1\}. \tag{A1.14}$$

Let $P_v(S)$ and $P_v^{(2)}(S)$ denote the spectral projectors of H_v and $H_v^{(2)}$ associated with the borel set S.

We complete the proof of Theorem 4.1 by showing that for k, $n=0,1,2,\ldots$

$$\lim_{v\to 0}\|P_v(D_k)-P_v^{(2)}(\lambda_v^k)\|=0, \tag{A1.15}$$

$$\lim_{v\to 0}\|P_v(S_n)-P_v^{(2)}(S_n)\|=0. \tag{A1.16}$$

(A1.15) follows from (A1.2) and Kato [10], p. 212 which together imply that for v small and all $z\in\partial D_k$

$$(z-H_v)^{-1}\ \text{exists} \tag{A1.17}$$

$$\sup_{z\in\partial D_k}\|(z-H_v)^{-1}-(z-H_v^{(2)})^{-1}\|\ \leqq v^{\alpha_0/2}. \tag{A1.18}$$

Hence for v small we may use the representation

$$P_v(D_k)-P_v^{(2)}(\lambda_v^k)=\frac{1}{2\pi i}\oint_{\partial D_k}(z-H_v)^{-1}-(z-H_v^{(2)})^{-1}dz \tag{A1.19}$$

and (A1.18) to prove (A1.15). (A1.16) is proven similarly.

We establish Theorem 4.1a) by observing that (A1.15) and (A1.16) imply (by Kato [10], p. 156) that for v small enough

$$\dim P_v(D_k)=\dim P_v^{(2)}(\lambda_v^k)=2 \tag{A1.20}$$

$$\dim P_v(S_n)=\dim P_v^{(2)}(S_n)=2n+2. \tag{A1.21}$$

Because the eigenvalues of H_v are non-degenerate and positive, (A1.21) implies that the first $2n+2$ of them are in S_n. However (A1.20) implies that for v small enough exactly two of them are within r of $\lambda_0^k=2^{1/2}(2k+1)$ (for $k\leqq n$). Letting $r\to 0$ [and $v=v(r)\to 0$] we conclude that

$$\lim_{v\to 0}E_v^{2j}=\lim_{v\to 0}E_v^{2j+1}=\lambda_0^j=2^{1/2}(2j+1). \tag{A1.22}$$

A complete proof of Theorem 4.1, including a discussion of cores for H_v, $H_v^{(1)}$, and $H_v^{(2)}$ is given in [19].

Appendix 2. The Scaling Limit of $I^{(n)}$

The n-point correlation function, $I^{(n)}$, of a one dimensional Ising model with spins $s_j = \pm 2^{-1/2}$, $j=0,\pm 1,\pm 2,\ldots$, lattice spacing ε, and temperature β^{-1} is given by

$$I^{(n)}(t_1,\ldots,t_n;\beta,\varepsilon)$$

$$\equiv \lim_{N\to\infty} z_N^{-1}\left(\sum_{s_{-N}=\pm 2^{-1/2}} \cdots \sum_{s_N=\pm 2^{-1/2}} s_{[t_1/\varepsilon]}\cdots s_{[t_n/\varepsilon]} e^{\beta\sum_{j=-N}^{j=N-1} s_j s_{j+1}}\right), \tag{A2.1}$$

where

$$z_N \equiv \sum_{s_{-N}=\pm 2^{-1/2}} \cdots \sum_{s_N=\pm 2^{-1/2}} e^{\beta\sum_{j=-N}^{N-1} s_j s_{j+1}}$$

and $[A]$ is the largest integer less than or equal to A.

The correlation length $\xi=\xi(\beta,\varepsilon)$ is given by

$$\xi^{-1} \equiv -\lim_{|t|\to\infty} \frac{\ln I^{(2)}(0,t;\beta,\varepsilon)}{|t|}. \tag{A2.2}$$

Theorem A2.1. *For every $\varepsilon>0$, one can choose $\beta=\beta(\varepsilon)$ so that $\xi(\beta,\varepsilon)=1$. If this is done then*

$$\lim_{\substack{\varepsilon\to 0\\ \xi=1}} I^{(n)}(t_1,\ldots,t_n;\beta,\varepsilon) = \left\{\begin{array}{ll} 0 & \text{if } n \text{ is odd} \\ 2^{-n/2}\prod_{j=1}^{n/2} e^{-|t_{2j}-t_{2j-1}|} & \text{if } n \text{ is even.} \end{array}\right\}. \tag{A2.3}$$

Proof. Assume n is even and $t_1 \leqq t_2 \leqq \ldots \leqq t_n$. Let $i_j \equiv [t_j/\varepsilon]$, and change variables in (A2.1) by setting $s_j = 2^{-1/2}\sigma_j$, $\eta_j = \sigma_j\sigma_{j+1}$. Use the facts that

$$\sum_{\sigma_{-N}=\pm 1} \cdots \sum_{\sigma_N=\pm 1} = \sum_{\sigma_{-N}=\pm 1}\ \sum_{\eta_{-N}=\pm 1} \cdots \sum_{\eta_{N-1}=\pm 1}, \tag{A2.4}$$

$$\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_n} = (\sigma_{i_1}\sigma_{i_2})(\sigma_{i_3}\sigma_{i_4})\ldots(\sigma_{i_{n-1}}\sigma_{i_n}), \tag{A2.5}$$

$$\sigma_i\sigma_j = \sigma_i\sigma_{i+1}\sigma_{i+1}\sigma_{i+2}\ldots\sigma_{j-1}\sigma_j = \eta_i\eta_{i+1}\ldots\eta_{j-1}. \tag{A2.6}$$

To conclude that

$$I^{(n)}(t_1,\ldots,t_n;\beta,\varepsilon) = 2^{-n/2}\prod_{j=1}^{n/2}\left[\tanh(\beta/2)\right]^{(i_{2j}-i_{2j-1})}. \tag{A2.7}$$

Observe that if n is odd, mapping s_j into $-s_j$ implies that $I^{(n)} = -I^{(n)}$, which implies $I^{(n)}=0$.

To prove Theorem (A2.1) use (A2.7) to find

$$\xi^{-1} = -(1/\varepsilon)\ln\tanh(\beta/2). \tag{A2.8}$$

Choosing $\beta(\varepsilon) = 2\,\mathrm{arc\,tanh}\, e^{-\varepsilon}$ yields the theorem.

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