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Asymptotic Behavior of Solutions to Certain Nonlinear Schrödinger-Hartree Equations*

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Abstract. The asymptotic behavior of solutions to the Cauchy problem for the equation

$$i\psi_t = \tfrac{1}{2}\Delta\psi - v(\psi)\psi\,, \quad v = r^{-1} * |\psi|^2\,,$$

and for systems of similar form, is studied. It is shown that the norms

$$\|\psi(t)\|^2_{L_2(|x| \leqq R)} + \|\nabla\psi(t)\|^2_{L_2(|x| \leqq R)}$$

are integrable in time for any fixed $R > 0$, from which it follows that

$$\lim_{t \to \infty} \|\psi(t)\|_{L_2(|x| \leqq R)} = 0\,.$$

Nevertheless, it is established that an L_2-scattering theory is impossible.

Introduction

We consider classical solutions to the Cauchy problem for the equations

$$\begin{aligned} &i\psi_t = \tfrac{1}{2}\Delta\psi - v(\psi)\psi \qquad (x \in \mathbb{R}^3,\ t > 0) \\ &v(\psi) = r^{-1} * |\psi|^2 = \int_{\mathbb{R}^3} |x-y|^{-1} |\psi(y,t)|^2 dy \qquad (r = |x|) \end{aligned} \tag{1}$$

and

$$i\partial\psi_j/\partial t = \tfrac{1}{2}\Delta\psi_j - \sum_{k=1}^{N} (\psi_j v_k - \psi_k v_{jk}) \qquad (j = 1, 2, \ldots, N) \tag{2}$$

where

$$v_{jk} = r^{-1} * \psi_j \overline{\psi_k}\,, \qquad v_k = v_{kk} = r^{-1} * |\psi_k|^2\,.$$

Equations (1), (2) are Coulomb-free versions of the time-dependent Hartree and Hartree-Fock equations. In [2] we have treated the existence question for

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Equations (1), (2) with coulomb terms present, and have shown that global solutions exist with the quantity [for (2)]

$$\sum_{j=1}^{N} \{\|\psi_j(t)\|_2^2 + \|\nabla\psi_j(t)\|_2^2\}$$

remaining uniformly bounded. The notation here is

$$\|\psi(t)\|_2 = \left(\int_{\mathbb{R}^3} |\psi(x,t)|^2 dx\right)^{1/2},$$

etc. A similar result is valid for solutions to (1).

In this paper we shall obtain the following results: Let ψ be a solution of (1) with finite energy norm (as above). Then for every fixed $R>0$ we have

$$\int_0^\infty \int_{|x|\leqq R} (|\psi|^2 + |\nabla\psi|^2) dx dt < \infty$$

from which we conclude that

$$\lim_{t\to\infty} \int_{|x|\leqq R} |\psi|^2 dx = 0 .$$

For spherically symmetric solutions ψ_j of (2) with finite energy, the same results are valid for each ψ_j, $1 \leqq j \leqq N$. However, we also show that an L_2-scattering theory for non-trivial solutions of (1) is impossible. It is plausible that solutions to (1) do decay uniformly to zero as $t\to\infty$, but if so, the rate of decay cannot be fast enough to insure the existence of asymptotic free states.

The desired estimates follow from an identity obtained essentially through use of the multiplier $\partial\bar{\psi}/\partial r$ [for (1)]. The resulting estimate is the exact analogue of Morawetz' estimate [5]. During the preparation of this work, the author learned that this multiplier was found independently and, in fact, first, by Lin. In his thesis [4], Lin studies the asymptotic behavior of solutions to equations of the form

$$iu_t = \Delta u - h(x) q(|u|^2) u$$

and shows that

$$\|u(t)\|_\infty = 0(t^{-3/2}) \quad \text{as} \quad t\to\infty$$

under certain conditions on h and q[1]. In addition, decay of the "local energy norm" is established, and a scattering theory is developed.

The reason for the restriction to spherically symmetric solutions of (2) involves the use of a radial derivative as a multiplier; this will be evident from the proof.

Although we dealt in [2] only with generalized solutions, it is easy to see that in the absence of coulomb terms [i.e. for (1), (2)], solutions will be smooth if the data is. By induction on k we can show that the norms $\sum_{|\alpha|\leqq k} \|D^\alpha \psi\|_2$ are finite for all $t \geqq 0$. For $k \leqq 2$, this was done in [2]. For higher values of k, we write the potential v [in the case of Equation (1)] as

$$v = \int_{\mathbb{R}^3} |z|^{-1} |\psi(x-z,t)|^2 dz$$

[1] The restrictions involving h can be removed, e.g. $h \equiv 1$ is admissible (private communication from Prof. Walter Strauss)

and take successively higher derivatives of $|\psi|^2$. The induction suceeds easily when we take into account the inequality

$$\int |x-y|^{-2}|\varphi(y)|^2 dy \leqq 4 \int_{\mathbb{R}^3} |\nabla\varphi|^2 dx \qquad (\varphi\in C_0^\infty(\mathbb{R}^3))$$

and apply Gronwall's inequality.

I. Time Decay

We shall work first with Equations (2); the corresponding results for (1) can be simply deduced from this. The system to be studied is

$$i\partial\psi_j/\partial t = \tfrac{1}{2}\Delta\psi_j - \sum_{k=1}^{N}(\psi_j v_k - \psi_k v_{jk}) \qquad (j=1,2,\ldots,N) \tag{2}$$

for $x\in\mathbb{R}^3$, $t>0$. Smooth initial values $\psi_j(x,0)=\varphi_j(x)$ are given, which belong, say, to $\mathcal{S}$.

Here

$$v_{jk} = \int |x-y|^{-1}\psi_j\bar{\psi_k}\,dy, \qquad v_k = \int |x-y|^{-1}|\psi_k|^2 dy$$

so that $\bar{v_{jk}} = v_{kj}$. From [2] we have that there is a constant M such that the *energy* satisfies

$$\sum_{j=1}^{N}[\|\psi_j(t)\|_2^2 + \|\nabla\psi_j(t)\|_2^2] \leqq M$$

for all $t\geqq 0$.

Let $\zeta=\zeta(r)$, $r=|x|$, be a smooth bounded real-valued function. We obtain our estimates from the following identity:

Lemma. *Let ψ_j, $j=1,2,\ldots,N$, be solutions to* (2) *with finite energy. Then*

$$\begin{aligned} -\frac{d}{dt}\,\mathrm{Im}\sum_{j=1}^{N}\int \zeta\psi_j\,\partial\bar{\psi_j}/\partial r\,dx &= -\sum_{j=1}^{N}\int \zeta'|\partial\psi_j/\partial r|^2 dx \\ &- \sum_{j=1}^{N}\int \zeta|x|^{-1}(|\nabla\psi_j|^2 - |\partial\psi_j/\partial r|^2)dx - 2\pi\sum_{j=1}^{N}\zeta(0)|\psi_j(0,t)|^2 \\ &+ \sum_{j=1}^{N}\int |\psi_j|^2[\tfrac{1}{2}\Delta\zeta|x|^{-1} - \zeta'|x|^{-2} + \tfrac{1}{4}\Delta\zeta']dx \\ &+ \sum_{k,j=1}^{N}\int \zeta[|\psi_j|^2(\partial v_k/\partial r) - \mathrm{Re}\,\bar{\psi_j}\psi_k\partial v_{jk}/\partial r\,dx \end{aligned} \tag{3}$$

We sketch the derivation of the lemma. We begin by multiplying each of Equations (2) by $\zeta\partial\bar{\psi_j}/\partial r$ (the use of ζ is an idea from [8,9]). The real part of the resulting expression is then summed over j and integrated over $\mathbb{R}^3$. The result can be written as

$$I_1 = I_2 + I_3$$

where

$$I_1 = i \int \sum_{j=1}^{N} \sum_{l=1}^{3} \zeta x_l |x|^{-1} \left(\frac{\partial \psi_j^-}{\partial x_l} \frac{\partial \psi_j}{\partial t} - \frac{\partial \psi_j}{\partial x_l} \frac{\partial \psi_j^-}{\partial t} \right) dx \,,$$

$$I_2 = \tfrac{1}{2} \sum_{j=1}^{N} \sum_{l=1}^{3} \int \zeta |x|^{-1} x_l \left(\frac{\partial \psi_j^-}{\partial x_l} \Delta \psi_j + \frac{\partial \psi_j}{\partial x_l} \Delta \psi_j^- \right) dx \,,$$

$$I_3 = -2 \sum_{k,j=1}^{N} \sum_{l=1}^{3} \int \zeta |x|^{-1} x_l \left[\mathrm{Re} \left(\frac{\partial \psi_j^-}{\partial x_l} \psi_j v_k \right) - \mathrm{Re} \left(\frac{\partial \psi_j^-}{\partial x_l} \psi_k v_{jk} \right) \right] dx \,.$$

We rewrite I_1 as

$$I_1 = i \int \sum_{j=1}^{N} \sum_{l=1}^{3} \zeta |x|^{-1} x_l \left[\frac{\partial}{\partial x_l} \left(\psi_j^- \frac{\partial \psi_j}{\partial t} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \psi_j}{\partial x_l} \psi_j^- \right) \right] dx$$

and integrate by parts; in doing so we find that

$$I_1 = (d/dt)\, \mathrm{Im} \int \sum_{j=1}^{N} \zeta \psi_j^- (\partial \psi_j / \partial r) dx - \mathrm{Re} \left(i \int \sum_{j=1}^{N} (\zeta' + 2|x|^{-1} \zeta) \psi_j^- \partial \psi_j / \partial t dx \right).$$

The second term appearing here may be calculated by appealing to the Equation (2) again. Integrating by parts several more times, we evaluate I_1 as

$$\begin{aligned} I_1 = & \frac{d}{dt} \mathrm{Im} \int \sum_{j=1}^{N} \zeta \psi_j^- \partial \psi_j / \partial r dx \\ & + \int \sum_{j,k=1}^{N} (\zeta' + 2|x|^{-1} \zeta)(|\psi_j|^2 v_k - \mathrm{Re}\, \psi_j^- \psi_k v_{jk}) dx \\ & + \tfrac{1}{2} \int \sum_{j=1}^{N} (\zeta' + 2|x|^{-1} \zeta) |\nabla \psi_j|^2 dx \\ & - \tfrac{1}{4} \int \sum_{j=1}^{N} |\psi_j|^2 (\Delta(\zeta' + 2|x|^{-1} \zeta)) dx \,. \end{aligned}$$

Again integrating by parts, we calculate I_2 in a straightforward fashion and obtain

$$I_2 = \int \sum_{j=1}^{N} [(\zeta |x|^{-1} - \zeta') |\partial \psi_j / \partial r|^2 + \tfrac{1}{2} \zeta' |\nabla \psi_j|^2] dx \,.$$

Finally, since the matrix (v_{jk}) is hermitean, I_3 can be written as

$$\begin{aligned} I_3 = & - \int \sum_{k,j=1}^{N} \sum_{l=1}^{3} \zeta |x|^{-1} x_l v_k (\partial |\psi_j|^2 / \partial x_l) dx \\ & + \mathrm{Re} \int \sum_{k,j=1}^{N} \sum_{l=1}^{3} \zeta |x|^{-1} x_l v_{jk} \partial (\psi_j - \chi_k) / \partial x_l dx \,. \end{aligned}$$

We integrate by parts once more and find that

$$\begin{aligned} I_3 = & \int \sum_{k,j=1}^{N} |\psi_j|^2 [(2\zeta |x|^{-1} + \zeta') v_k + \zeta \partial v_k / \partial r] dx \\ & - \mathrm{Re} \int \sum_{k,j=1}^{N} \psi_j^- \psi_k [(\zeta' + 2\zeta |x|^{-1}) v_{jk} + \zeta \partial v_{jk} / \partial r] dx \,. \end{aligned}$$

When we combine these expressions we obtain the lemma.

Following [9] we now choose ζ as

$$\zeta(r)=\tfrac{1}{2}(r+1)^{-1}-1=-(2r+1)/2(r+1). \tag{4}$$

Then $\zeta \leqq 0$, $\zeta'=-\tfrac{1}{2}(r+1)^{-2}\leqq 0$ and a direct calculation shows that

$$\tfrac{1}{2}\Delta\zeta|x|^{-1}-\zeta'|x|^{-2}+\tfrac{1}{4}\Delta\zeta'=(r+4)/4r(r+1)^4>0.$$

We then have

Theorem 1. *Let ψ_j, $j=1,\ldots,N$, be spherically symmetric solutions of* (2) *with finite energy, and let $R>0$ be arbitrary. Then for each $j=1,2,\ldots,N$ we have*

i) $\int_0^\infty |\psi_j(0,t)|^2 dt<\infty$,

ii) $\int_0^\infty \int_{|x|\leqq R} [|\psi_j|^2+|\partial\psi_j/\partial r|^2]dxdt<\infty$,

iii) $\lim_{t\to\infty} \int_{|x|\leqq R} |\psi_j|^2 dx=0$.

Proof. Let $T>0$ be arbitrary and integrate (3) over the interval $[0,T]$. We find that

$$\begin{aligned}
&\operatorname{Im}\sum_{j=1}^N \int (2r+1)2^{-1}(r+1)^{-1}\psi_j\frac{\partial\overline{\psi_j}}{\partial r}dx\Bigg]_{t=0}^{t=T}\\
&=\tfrac{1}{2}\sum_{j=1}^N \int_0^T\int (r+1)^{-2}|\partial\psi_j/\partial r|^2 dxdt\\
&\quad+\pi\sum_{j=1}^N\int_0^T|\psi_j(0,t)|^2dt\\
&\quad+\tfrac{1}{4}\int_0^T\int\sum_{j=1}^N|\psi_j|^2(r+4)r^{-1}(r+1)^{-4}dxdt\\
&\quad+\int_0^T\int\zeta\sum_{k,j=1}^N[|\psi_j|^2\partial v_k/\partial r-\operatorname{Re}\psi_j\overline{\psi_k}\partial v_{jk}/\partial r]dxdt.
\end{aligned}$$

Let us denote by L the last term appearing here. If we could show that $L\geqq 0$, then, since ψ_j and $\partial\psi_j/\partial r$ are bounded in $L_2(\mathbb{R}^3)$, conclusions i) and ii) of the theorem would follow immediately. To accomplish this, we recall that in the spherically symmetric case the potentials v_k, v_{jk} are expressible as

$$v_k=4\pi\left[r^{-1}\int_0^r s^2|\psi_k(s,t)|^2ds+\int_r^\infty s|\psi_k(s,t)|^2ds\right],$$

$$v_{jk}=4\pi\left[r^{-1}\int_0^r s^2\psi_j\overline{\psi_k}(s,t)ds+\int_r^\infty s\psi_j\overline{\psi_k}(s,t)ds\right].$$

Hence we have

$$\partial v_k/\partial r=-4\pi r^{-2}\int_0^r s^2|\psi_k(s,t)|^2ds$$

and

$$\partial v_{jk}/\partial r = -4\pi r^{-2} \int_0^r s^2 \psi_j \bar{\psi_k}(s,t)\,ds\,.$$

Thus the expression for L can be written as

$$(4\pi)^{-1}L = \int_0^T\int (-\zeta r^{-2}) \left[\sum_{j=1}^N |\psi_j|^2 \sum_{k=1}^N \int_0^r s^2 |\psi_k|^2 ds - \sum_{j,k=1}^N \operatorname{Re} \bar{\psi_j} \psi_k \int_0^r s^2 \psi_j \bar{\psi_k}\, ds \right] dx\,dt\,.$$

Recall that $\zeta \leqq 0$. Let J be the expression in the brackets above. We show that $J \geqq 0$. We have by the Schwarz inequality

$$\begin{aligned} J &\geqq \sum_{j=1}^N |\psi_j|^2 \sum_{k=1}^N \int_0^r s^2 |\psi_k|^2 ds \\ &\quad - \sum_{j,k=1}^N |\bar{\psi_j} \psi_k| \left(\int_0^r s^2 |\psi_j|^2 ds \right)^{\frac{1}{2}} \left(\int_0^r s^2 |\psi_k|^2 ds \right)^{\frac{1}{2}} \\ &\geqq \sum_{j=1}^N |\psi_j|^2 \sum_{k=1}^N \int_0^r s^2 |\psi_k|^2 ds - \left(\sum_{j,k=1}^N |\psi_j|^2 \int_0^r s^2 |\psi_k|^2 ds \right)^{\frac{1}{2}} \cdot \\ &\quad \left(\sum_{j,k=1}^N |\psi_k|^2 \int_0^r s^2 |\psi_j|^2 ds \right)^{\frac{1}{2}} \\ &\geqq 0\,. \end{aligned}$$

Hence L is nonnegative, which proves i) and ii).

It remains to establish iii). We put

$$Q_R[\psi_j(t)] = \int_{|x| \leqq R} |\psi_j(x,t)|^2 dx\,.$$

From [2] we have the conservation law

$$\tfrac{1}{2}\, \partial |\psi_j|^2/\partial t = \operatorname{Im} \sum_{k=1}^3 (\partial/\partial x_k) \left\{ \frac{1}{2} \bar{\psi_j}\, \partial \psi_j/\partial x_k - (4\pi)^{-1} \sum_{l=1}^N v_{jl}\, \partial \bar{v_{jl}}/\partial x_k \right\}.$$

We sum this over j and integrate over $|x| \leqq \varrho$:

$$\begin{aligned} \tfrac{1}{2} \sum_{j=1}^N (\partial/\partial t) Q_\varrho[\psi_j(t)] &= \tfrac{1}{2} \operatorname{Im} \sum_{j=1}^N \int_{|x|=\varrho} \bar{\psi_j}\, \partial \psi_j/\partial r\, dS_x \\ &\quad - \frac{1}{4\pi} \operatorname{Im} \sum_{j,l=1}^N \int_{|x|=\varrho} v_{jl}\, \partial \bar{v_{jl}}/\partial r\, dS_x\,. \end{aligned}$$

The last expression vanishes identically since $\bar{v_{jl}} = v_{lj}$. This relation is integrated with respect to ϱ over $R_1 < \varrho < R_2$, where $R_1 > 0$; we get

$$\sum_{j=1}^N (\partial/\partial t) \int_{R_1}^{R_2} Q_\varrho[\psi_j(t)]\, d\varrho = \operatorname{Im} \sum_{j=1}^N \int_{R_1 \leqq |x| \leqq R_2} \bar{\psi_j} (\partial \psi_j/\partial r)\, dx\,.$$

Next, we integrate this with respect to t over $t_1 \leqq t \leqq \tau$, obtaining

$$\sum_{j=1}^{N}\left(\int_{R_1}^{R_2} Q_\varrho[\psi_j(\tau)]d\varrho - \int_{R_1}^{R_2} Q_\varrho[\psi_j(t_1)]d\varrho\right)$$

$$= \sum_{j=1}^{N} \operatorname{Im}\int_{t_1}^{\tau} \int_{R_1 \leqq |x| \leqq R_2} \psi_j^-(\partial\psi_j(x,s)/\partial r)dxds\,.$$

This is integrated once again with respect to τ over $t_1 \leqq \tau \leqq t$. The result can be written as

$$(t-t_1)\int_{R_1}^{R_2}\sum_{j=1}^{N} Q_\varrho[\psi_j(t_1)]d\varrho = \sum_{j=1}^{N}\int_{t_1}^{t}\int_{R_1}^{R_2} Q_\varrho[\psi_j(\tau)]d\varrho d\tau$$

$$-\operatorname{Im}\int_{t_1}^{t}(t-s)\sum_{j=1}^{N}\int_{R_1 \leqq |x| \leqq R_2} \psi_j^-(\partial\psi_j(x,s)/\partial r)(x,s)dxds$$

which leads to the inequality

$$(t-t_1)(R_2-R_1)\sum_{j=1}^{N} Q_{R_1}[\psi_j(t_1)]$$

$$\leqq (R_2-R_1)\int_{t_1}^{t}\sum_{j=1}^{N} Q_{R_2}[\psi_j(\tau)]d\tau$$

$$+\tfrac{1}{2}\sum_{j=1}^{N}\int_{t_1}^{t}(t-s)\int_{R_1 \leqq |x| \leqq R_2}(|\psi_j|^2 + |\partial\psi_j/\partial r|^2)dxds\,.$$

We now choose $t_1 = t-1$ and apply ii) to conclude that iii) is valid.

We remark that the assumption of spherical symmetry was only needed to show that the last term in the expression (3) was nonnegative (for $\zeta < 0$).

For the single Equation (1) no such assumption is necessary. In fact, in this case we have the following identity:

Lemma. *A solution ψ of finite energy of* (1) *satisfies the identity*

$$-(d/dt)\operatorname{Im}\int \zeta\psi\psi_r^- dx = -\int \zeta'|\psi_r|^2 dx$$

$$-\int \zeta|x|^{-1}(|\nabla\psi|^2 - |\psi_r|^2)dx - 2\pi\zeta(0)|\psi(0,t)|^2$$

$$+\int |\psi|^2[\tfrac{1}{2}\Delta\zeta|x|^{-1} - \zeta'|x|^{-2} + \tfrac{1}{4}\Delta\zeta']dx$$

$$-(8\pi)^{-1}\int \zeta'|\nabla v|^2 dx - (4\pi)^{-1}\int(\zeta|x|^{-1} - \zeta')(\partial v/\partial r)^2 dx\,, \tag{5}$$

for ζ as above.

Now again choose $\zeta < 0$ as in (4). Since $\zeta' < 0$ and since by direct calculation

$$\zeta|x|^{-1} - \zeta' = -(2r^2 + 2r + 1)/2r(r+1)^2 \leqq 0$$

we conclude, as in ii) of Theorem 1, that for any $R > 0$,

$$\int_0^\infty \int_{|x| \leqq R} [|\psi|^2 + |\partial\psi/\partial r|^2]dxdt < \infty\,. \tag{6}$$

To bound the gradient of ψ, we put $\zeta \equiv -1$ in (5) and obtain

$$-(d/dt)\,\mathrm{Im}\int \psi\bar{\psi}_r\,dx = \int |x|^{-1}(|\nabla\psi|^2 - |\psi_r|^2)dx + 2\pi|\psi(0,t)|^2$$
$$+(4\pi)^{-1}\int |x|^{-1}v_r^{\,2}dx\,.$$

Thus by (6), $\int_0^\infty \int_{|x|\leqq R} |\nabla\psi|^2 dx dt$ is finite. Hence we have

Theorem 2. *Let ψ be a solution of* (1) *with finite energy. Then*

i) $\int_0^\infty |\psi(0,t)|^2 dt < \infty\,,$

ii) $\int_0^\infty \int_{|x|\leqq R} [|\psi|^2 + |\nabla\psi|^2] dx dt < \infty\,.$

iii) $\int_0^\infty \int_{\mathbb{R}^3} |x|^{-1} v_r^{\,2} dx dt < \infty\,,$

iv) $\lim_{t\to\infty} Q_R[\psi(t)] = \lim_{t\to\infty} \int_{|x|\leqq R} |\psi|^2 dx = 0\,.$

Of course, iii) is the direct analogue of Morawetz' estimate [5] (cf. also [6]). Conclusion iii) seems to be very weak here. In fact, it is not difficult to show, by considering the spherical means of v that iii) implies

$$\int_0^\infty Q_R^2[\psi(t)]dt < \infty$$

which is a trivial result in view of ii), iv) above.

II. Nonexistence of Scattering

Consider again the equation

$$i\partial\psi/\partial t = \Delta\psi - v(\psi)\psi \tag{1'}$$

where

$$v(\psi) = \int |x-y|^{-1}|\psi(y,t)|^2 dy$$

and where we have changed the coefficient of $\Delta\psi$ to unity for simplicity. We call a solution ψ of (1') *asymptotically free* if there exists a free solution ψ_+ (a solution of the linear Schrödinger equation) satisfying

$$\|\psi(t) - \psi_+(t)\|_2 \to 0 \quad \text{as} \quad t\to\infty\,.$$

We assume the solution ψ itself has data in L_2 and that the free state ψ_+ has data in $L_1 \cap L_2$.

Following [3], [7] we then have

Theorem 3. *The only solution of* (1') *which is asymptotically free is* $\psi \equiv 0$.

Proof. Suppose that $\psi \not\equiv 0$ is a solution of (1′) which has an asymptotic free state ψ_+. We form the expression

$$H(t) = \operatorname{Im} \int \psi \bar{\psi}_+ dx$$

and calculate directly

$$\begin{aligned} \dot{H}(t) &= \operatorname{Re} \int v(\psi)\psi \bar{\psi}_+ dx \\ &= \int v(\psi_+)|\psi_+|^2 dx + \int |\psi_+|^2 (v(\psi) - v(\psi_+)) dx \\ &\quad + \operatorname{Re} \int \bar{\psi}_+ v(\psi)(\psi - \psi_+) dx . \end{aligned} \tag{7}$$

Put

$$J_1 = \int |\psi_+|^2 (v(\psi) - v(\psi_+)) dx$$
$$J_2 = \operatorname{Re} \int \bar{\psi}_+ v(\psi)(\psi - \psi_+) dx .$$

As is well-known, ψ_+ satisfies the estimate

$$\|\psi_+(t)\|_\infty = 0(t^{-3/2}) \quad \text{as} \quad t \to \infty .$$

Hence we have, for $t \geqq 1$, say,

$$\begin{aligned} |J_1| &\leqq \|\psi_+(t)\|_\infty^{2/3} \int |\psi_+|^{4/3} |v(\psi) - v(\psi_+)| dx \\ &\leqq \text{const.}\, t^{-1} \|\psi_+^{4/3}(t)\|_{3/2} \|v(\psi) - v(\psi_+)\|_3 \\ &\leqq \text{const.}\, t^{-1} \|\psi_+(t)\|_2^{4/3} \|\psi(t) - \psi_+(t)\|_2 \end{aligned}$$

where we have used a Sobolev-type inequality from [1] to estimate the L_3-norm of the expression $v(\psi) - v(\psi_+)$. Thus

$$|J_1| = o(t^{-1}) \quad \text{as} \quad t \to \infty .$$

Similarly we have

$$\begin{aligned} |J_2| &\leqq \|\psi_+(t)\|_\infty^{2/3} \int |\psi_+|^{1/3} v(\psi) |\psi - \psi_+| dx \\ &\leqq \text{const.}\, t^{-1} \|\psi_+^{1/3}(t)\|_6 \|v(\psi)(t)\|_3 \|\psi(t) - \psi_+(t)\|_2 \\ &\leqq \text{const.}\, t^{-1} \|\psi_+(t)\|_2^{1/3} \|\psi(t)\|_2 \|\psi(t) - \psi_+(t)\|_2 \\ &= o(t^{-1}) \quad \text{as} \quad t \to \infty . \end{aligned}$$

Hence from (7) we have

$$\dot{H}(t) \geqq \int v(\psi_+)|\psi_+|^2 dx - o(t^{-1}) \tag{8}$$

for all $t \geqq 1$, say. Now let $k > 0$ be arbitrary. Then

$$v(\psi_+) = \int |x-y|^{-1} |\psi_+(y,t)|^2 dy \geqq \int_{|y|<kt} |x-y|^{-1} |\psi_+(y,t)|^2 dy .$$

Hence for $|x| < kt$ we certainly have

$$v(\psi_+) \geqq (2kt)^{-1} \int_{|y|<kt} |\psi_+(y,t)|^2 dy .$$

It follows that

$$\int v(\psi_+)|\psi_+|^2 dx \geqq \int_{|x|<kt} v(\psi_+)|\psi_+|^2 dx$$
$$\geqq (2kt)^{-1} \left(\int_{|x|<kt} |\psi_+(x,t)|^2 dx \right)^2 .$$

However, from [7] we have the result

$$\lim_{t\to\infty} \int_{|x|<kt} |\psi_+|^2 \, dx = \int_{|\xi|<k/2} |\hat{\varphi}_+(\xi)|^2 d\xi$$

where $\hat{\varphi}_+$ denotes the Fourier transform of the initial data φ_+ of ψ_+. Since ψ was assumed nontrivial, there is some value of k for which this limit does not vanish. Then from (8) we have that there exists a positive constant c_0 such that

$$\dot{H}(t) \geqq c_0/t$$

for all sufficiently large t. Since the left-hand side here is integrable in time, we have a contradiction, which proves the theorem.

We remark finally that for solutions of Equation (2), a simple corresponding theorem does not seem to be readily available. This is due to the fact that the dominant term in the proof of Theorem 3 vanishes identically in equations (2), i.e. when $k=j$ in Equation (2), the resulting nonlinear term disappears.

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