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Implementation of Automorphisms and Derivations of the CAR-Algebra

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Abstract. The implementability of automorphisms and derivations of the CAR-algebra in a pure quasifree state is discussed in detail. Especially the properties of the implementing operators are investigated, and an explicit construction is given. Extending a result from [2] for the CCR-algebra, we get a new necessary condition for a derivation to be implementable by a selfadjoint operator.

1. Introduction

Whereas there are many results about the implementability of automorphisms and derivations of the form $\psi(f)\to\psi(Sf)$, little is known about the properties of the implementing operators and their construction. These questions can be answered by applying the methods of constructive quantum field theory. Some useful and interesting new results come out. Among them there are: Estimates, corresponding to the N_τ-estimates in constructive quantum field theory, criteria for selfadjointness and an explicit construction of a bilinear form which implements a given automorphism. If this automorphism satisfies the criterion of Shale and Stinespring [3], the implementing form can be extended to an unique bounded operator, which differs from an unitary only by a constant factor. If the automorphism does not map any annihilation operator on a pure creation operator, the bilinear form can be given in a closed form, in which it can be compared with the formal expression resulting from the linked cluster theorem.

In the last section a new necessary condition is found for a derivation to be implementable by a selfadjoint operator, analog to Hochstenbach's result in the scalar field case [2]. It proves the conjecture that the first term in the perturbation serie decides over the implementability. It follows the explicit construction of the implementing operators with the method of dressing transformations and a comparison of this method with approximation techniques.

The results presented in this paper are essentially contained in the authors thesis [1]. Some of these results have been obtained also by Ruijsenaars [4].

2. Notation

H complex Hilbert space, $\mathscr{C}_0(H)$ Clifford algebra over H, $\mathscr{C}(H)$ C^*-norm-completion of $\mathscr{C}_0(H)$, $\psi: H \to \mathscr{C}(H)$ antilinear injection with

$$(\psi(f)+\psi(g)^*)^2=(f,g) \quad \text{(Canonical Anticommutation Relations)}.$$

To any invertible operator $S \in \mathscr{B}(H)$ corresponds an unique automorphism α_S of $\mathscr{C}_0(H)$ with the properties

$$\alpha_S\psi(f)^*=\psi(Sf)^*, \quad \alpha_S\psi(f)=\psi(S^{-1*}f) \quad (f\in H).$$

In the same way d_T, $T\in\mathscr{B}(H)$ denotes the derivation of $\mathscr{C}_0(H)$ with

$$d_T\psi(f)^*=\psi(Tf)^*, \quad d_T\psi(f)=-\psi(T^*f).$$

ω_P, P projector in H, is the pure quasifree gauge invariant state of $\mathscr{C}(H)$ with the two point function

$$\omega_P(\psi(f)\psi(g)^*)=(f,Pg).$$

Let $(\mathscr{H},\pi,\Omega)$ be the GNS-construction to ω_P. The one particle space is

$$\mathscr{H}_1=PH\oplus\overline{QH} \quad (Q=1-P, \overline{} \text{ denotes the conjugate Hilbert space}),$$

and $\mathscr{H}$ is the antisymmetric tensor space over $\mathscr{H}_1$:

$$\mathscr{H}=\Lambda\mathscr{H}_1=\bigoplus_{n=0}^{\infty}\Lambda_n\mathscr{H}_1.$$

Denoting the identical mapping from H into $\mathscr{H}_1$ with I, we get the annihilation operators

$$a(If):=\psi(Pf)+\psi(Qf)^* \quad (f\in H).$$

For $p\geqq 1$, C_p denotes the trace ideal $\{A\in\mathscr{B}(H) \mid \|A\|_p^p=\operatorname{Tr}|A|^p<\infty\}$. Especially C_2 is the Hilbert-Schmidt class, and C_1 is the trace class.

3. Invariance of the Ground State

Let S be an invertible bounded operator in H, which commutes with P. Then ω_P is invariant under α_S, and α_S is implemented by the densely defined closable operator

$$\Gamma(S): A\Omega \to \alpha_S(A)\Omega, \quad A\in\mathscr{C}_0(H).$$

Relating $\Gamma(S)$ to the tensor product structure of $\mathscr{H}=\Lambda\mathscr{H}_1$, we get

$$\Gamma(S)=\Lambda(I(PS+QS^{-1*})I^{-1}).$$

The corresponding result for derivations d_T, $T\in\mathscr{B}(H)$, $PT=TP$, is:

$$d\Gamma(T)=d\Lambda(I(PT-QT^*)I^{-1}),$$

where $d\Gamma(T)$ denotes the densely defined closable operator

$$A\Omega\to d_T(A)\Omega, \quad A\in\mathscr{C}_0(H).$$

4. Weak Implementability of Derivations

Let $T \in \mathscr{B}(H)$. The formal expression

$$d\Gamma_0(T) = \sum_{i,j} (f_j, Tf_i)\{(f_i, Pf_j) - \psi(f_i)\psi(f_j)^*\},$$

$$(f_i) \quad \text{basis of} \quad H,$$

implements the derivation d_T. $d\Gamma_0(T)$ has a well defined meaning as a bilinear form on $\overline{\mathscr{C}_0(H)\Omega} \times \mathscr{C}_0(H)\Omega$. If $PT = TP$, $d\Gamma_0(T)$ is the restriction of $d\Gamma(T)$.

By translating the N_τ-estimates of constructive quantum field theory (Glimm and Jaffe [5]) into the language of CAR-algebras, we get the following results:

4.1. Proposition. *Let τ be a selfadjoint operator in H with $P\tau = \tau P$ and $\tau(P-Q) > 0$. Set for $\alpha \in \mathbb{R}$*

$$N_\tau^\alpha := 0_{|\{\lambda\Omega | \lambda \in \mathbb{C}\}} + (d\Gamma(\tau)_{|\{\Omega\}^\perp})^\alpha .$$

a) Let T be an operator in H which commutes with P. Then:

$$\|N_\tau^{-1/2} d\Gamma(T) N_\tau^{-1/2}\| = \||\tau|^{-1/2} T |\tau|^{-1/2}\| .$$

b) Let $T \in \mathscr{B}(H)$. Then:

$$\|N_\tau^{-1/2} d\Gamma_0(PT)\| \leqq \||\tau|^{-1/2} PT\|_2$$

and

$$\|d\Gamma_0(TP) N_\tau^{-1/2}\| \leqq \|TP|\tau|^{-1/2}\|_2 .$$

It follows from this proposition, that the annihilation term $d\Gamma_0(QTP)$ can be extended to an operator on $D(N_\tau^{-1/2})$ if $QTP|\tau|^{-1/2} \in C_2$. As is easily seen, this operator is closable if and only if $QTP \in C_2$. This corresponds to the fact, that the creation term $d\Gamma_0(PTQ)$ can be extended to an operator on $\mathscr{C}_0(H)$ if and only if $PTQ \in C_2$. These operators will be denoted with $d\Gamma(T)$ too.

The estimates in Proposition 4.1 are useful for the discussion of selfadjointness of $N_\tau + d\Gamma_0(T)$:

4.2. Proposition. *Let T be a bounded selfadjoint operator in H with*

(1) $|\tau|^{-1/2} PTQ \in C_2$,

(2) $|\text{ess spec}(|\tau|^{-1/2}(PTP + QTQ)|\tau|^{-1/2})| < 1$.

Then the form sum $N_\tau + d\Gamma_0(T)$ defines an unique selfadjoint operator, bounded from below.

Proof. We apply [6, Th. II.7] and have to show that $d\Gamma_0(T)$ is bounded relative to N_τ with bound less than 1 in the sense of quadratic forms.

Take $\varepsilon = \mathrm{dist}(1, |\mathrm{ess\,spec}(V)|)$, $V = |\tau|^{-1/2}(PTP+QTQ)|\tau|^{-1/2}$. Let P_ε be the eigenprojector to eigenvalues of $|V|$ greater than $1-2\varepsilon/3$. $P_\varepsilon H$ is finite dimensional and lies in the domain of $|\tau|^{1/2}$. Now:

$$\begin{aligned}(3)\ |(\phi, d\Gamma_0(PTP+QTQ)\phi)| &\leqq |(\phi, d\Gamma_0(|\tau|^{1/2}P_\varepsilon V|\tau|^{1/2})\phi)| \\ &\quad + |(\phi, d\Gamma_0(|\tau|^{1/2}(1-P_\varepsilon)V|\tau|^{1/2})\phi) \\ &\leqq \mathrm{Tr}\,||\tau|^{1/2}P_\varepsilon V|\tau|^{1/2}|\,\|\phi\|^2 + (1-2\varepsilon/3)|(\phi, N_\tau\phi)|,\end{aligned}$$

$$\begin{aligned}(4)\ |(\phi, d\Gamma_0(PTQ)\phi)| &\leqq \|N_\tau^{1/2}\phi\|\,\||\tau|^{-1/2}PTQ\|_2\|\phi\| \\ &\leqq (\varepsilon/6)(\phi, N_\tau\phi) + \mathrm{const}\,\|\phi\|^2\end{aligned}$$

and

$$|(\phi, d\Gamma_0(QTP)\phi)| = |(\phi, d\Gamma_0(PTQ)\phi)|.$$

It follows:

$$|(\phi, d\Gamma_0(T)\phi)| \leqq (1-\varepsilon/3)(\phi, N_\tau\phi) + \mathrm{const}\,\|\phi\|^2 \qquad \text{q.e.d.}$$

4.3. Proposition. *Let T be a bounded selfadjoint operator in H with the properties:*

(1) $PTQ \in C_2$

(2) $|\mathrm{ess\,spec}(PTP+QTQ)|\tau|^{-1/2}| < 1$

Then the operator $N_\tau + d\Gamma(T)$ is selfadjoint and bounded from below.

Proof. It suffices to show that $d\Gamma(T)$ is bounded relative to N_τ in the operator sense with a bound less than 1 [7, V. Th. 4.3, 4]. A slight improvement of 4.1.a gives

$$\|N^{-1/2}d\Gamma(PTP+QTQ)N_\tau^{-1/2}\| \leqq \|(PTP+QTQ)|\tau|^{-1/2}\|$$

($N = N_{P-Q}$ particle number operator in $\mathscr{H}$). $d\Gamma(PTP+QTQ)$ commutes with N, therefore:

$$\|d\Gamma(PTP+QTQ)N_\tau^{-1/2}N^{-1/2}\| \leqq \|(PTP+QTQ)|\tau|^{-1/2}\|.$$

Now we proceed as in the proof of Proposition 4.2. q.e.d.

5. Implementation of Automorphisms

5.1. Theorem. *Let U be an unitary operator in H with $0 \notin \mathrm{ess\,spec}(PUP+QUQ)$. Then there exists a bilinear form $\mathscr{U}$ on $\overline{\mathscr{C}_0(H)\Omega} \times \mathscr{C}_0(H)\Omega$ with the property*

(1) $\mathscr{U}\psi(f)^* = \psi(Uf)^*\mathscr{U}$.

$\mathscr{U}$ is explicitly given by the formula

$$(2)\ (A\Omega, \mathscr{U}B\Omega) = \left(e^{d\Gamma(L)}\prod_{i=1}^{n} a(If_i)\alpha_U(B)^*A\Omega, \Omega\right)$$

with $A, B \in \mathscr{C}_0(H)$, $(f_i)_{i=1,\dots,n}$ *orthonormal basis of* $K = U\,\mathrm{Ker}(PUP+QUQ)$ *and* $L = -U_{21}U_{11}^{-1}$, *where the matrix representation*

$$U = \begin{pmatrix} U_0 & 0 & 0 \\ 0 & U_{11} & U_{12} \\ 0 & U_{21} & U_{22} \end{pmatrix} \text{ corresponds to the decomposition}$$

$$U : U^*K \oplus U^*PK^\perp \oplus U^*QK^\perp \to K \oplus PK^\perp \oplus QK^\perp .$$

Proof. If $\mathscr{U}$ is well defined, (1) is automatically fulfilled. Therefore it suffices to show that the right hand side of (2) vanishes if $B\Omega = 0$. The case of general B can be reduced to the case $B = a(If)$, $f \in H$.

a) Let $f \in U^*K$. Then:

$$\begin{aligned}\alpha_U(a(If))^* &= \alpha_U(\psi(Pf)+\psi(Qf)^*)^* = \psi(UPf)^* + \psi(UQf) \\ &= \psi(QUf)^* + \psi(PUf) = a(IUf) \quad \text{with} \quad Uf \in K .\end{aligned}$$

Therefore: $\prod_{i=1}^{n} a(If_i)a(IUf) = 0$, i.e. the right hand side of (2) vanishes.

b) Let $f \in U^*K^\perp$. Then:

$$\begin{aligned}&e^{d\Gamma(L)} \prod_{i=1}^{n} a(If_i)\alpha_U(a(If)) \\ &= e^{d\Gamma(L)} \prod_{i=1}^{n} a(If_i)\,(\psi(UPf)^* + \psi(UQf)) \\ &= e^{d\Gamma(L)}(\psi(UPf)^* + \psi(UQf)) \prod_{i=1}^{n} a(If_i) \\ &= (\psi((1+L)UPf)^* + \psi((1-L^*)UQf))e^{d\Gamma(L)} \prod_{i=1}^{n} a(If_i) .\end{aligned}$$

We have

$$\begin{aligned}(1+L)UPf &= (U_{11} + U_{21} - U_{21}U_{11}^{-1}U_{11})f = U_{11}f , \\ (1-L^*)UQf &= (U_{12} + U_{22} + U_{11}^{-1}U_{21}^*U_{22})f = U_{22}f ,\end{aligned}$$

where the last equation comes from the unitarity relation

$$U_{21}^*U_{22} + U_{11}^*U_{12} = 0 .$$

It follows that $\psi((1+L)UPf)^* + \psi((1-L^*)UQf)$ is a creation operator. So the r.h.s. of (2) vanishes for $f \in U^*K^\perp$ too. q.e.d.

Remark. If L or equivalently QUP is a Hilbert-Schmidt operator, $\mathscr{U}$ can be extended to an operator on $\mathscr{C}_0(H)\Omega$, and $\mathscr{U}\Omega$ is annihilated by the transformed annihilation operators $\alpha_U(a(If))$, $f \in H$. Therefore:

$$(\omega_P \circ \alpha_U)(A) = \|\mathscr{U}\Omega\|^{-2}(\mathscr{U}\Omega, A\mathscr{U}\Omega) , \qquad \|\mathscr{U}\Omega\|^2 = \det(1+LL^*)$$

and $\|\mathscr{U}\Omega\|^{-1}\mathscr{U}$ is unitary; i.e. one gets the well known implementability criterion of Shale and Stinespring [3]. The explicit form of the transformed vacuum vector, given by the theorem, has been found independently also by Ruijsenaars [4]. For the case $K=\{0\}$ this result goes back to Schroer, Seiler and Swieca [8].

In the case $K=\{0\}$ no annihilation operator becomes a pure creation operator. Then for $U=e^{tA}$, $t\in\mathbb{R}$, $A=-A^*\in\mathscr{B}(H)$, a candidate for $\mathscr{U}$ is the formal expression

$$\mathscr{U}_{\text{formal}}(t)=(\Omega, e^{td\Gamma_0(A)}\Omega)^{-1}e^{td\Gamma_0(A)}\,.$$

$\mathscr{U}_{\text{formal}}$ can be defined as a bilinear form in the sense of formal power series in t. The linked cluster theorem [9] states the following identity (in the sense of formal power series):

$$\begin{aligned}&(\Omega, e^{td\Gamma_0(A)}\Omega)^{-1}e^{td\Gamma_0(A)}\\ &=:\exp\left\{\sum_{n=0}\frac{t^n}{n!}[d\Gamma_0(A)^n_c-(\Omega, d\Gamma_0(A)^n_c\Omega)]\right\}:\end{aligned}$$

Here $d\Gamma_0(A)^n_c$ is the connected part of $d\Gamma_0(A)^n$, and the double dots denote normal ordering with respect to the annihilation and creation operators $a(If)$ respectively $a(If)^*$, $f\in H$. To discuss the relations between $\mathscr{U}_{\text{formal}}$ and $\mathscr{U}$ we need the following result:

5.2. Proposition. *Let U be an unitary operator in H with $PUP+QUQ$ invertible in $\mathscr{B}(H)$. Then the implementing bilinear form $\mathscr{U}$ from* 5.1 *is expressible in closed form:*

$$\mathscr{U}=:\exp\{d\Gamma_0((1+(U-1)Q)^{-1}(U-1))\}:.$$

Remark. This result too has been found independently by Ruijsenaars [4]. Nevertheless, we present a proof, because our methods are quite different.

Proof. Normal ordering of an exponential of $d\Gamma_0(A)$, $A\in\mathscr{B}(H)$, can be performed in the following steps: ($A=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}$ matrix representation with respect to the decomposition $H=PH\oplus QH$)

$$(1)\ \ :e^{d\Gamma_0(A)}:=e^{d\Gamma_0\begin{pmatrix}0 & A_{12}\\ 0 & 0\end{pmatrix}}:e^{d\Gamma_0\begin{pmatrix}A_{11} & 0\\ 0 & A_{22}\end{pmatrix}}:e^{d\Gamma_0\begin{pmatrix}0 & 0\\ A_{12} & 0\end{pmatrix}},$$

$$(2)\ \ d\Gamma_0\begin{pmatrix}A_{11} & 0\\ 0 & A_{22}\end{pmatrix}=dA\left(I\begin{pmatrix}A_{11} & 0\\ 0 & -A_{22}^*\end{pmatrix}I^{-1}\right),$$

$$(3)\ \ :e^{dA(B)}:=A(1+B),\qquad B\in\mathscr{B}(\mathscr{H}_1),$$

$$(4)\ \ A\left(1+I\begin{pmatrix}A_{11} & 0\\ 0 & -A_{22}^*\end{pmatrix}I^{-1}\right)=\Gamma\begin{pmatrix}1+A_{11} & 0\\ 0 & (1-A_{22})^{-1}\end{pmatrix}.$$

Using $(1+(U-1)Q)^{-1}(U-1)=\begin{pmatrix}U_{11}-1-U_{12}U_{22}^{-1}U_{21} & U_{12}U_{22}^{-1}\\ U_{22}^{-1}U_{21} & 1-U_{22}^{-1}\end{pmatrix}$ *we get:*

$$\begin{aligned}\mathscr{U}=e^{d\Gamma_0\begin{pmatrix}0 & U_{12}U_{22}^{-1}\\ 0 & 0\end{pmatrix}}\Gamma\begin{pmatrix}U_{11}-U_{12}U_{22}^{-1}U_{21} & 0\\ 0 & U_{22}\end{pmatrix}\\ \cdot e^{d\Gamma_0\begin{pmatrix}0 & 0\\ U_{22}^{-1}U_{21} & 0\end{pmatrix}}.\end{aligned}$$

$\mathscr{U}$ implements the automorphism

$$\begin{pmatrix} 1 & U_{12}U_{22}^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} U_{11}-U_{12}U_{22}^{-1}U_{21} & 0 \\ 0 & U_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ U_{22}^{-1}U_{21} & 1 \end{pmatrix}=U.$$

q.e.d.

Now the formal power serie $B_f(t)$ with $d\Gamma_0(B_f(t))=\sum_{n=0}\frac{t^n}{n!}\{d\Gamma_0(A)_c^n - (\Omega, d\Gamma_0(A)_c^n\Omega)\}$ satisfies the following integral equation [10]:

$$B_f(t)=\int_0^t ds(1-B_f(s)Q)A(1+PB_f(s)).$$

Assume that A is bounded and that $(e^{tA})_{22}$ has a bounded inverse for all t between 0 and t_0 for some $t_0 \in \mathbb{R}$. Then

$$B(t)=(1+(e^{tA}-1)Q)^{-1}(e^{tA}-1), \quad t\in[0,t_0]$$

is a solution of this integral equation.

Proof.

$$B=(1+KQ)^{-1}K=K(1+QK)^{-1}, \quad K=e^{tA}-1,$$

$K'=A(K+1)$ (′ denotes differentitation with respect to t),

$$\begin{aligned} B'&=A(K+1)(1+QK)^{-1}-K(1+QK)^{-1}QA(K+1)(1+QK)^{-1} \\ &=(1-BQ)A(B+(1+QK)^{-1}) \end{aligned}$$

with

$$(1+QK)^{-1}=1-QK(1+QK)^{-1}=1-QB$$

follows:

$$B'=(1-BQ)A(1+PB).$$

Together with $B(0)=0$ the proposition follows.

Now standard results from the theory of ordinary differential equations give the uniqueness of the solution. B_f clearly is asymptotic to B, and because B is analytic in $t=0$, $B_f(t)$ converges in norm to $B(t)$ for sufficiently small t. If the condition $-(e^{tA})_{22}$ has a bounded inverse—is not fulfilled for all t between 0 and t_0, then there exists an example, where $B_f(t_0)$ converges as a bilinear form on a dense set and differs from $B(t_0)$ [10].

6. Strong Implementation of Derivations

Whereas the discussion about the implementability of automorphisms is completed (the theorem of Shale and Stinespring says, that an automorphism α_U is unitarily implementable, if and only if $[P, U]$ is a Hilbert-Schmidt operator), there are more possibilities to implement a derivation d_T than to extend the bilinear form $d\Gamma_0(T)$.

By adding infinite constants one can try to find implementing operators on other domains than $\mathscr{C}_0(H)\Omega$.

Now according to Shale and Stinespring a derivation d_T, T selfadjoint in H, is implementable by a selfadjoint operator $\mathscr{T}$ in $\mathscr{H}$ if and only if $Pe^{itT}Q \in C_2$ for all real t. In practical cases the investigation of e^{itT} is difficult, and it is desirable to have a criterion which involves T directly. If $T = T_0 + V$, T_0, V selfadjoint, V bounded, $T_0P = PT_0$, there exists a criterion which involves the first term in the perturbation expansion of $e^{-itT_0}e^{itT}$:

6.1. Theorem. *If* $P\int_0^t ds\, e^{-itT_0} V e^{itT_0} Q$ *is a Hilbert-Schmidt operator for all real* t, *then there exists a selfadjoint operator* $\mathscr{T}$ *in* $\mathscr{H}$ *with* $\mathrm{ad}\,\mathscr{T} = d_T$.

This criterion can be found in a paper of Bongaarts [11] and in [3]. Recently Hochstenbach [2] proved, that in the CCR-case this criterion is also necessary. A slight modification of his argument gives the necessity also for the case of anticommutation relations (this result has been found independently also by Ruijsenaars [4]):

6.2. Theorem. *Let* T *be a selfadjoint operator in* H. *The derivation* d_T *is implementable by a selfadjoint operator* $\mathscr{T}$ *only if for each decomposition* $T = T_0 + V$, T_0, V *selfadjoint,* V *bounded,* $T_0P = PT_0$, *the first term in the perturbation expansion of*

$$P e^{-itT_0} e^{itT} Q,$$

namely

$$P\int_0^t ds\, e^{-itT_0} V e^{itT_0} Q,$$

is a Hilbert-Schmidt operator for all real t.

Remark. If T is bounded, choose $T_0 = 0$. Then a necessary and sufficient condition for d_T to be implementable is $PTQ \in C_2$.

Proof. If $\alpha_{e^{itT}}$ is unitarily implementable, then also $\alpha_{R(t)}$ with

$$(1)\quad R(t) = e^{-itT_0}e^{itT}.$$

The criterion of Shale and Stinespring gives $[P, R(t)]$, $[P, R(t)^{-1}] \in C_2$.

On a dense set in H holds:

$$(2)\quad R'(t) = iV(t)R(t), \qquad V(t) := e^{-itT_0} V e^{itT_0},$$

$$(3)\quad (R(t)^{-1})' = -iR(t)^{-1}V(t) \quad (' \text{ denotes differentiation w.r.t. } t).$$

The right hand sides exist on H and are strongly continuous.
It follows:

$$(4)\quad iV(t) = R'(t)R(t)^{-1},$$

$$(5)\quad i[P, V(t)] = [P, R'(t)]R(t)^{-1} + R'(t)[P, R(t)^{-1}].$$

It must be shown:

(6) $\int_0^t ds[P, V(s)] \in C_2$ for all real t.

Integration and partial integration of the first term on the right hand side of (5) gives:

$$(7)\quad i\int_0^t ds[P, V(s)] = [P, R(t)]R(t)^{-1}$$

$$+ \int_0^t ds\, R'(s)\,[P, R(s)^{-1}] - [P, R(s)]\,(R(s)^{-1})'.$$

The first term is a Hilbert-Schmidt operator. The integrand of the second term is strongly continuous and $\in C_2$ for all real s. It suffices to show that its Hilbert-Schmidt-norm is uniformly bounded [11, Lemma 2]. This follows from the strong continuity of $e^{it\mathcal{T}}\Omega$ by using the representation of Theorem 5.1. For details see [1]. q.e.d.

When the existence of an implementing operator is known, there arise questions about its properties. One wishes to have an explicit representation, which allows to study domains of essential selfadjointness, lower and upper bounds and relations to other operators. Also the connection between the implementing selfadjoint operator and the implementing bilinear form on $\overline{\mathscr{C}_0(H)\Omega} \times \mathscr{C}_0(H)\Omega$ should be investigated.

Let T_0 be a selfadjoint operator in H commuting with P and with $(P-Q)T_0 > 0$. Then $d\Gamma(T_0) = d\Lambda(I(P-Q)T_0 I^{-1}) \geqq 0$.

Let V be a selfadjoint bounded operator in H. Define W as the unique bounded operator from QH into PH with

$$T_0 W = W T_0 + V_C, \qquad V_C = PVQ, \qquad V_A = QVP.$$

Assume that W is a Hilbert-Schmidt operator. Then

$$\int_0^t ds\, e^{-isT_0} V e^{isT_0} = i(e^{-isT_0} W e^{isT_0} - W),$$

and d_T with $T = T_0 + V$ is implementable by a selfadjoint operator. The contrary is also true:

6.3. Lemma. *If T is implementable by a selfadjoint operator, W is a Hilbert-Schmidt operator.*

Proof. Let $c > 0$ be the greatest lower bound of $(P-Q)T_0$. We know that

$$\|(e^{-it\,\mathrm{ad}\,T_0} - 1)(W)\|_2$$

is uniformly bounded for t between o and $a > 2c^{-1}$. Then

$$\int_0^a dt (e^{-it\,\mathrm{ad}\,T_0} - 1)(W) \in C_2.$$

But $\int_0^a dt(e^{-it\,\mathrm{ad}\,T_0}-1)$ as an operator on the Hilbert space $C_2(QH, PH)$ has a bounded inverse, because

$$\left\|1+a^{-1}\int_0^a dt(e^{-it\,\mathrm{ad}\,T_0}-1)\right\|=\left\|a^{-1}\int_0^a dt\,e^{-it\,\mathrm{ad}\,T_0}\right\|=\left\|\frac{e^{-ia\,\mathrm{ad}\,T_0}-1}{-ia\,\mathrm{ad}\,T_0}\right\|$$
$$\leqq 2(ac)^{-1}<1. \qquad \text{q.e.d.}$$

Define

$$U=(1-(W-W^*))(1+WW^*+W^*W)^{-1/2}.$$

U is unitary, and it transforms T into

$$T^\wedge=U^{-1}TU=(1+WW^*+W^*W)^{-1/2}\{T_0+V_0+[W-W^*,V]-(W-W^*)T(W-W^*)\}$$
$$\cdot(1+WW^*+W^*W)^{-1/2} \qquad (V_0=PVP+QVQ).$$

Now $T^\wedge$ is the sum of a selfadjoint operator, which commutes with P, and a Hilbert-Schmidt operator. Therefore $d\Gamma(T^\wedge)$ is a densely defined closable operator.

Moreover, $T^\wedge$ can be decomposed in the following way:

$$T^\wedge=T_0+V_0+A \quad \text{with} \quad A\in C_2.$$

Applying Proposition 4.3 we see that $d\Gamma(T)$ is a selfadjoint, semibounded operator if the essential spectrum of $V_0|T_0|^{-1/2}$ is contained in the open unit disc. Also we see that U satisfies the criterion of Shale and Stinespring and that the operator implementing α_U acts on the vacuum like

$$\mathscr{U}\Omega=det(1+W^*W)^{-1/2}e^{-d\Gamma(W)}\Omega.$$

Now we have proved the following theorem:

6.4. Theorem. *Let $T=T_0+V$, T_0, V selfadjoint, V bounded, $PT_0=T_0P$. Assume that d_T is implementable by a selfadjoint operator $\mathscr{T}$ and $(P-Q)T_0>0$, $|\text{ess spec}(PVP+QVQ)|T_0|^{-1/2}|<1$.*

Then $\mathscr{T}$ is the unique (up to an additive constant) semibounded operator

$$\mathscr{T}=\mathscr{U}d\Gamma(U^{-1}TU)\mathscr{U}^{-1}$$

with U and $\mathscr{U}$ as defined above.

To discuss the connection between $\mathscr{T}$ and $d\Gamma_0(T)$, we approximate V by a net of Hilbert-Schmidt operators (V_κ) such that

(1) $\|V_\kappa\|\leqq\|V\|$,

(2) $\|W_\kappa-W\|_2\to 0$,

(3) $\|(V_{\kappa 0}-V_0)|T_0|^{-1/2}\|\to 0$.

According to Proposition 4.3, $d\Gamma(T_\kappa)=d\Gamma(T_0)+d\Gamma(V_\kappa)$ is selfadjoint and bounded from below and implements the derivation d_{T_κ}. Applying the construction of

Theorem 6.4, we get another selfadjoint operator $\mathscr{T}_\kappa = \mathscr{U}_\kappa d\Gamma(U_\kappa^{-1} T_\kappa U_\kappa)\mathscr{U}_\kappa^{-1}$ which implements the same derivation. Therefore the both operators differ by an additive constant.

6.5. Lemma.

$$d\Gamma(T_\kappa) = \mathscr{T}_\kappa + E_\kappa^{(1)} + E_\kappa^{(2)},$$

where

$$E_\kappa^{(1)} = -\operatorname{Tr}\{(1 + W_\kappa W_\kappa^*)^{-1} W_\kappa V_{\kappa A}\}$$

and

$$E_\kappa^{(2)} = \operatorname{Tr}\{(1 + W_\kappa W_\kappa^* + W_\kappa^* W_\kappa)^{-1}(W_\kappa W_\kappa^* - W_\kappa^* W_\kappa)V_{\kappa 0}\}.$$

Proof.

(1) $(\mathscr{U}_\kappa \Omega, (d\Gamma(T_\kappa) - \mathscr{T}_\kappa)\mathscr{U}_\kappa \Omega) = (\mathscr{U}_\kappa \Omega, d\Gamma(T_\kappa)\mathscr{U}_\kappa \Omega)$,

(2) $\det(1 + W_\kappa^* W_\kappa)^{+1/2} d\Gamma(T_\kappa)\mathscr{U}_\kappa \Omega = d\Gamma(T_\kappa) e^{-d\Gamma(W_\kappa)}\Omega$
$= e^{-d\Gamma(W_\kappa)}\{d\Gamma(T_\kappa) + [d\Gamma(W_\kappa), d\Gamma(T_\kappa)] + \frac{1}{2}[d\Gamma(W_\kappa), [d\Gamma(W_\kappa), d\Gamma(T_\kappa)]]\}\Omega$,

(3) $[d\Gamma(W_\kappa), d\Gamma(T_\kappa)] = -d\Gamma(V_{\kappa C}) + d\Gamma([W_\kappa, V_\kappa]) - \operatorname{Tr}\{V_{\kappa A} W_\kappa\}$,

(4) $\frac{1}{2}[d\Gamma(W_\kappa), [d\Gamma(W_\kappa), d\Gamma(T_\kappa)]] = -d\Gamma(W_\kappa V_{\kappa A} W_\kappa)$.

(5) Let P_2 be the projector on the two particle space $\mathscr{H}_2 = \Lambda_2 \mathscr{H}_1$. Then:
$P_2(e^{-d\Gamma(W_\kappa)})^* e^{-d\Gamma(W_\kappa)}\Omega = \det(1 + W_\kappa^* W_\kappa) d\Gamma(-W_\kappa(1 + W_\kappa^* W_\kappa)^{-1})\Omega$.

Now by using the following formula, valid for Hilbert-Schmidt operators A, B:

(6) $(d\Gamma(A)\Omega, d\Gamma(B)\Omega) = \operatorname{Tr} QA^*PB$

we get the lemma after some straightforward calculations. q.e.d.

$(E_\kappa^{(2)})$ converges under our assumptions to

$$E^{(2)} = \operatorname{Tr}\{(1 + WW^* + W^*W)^{-1}(WW^* - W^*W)V_0\}.$$

Since $V_A \notin C_2$ in general, $(E_\kappa^{(1)})$ may not converge. But after subtracting the possibly divergent net $(\delta_\kappa) = (-\operatorname{Tr} W_\kappa V_{\kappa A})$ we get a converging net

$$(E_\kappa^{(1)} - \delta_\kappa) \to E_{\text{ren}}^{(1)} = \operatorname{Tr}\{(1 + WW^*)^{-1} WW^* W V_A\}.$$

Now we are prepared to prove the following proposition:

6.6. Proposition. *The unitary groups* $(e^{it(d\Gamma(T_\kappa) - \delta_\kappa)})$ *converge strongly to*

$$e^{it(\mathscr{T} + E_{\text{ren}}^{(1)} + E^{(2)})}.$$

Proof. It suffices to show: $e^{it\mathscr{T}_\kappa} \underset{s}{\to} e^{it\mathscr{T}}$. Now

$$\mathscr{T}_\kappa = \mathscr{U}_\kappa d\Gamma(T^\wedge{}_\kappa)\mathscr{U}_\kappa^{-1}$$

and $\mathscr{U}_\kappa \underset{s}{\to} \mathscr{U}$, $e^{itd\Gamma(T^\wedge{}_\kappa)} \underset{s}{\to} e^{itd\Gamma(T^\wedge)}$. So the proposition follows.

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