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Mixing Properties in Lattice Systems

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Abstract. We study mixing or spatial cluster properties and some of their consequences in classical lattice systems, in particular complete regularity and the weaker notion of strong mixing. Introducing the notion of reflection positivity as a generalization of T-positivity of [1], we construct a generalized transfer matrix P and relate complete regularity to a spectral gap in P. It is shown that all reflection invariant Ising systems with n.n. and ferromagnetic n.n.n. interaction satisfy reflection positivity. For Ising ferromagnets with reflection positivity, exponential decay of the truncated 2-point function implies complete regularity. In particular, the 2-dimensional spin-1/2 Ising model is completely regular, except at the critical point. This complements a result of [2] that strong mixing fails at the critical point of this model and in this case verifies the suggestion of Jona-Lasinio [3] that critical behaviour should be linked with failure of strong mixing. We then show that strong mixing imposes severe restrictions on the possible form of limits of block spins. Strong mixing in each direction allows only *independent Gaussians* as non-zero limit if the 2-point function exists; strong mixing in a single direction only will allow infinitely divisible distributions.

1. Introduction

A classical square lattice system is a set of random variables X_k, $k \in \mathbb{Z}^d$ ("spin at site k"). It can be described by a probability measure μ on a measure space with σ-algebra Σ. We always assume translation invariance. We put $\mathfrak{H} = L^2(\mu)$ and denote by Ω the function identically 1. The smallest sub-σ-algebra of Σ generated by $\{X_k; m \leqq k_1 \leqq n\}$ is denoted by Σ_m^n, by $\mathfrak{H}_m^n$ the corresponding subspace of $\mathfrak{H}$ and by E_m^n the projector onto it. We take $\Sigma = \Sigma_{-\infty}^{\infty}$.

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We define the "mixing coefficient" $\alpha(n)$ by

$$\alpha(n) \equiv \sup_{A \in \Sigma^{0}_{-\infty},\, B \in \Sigma^{\infty}_{n}} |\mu(A \cap B) - \mu(A)\mu(B)|. \tag{1.1}$$

The system is *strongly mixing* (in the 1-direction) if $\alpha(n) \to 0$ for $n \to \infty$. Alternatively, one can consider

$$\hat{\alpha}(n) \equiv \sup_{\xi, \eta} |\langle \xi\eta \rangle - \langle \xi \rangle \langle \eta \rangle| \, \|\xi\|_{\infty}^{-1} \, \|\eta\|_{\infty}^{-1} \tag{1.2}$$

where ξ, η are bounded and in $\mathfrak{H}^{0}_{-\infty}, \mathfrak{H}^{\infty}_{n}$ respectively. Then [4, Th. 17.2.1] $\alpha(n) \leqq \hat{\alpha}(n) \leqq 16\,\alpha(n)$.

Replacing $\|\cdot\|_{\infty}$ by $\|\cdot\|_2$ in Equation (1.2) one obtains the "maximal correlation coefficient" $\varrho(n)$. The system is *completely regular* (in the 1-direction) if $\varrho(n) \to 0$ for $n \to \infty$. The latter implies strong mixing.

This extends the well-known 1-dimensional concepts [4, 5]. Dobrushin [6] has given a slightly different extension using sets different from half spaces.

Definition. Let a system be invariant under the reflection $(k_1, \boldsymbol{k}) \mapsto (-k_1, \boldsymbol{k})$ and let ϱ denote the corresponding operator in $\mathfrak{H}$. If, for some $r \geqq 0$,

$$E_r^{\infty} \varrho E_r^{\infty} \geqq 0 \tag{1.3}$$

the system is said to satisfy *reflection positivity* (r.p.) (strong r.p. if $r = 0$).

This notion generalizes the concept of T-positivity introduced in [1] as a sharpening of the positivity condition of [7]. It is easy to see that Markov property plus reflection invariance implies strong r.p. In Section 2 we prove our main result on mixing properties.

Theorem 1. *Let a lattice system on $\mathbb{Z}^d$ satisfy the FKG inequalities and reflection positivity. If the truncated 2-point-function decreases exponentially (in the 1-direction), then the system is completely regular (in the 1-direction) with exponentially decreasing $\varrho(n)$.*

For Ising ferromagnets with reflection invariant next nearest neighbour (n.n.n.) interaction the assumptions are fulfilled, and for n.n. interaction also the converse holds.

In Section 3 we derive a criterion for strong r.p. in Ising models in terms of the interaction matrix, and Section 4 deals with consequences of strong mixing for the possible form of the limit distributions of block spins.

2. Generalized Transfer Matrix and Complete Regularity

Let r.p. hold, and let $\mathfrak{H}_{\mathfrak{N}}$ be the null space of $E_r^{\infty} \varrho E_r^{\infty}$ in $\mathfrak{H}_r^{\infty}$. By the spectral theorem one can write $\mathfrak{H}_r^{\infty} = \mathfrak{H}_0 \oplus \mathfrak{H}_{\mathfrak{N}}$ such that

$$\varrho_0 \equiv E_r^{\infty} \varrho E_r^{\infty} \restriction \mathfrak{H}_0 > 0. \tag{2.1}$$

Thus ϱ_0^{-1} exists as a (possibly unbounded) operator in $\mathfrak{H}_0$. Denote by E_0 the projector onto $\mathfrak{H}_0$. Similarly as in Section 2 of [1] one finds

Proposition 1. *Let T_τ, $\tau \in \mathbb{Z}$, be the translation operator in the 1-direction and define, on $\varrho_0^{1/2}\mathfrak{H}_0$,*

$$P_\tau \equiv \varrho_0^{1/2} E_0 T_\tau \varrho_0^{-1/2}. \tag{2.2}$$

Then $\{P_\tau\}_0^\infty$ can be extended by continuity to a self-adjoint contraction semi-group. We call $P=P_1$ the generalized transfer matrix[1].

In principle the spectrum of P can lie in $[-1,1]$. Ω belongs to the eigenvalue 1, and if ordinary clustering (mixing) holds, -1 cannot be an eigenvalue. E_Ω denotes the projector onto $\{\lambda\Omega\}$.

Theorem 2. *Let a lattice system on $\mathbb{Z}^d$ satisfy* r.p. *If the spectrum of $P-E_\Omega$ lies in $[-a,a]$ for some $0<a<1$ ("spectral gap"), then the system is completely regular (in the 1-direction) with exponentially decreasing $\varrho(n)$. If, conversely, the system is completely regular and if ϱ_0^{-1} is bounded then P has a spectral gap.*

Proof. Let $\xi\in\mathfrak{H}_{-\infty}^0, \eta\in\mathfrak{H}_n^\infty, n>2r$. With Lemma 2.1 of [1] one shows

$$\begin{aligned}&\langle\xi\eta\rangle-\langle\xi\rangle\langle\eta\rangle\\&=\langle\varrho_0^{1/2}E_0\varrho T_{-r}\bar{\xi},\,(P-E_\Omega)^{n-2r}\varrho_0^{1/2}E_0T_{r-n}\eta\rangle.\end{aligned}\tag{2.3}$$

The r.h.s. is bounded by $\|\xi\|_2\,\|\eta\|_2\,\|P-E_\Omega\|^{n-2r}$, and this implies the first part of the theorem. The second part also follows from Equation (2.3) by using $\|\varrho_0^{1/2}u\|_2\geqq b\|u\|_2$ for some $b>0$. QED.

Theorem 1 now follows from Theorem 2 and the next lemma.

Lemma 1. *Let* r.p., *FKG inequalities and exponential decay of the truncated 2-point function in the 1-direction hold. Then P has a spectral gap.*

Proof. As in [8, Th. VIII. 35] one shows that, for a dense set in $\mathfrak{H}_r^\infty$, $\langle\varrho u, T_\tau u\rangle-|\langle u\rangle|^2$ is bounded by truncated 2-point functions. From this and Equations (2.1/2) one concludes that $\langle v,(P-E_\Omega)^\tau v\rangle$ is exponentially decreasing in τ for a dense set in $\mathfrak{H}_0$. A similar argument as in [9, p. 158] then yields the spectral gap. QED.

A partial converse of Theorem 1 follows from the second part of Theorem 2. If ϱ_0^{-1} is bounded then exponential decay of the truncated 2-point function is also necessary, in particular for n.n. interactions since there $\varrho_0\equiv 1$ by the Markov property.

3. Reflection Positivity in General Ising Models

We consider Hamiltonians of the form $H=sJs+hs$, where

$$sJs=\sum_{j,k\in\mathbb{Z}^d}s(j)J_{jk}s(k),\qquad hs=\sum_k h(k)s(k),$$

and any single spin distribution. We put $\theta j=(-j_1,\mathbf{j})$ and $(\vartheta h)(k)=h(\theta k)$.

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[1] In case of strong r.p. P has the usual properties of a transfer matrix. Indeed, if $Y^{(1)},\ldots,Y^{(n)}$ and their products are in $\mathfrak{H}_0^0$ and if $Y_{i_k}^{(k)}\equiv T_{i_k}Y^{(k)}$, $i_1<\ldots<i_n$, then

$$\langle Y_{i_1}^{(1)}\ldots Y_{i_n}^{(n)}\rangle=\langle Y^{(1)}P^{i_2-i_1}Y^{(2)}\ldots P^{i_n-i_{n-1}}Y^{(n)}\rangle.$$

The proof is similar to that of Corollary 3.3 in [1]

Theorem 3. *A general Ising model with reflection invariant* $\{J_{jk}\}$, *external field and boundary conditions satisfies strong* r.p. *if the matrix* $\{J_{\theta jk}; j_1, k_1 \geqq 1\}$ *is negative semi-definite*[2].

Proof. Absorbing $\pm$ boundary conditions in h we consider the free case; the periodic case is similar. We first treat equally distributed finite discrete spins. It suffices to show that $\langle \exp\{i(f_\alpha - \vartheta f_{\alpha'})\sigma\}\rangle$ is a positive semi-definite matrix in α, α' whenever f_α has compact support in $\{k; k_1 \geqq 0\}$, $\alpha = 1, \ldots, n$. We use finite volume approximations (reflection invariant) and decompose

$$s = s_- + s^0 + s_+ \equiv \vartheta s'_+ + s^0 + s_+$$

where $\operatorname{supp} s_\pm \subset \{k \in \mathbb{Z}^d \cap V; \pm k_1 \geqq 1\}$ and $s^0(k) = 0$ for $k_1 \neq 0$. Then one gets

$$Z_V^{-1} \sum_s \sum_{\alpha\alpha'} \lambda_\alpha \bar{\lambda}_{\alpha'} e^{-\beta H + i(f_\alpha - \vartheta f_{\alpha'})s}$$

$$= Z_V^{-1} \sum_{s^0} e^{-\beta(s^0 J s^0 + h s^0)} \sum_{s'_+ s_+}$$

$$\Big[\sum_{\alpha'} \bar{\lambda}_{\alpha'} \exp\{-i f_{\alpha'}(s^0 + s'_+)$$

$$-\beta(s'_+ J s'_+ + 2 s'_+ J s^0 + h s'_+)\}\Big]$$

$$\exp\{-2\beta(\vartheta s'_+) J s_+\}$$

$$\Big[\sum_\alpha \lambda_\alpha \exp\{i f_\alpha(s^0 + s_+)$$

$$-\beta(s_+ J s_+ + 2 s_+ J s^0 + h s_+)\}\Big].$$

The last square bracket is of the form $g(s_+, s^0)$, the first then is $\bar{g}(s'_+, s^0)$. Hence the r.h.s. is non-negative if $\{\exp[-2\beta(\vartheta s'_+) J s_+]\}$ is a positive semi-definite matrix in s'_+, s_+, and this holds if $\{(\vartheta s'_+) J s_+\}$ is negative semi-definite [10]. The latter follows from the assumption.

For $V \to \infty$ the result follows. In the general spin case matrices in s'_+, s_+ are replaced simply by integral kernels. Q.E.D.

Examples. 1) N. n. Interaction: $J_{\theta jk} = 0$ for j_1, $k_1 \geqq 1$.

2) *Ferromagnetic n.n.n. Interaction:* $J_{jk} \leqq 0$ for $|j-k| = 2$, $J_{jk} = 0$ for $|j-k| > 2$. Then

$$\sum_{j_1, k_1 \geqq 1} \bar{\lambda}_j \lambda_k J_{\theta jk} = \sum_k |\lambda_{(1,k)}|^2 J_{(-1,k)(1,k)} \leqq 0 .$$

3) *General Solution for* $d = 1$. By translation invariance, $J_{jk} \equiv J_{k-j}$. Then the conditon resembles a moment problem, yielding

$$J_k = -c\delta_{k2} - \int_{\mathbb{R}} x^k d\nu(x), \qquad k \geqq 2, \tag{3.1}$$

[2] A similar result was found independently by J. Fröhlich (private communication)

where $c \geqq 0$ and where ν is a positive measure (possibly infinite at 0). A simple example for Equation (3.1) is given by

$$J_{jk} = -J_0 |j-k|^{-\alpha}, \quad \alpha > 0, \quad j \neq k, \quad J_0 > 0. \tag{3.2}$$

Dyson [11] has proved existence of phase transitions for such models in the case of $1 < \alpha < 2$ and spin 1/2.

4. Strong Mixing and Limit Distributions of Block Spins

For given $k \in \mathbb{Z}^d$ and $N \in \mathbb{N}$ let V_k^N be the cube of side length N and corner point Nk, $V_k^N \equiv \{j \in \mathbb{Z}^d;\ Nk_\alpha \leqq j_\alpha < N(k_\alpha + 1),\ \alpha = 1, \ldots, d\}$. Let $\{B_N\}$ be a sequence of positive numbers, $B_N \to \infty$, and define "block spins" by

$$\zeta_k^N \equiv B_N^{-1} \sum_{j \in V_k^N} (X_j - \langle X_j \rangle). \tag{4.1}$$

In statistical mechanics one takes $B_N = N^{d\varrho/2}$, $1 \leqq \varrho \leqq 2$.

It is known for $d=1$ that if $\{\zeta_k^N\}$ converges weakly for $N \to \infty$ the limits must be independent stable distributions if strong mixing holds [4]. If one tries to carry over the proof to $d \geqq 2$ one gets difficulties with the surface contributions in Equation (4.1). For $d=1$ the latter arise from a single point and converge weakly to zero for $B_N \to \infty$. An analysis of the proof in [4] shows that this fact, together with strong mixing, is responsible for stability and independence. For $d > 1$, however, the number of surface points increases with N and one cannot, a priori, conclude weak convergence to zero. Another complication is that one may have strong mixing in one direction only. With a variation of the methods used in [4] we can prove the following results.

Lemma 2. *Let strong mixing in the 1-direction hold, let B_N/B_{mN} be bounded in N, for each m, and assume that the block spins $\{\zeta_k^N\}$ converge weakly to some $\{\zeta_k\}$. If the contribution in Equation* (4.1) *from the surface $j_1 = Nk_1$ converges weakly to zero as $N \to \infty$ then each ζ_k is infinitely divisible and different hyperplanes $k_1 = const$ are independent (i.e. $\Sigma \lambda_j \zeta_{(k_1, j)}$ and $\Sigma \lambda'_j \zeta_{(k'_1, j)}$ are independent for $k_1 \neq k'_1$).*

Lemma 3. *Let strong mixing hold in each of the d directions and assume that $\{\zeta_k^N\}$ converges weakly to some $\{\zeta_k \neq 0\}$. Then ζ_k and ζ_j are independent if $|k-j| \geqq 2$. If the surface contributions in Equation* (4.1) *converge weakly to zero then the $\{\zeta_k\}$ are independent and there is an α, $0 < \alpha \leqq 2$ such that each ζ_k is stable with exponent α and $B_N = N^{d/\alpha} h(N)$ where $h(N)$ is slowly varying.*

Under additional assumptions the surface contributions can be controlled, as in the following results.

Theorem 4. *Let a lattice system $\{X_k\}$ on $\mathbb{Z}^d$ be strongly mixing in each of the d directions. Let $B_N = N^{d\varrho/2} h(N)$, where $\varrho \geqq 1$ and where $h(N)$ is slowly varying, and let the block spins in Equation* (4.1) *converge weakly to some $\{\zeta_k \neq const\}$. If the second moment of ζ_0 exists then the $\{\zeta_k\}$ are independent Gaussians and $\varrho = 1$.*

Proof. The system $\{\zeta_k\}$ is seen to be strongly mixing and to be stable under forming block spins if B_N is replaced by $N^{d\varrho/2}$. This and the first independence statement in Lemma 3 yield $\langle\zeta_k\rangle=0$, and one estimates the second moment of the surface contribution (to the block spins of $\{\zeta_k\}$) by $N^{-d\varrho}N^{d-1}3^{d-1}\langle\zeta_0\zeta_0\rangle$. This implies weak convergence to zero, and thus Lemma 3 applies to $\{\zeta_k\}$. QED.

In contrast to the case $d=1$, Theorem 4 assumes a special form of B_N and existence of $\langle\xi_0\zeta_0\rangle$. If the latter is dropped one cannot apply Lemma 3.

If $\langle X_0X_k\rangle^t$ has at least a power-like fall-off in $|k|$ with exponent $-d+2-\eta$ for some $\eta>0$ then it is easy to see that the surface contributions tend to zero if B_N is as in Theorem 4. So Lemmas 2 and 3 can be applied. The same holds if $\langle X_0X_k\rangle^t\geqq 0$ and $\langle\zeta_0^N\zeta_0^N\rangle\to\langle\zeta_0\zeta_0\rangle<\infty$ where no assumption on B_N is needed.

5. Discussion

A general Ising ferromagnet satisfies the FKG inequalities [8]. If in addition the interaction is n.n. or n.n.n. also r.p. holds. For these models the truncated n-point functions decay exponentially for all β when $h\neq 0$ and for $\beta<\beta_0\leqq\beta_c$ when $h=0$ [12]. Hence in these cases one has complete regularity, by Theorem 1.

The 2-dimensional spin $-1/2$ n.n. Ising model is completely regular for $\beta<\beta_c$ and for $\beta>\beta_c$ in the two pure phases. At $\beta=\beta_c$ complete regularity fails. This follows from the decay properties of the truncated 2-point function [13, 14]. By the results of [2], which complement ours, also strong mixing fails at $\beta=\beta_c$, and hence the proposal of [3] is verified for this model. It may be that for physically interesting systems complete regularity and strong mixing are equivalent; it would be interesting to prove this or to find a counter example. The last model in Section 3 might be a candidate.

For Ising ferromagnets with at most n.n.n. interaction Section 4 shows that the block spins can only converge to independent stable laws for all β if $h\neq 0$ and for $\beta<\beta_0<\beta_c$ if $h=0$. Whenever the 2-point function of the limit exists one only gets Gaussians. For a class of 2-dimensional spin $-1/2$ Ising models convergence of the block spins has actually been proved [15, 16]. We take the results of Section 4 as additional support for the proposal of [3] to link critical behaviour with failure of strong mixing since otherwise the idea of block spins and renormalization group would not work in the expected way. Another approach to critical behaviour has been taken by Klauder [17] who essentially suggested to link critical behaviour to a metric introduced in [18].

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