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# A Note on Hunziker's Theorem

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**Abstract.** We prove Hunziker's theorem that the essential spectrum of the internal Hamiltonian of an $N$-particle system is bounded below by the lowest 2-particle threshold. Our assumptions are given in terms of the Hamiltonian alone without reference to potentials. They include the previously treated cases.

## I. Introduction and Results

We give a simple proof of Hunziker's theorem [4] which says that the essential spectrum of an $N$-particle Hamiltonian (after separating off the free centre of mass motion) is bounded below by the lowest possible energy which two independent subsystems can have. Our assumptions are so weak that they include the previously treated cases. But we think the main point is that we can express our assumptions in terms of the Hamiltonian alone without reference to the potential. So one can hope that the method will be applicable also to relativistic quantum field theory.

We consider a quantum mechanical system of $N$ particles with masses $m_i$, possibly with spin, moving in $\nu$-dimensional space. Exterior forces are absent so the whole system is translation invariant and we separate off the centre of mass motion. Let $\boldsymbol{x}_i$, $i=1, \ldots, N$ be the coordinates of the particles, and let $\xi$ denote the $\nu(N-1)$-dimensional vector representing a suitable set of $N-1$ relative coordinates. In what follows subsets of $\xi$-space will be given by conditions on the relative particle coordinates $\boldsymbol{x}_i - \boldsymbol{x}_j$. The (reduced) state space $\mathscr{H}$ is the subspace of those vector valued wave functions from $L^2(\mathbb{R}^{\nu(N-1)}, d\xi, \mathbb{M})$ which obey the symmetry or antisymmetry requirements for identical particles, $\mathbb{M}$ is the finite dimensional vector space of spin variables.

We assume the following conditions on the dynamics:

i) *The Hamiltonian $H$ is selfadjoint on $\mathscr{H}$ and bounded below by $c$.*

ii) *There is a core $\mathscr{D} \subset \mathscr{H}$ of $H$ such that for all $\Psi \in \mathscr{D}$: $f(\xi)\Psi \in \mathscr{Q}(H)$ (the form domain of $H$) and*

$$|\langle \Psi, [f(\xi/d), [H, g(\xi/d)]]\Psi\rangle| < h_d(f, g)\|(\mathbb{1} + |H|)\Psi\|^2 \tag{1}$$

*with* $\lim_{d\to\infty} h_d(f,g)=0$, *where* $f, g$ *are multiplication operators with arbitrary bounded* $C^\infty$*-functions of the internal coordinates with bounded derivatives.*

We denote by $F_R$ multiplication with the characteristic function of the set where $|\boldsymbol{x}_i-\boldsymbol{x}_j|\leqq R\,\forall i,j=1,\dots,N$. Thus in any state of $F_R\mathscr{H}$ the distance of any pair of particles is at most $R$. Our main assumption is:

iii) *Compactness criterion:* $F_R(i\mathbb{1}+H)^{-1}$ *is a compact operator on* $\mathscr{H}$ *for all* $R<\infty$.

The physical content of assumption ii) is that the energy density of the state $\Psi$ changes little if $\psi(\xi)$ is multiplied with a slowly varying function. The compactness criterion can be stated equivalently in the analogous form for the full Hamiltonian (including the centre of mass motion). It says roughly that there is only a finite number of states localized in a region with a given energy. One expects that it is connected with the existence of an asymptotic particle interpretation of the theory (see [3]), it was used to characterize bound states (particles) by their behaviour in space and time in [1, 2, 6].

If $\Delta_q$ and its complement $\Delta'_q$ are non void subsets of $1,2,\dots,N$ they define a partition of $1,\dots,N$; the index $q$ labels the $2^{N-1}-1$ different partitions. We shall characterize states consisting of two subsystems which are separated at least by $d$. To express this mathematically let $P_q^0(d)$ be the multiplication with $\prod_{i\in\Delta_q,\, j\in\Delta'_q}[1-\Theta(d-|\boldsymbol{x}_i-\boldsymbol{x}_j|)]$ which is a projection in $L^2$-space. By $P_p(d)$ we denote the range projection of the sum of those $P_q^0(d)$ which belong to equivalent partitions (i.e. they differ only by a permutation of identical particles), $p=1,2,\dots K\leqq 2^{N-1}-1$ labels the equivalence classes of partitions,

$$P_p(d)=P_q^0(d)\vee P_r^0(d)\vee\dots,\qquad q\sim r.$$

$P_p(d)$ are projection operators on the state space $\mathscr{H}$. Then

$$b_p(d)=\inf\{\langle\Psi,H\Psi\rangle\,|\,\Psi\in\mathscr{D}(H)\cap P_p(d)\mathscr{H},\ \|\Psi\|=1\}.$$

Since all $b_p(d)$ are monotone non decreasing in $d$ the limits in

$$b=\min_p\lim_{d\to\infty} b_p(d) \tag{2}$$

either exist or are infinite. $b$ is the minimal energy of (superpositions of) states which consist of two far separated subsystems.

Now we can state our theorems:

**Theorem 1.** *The essential spectrum of the internal Hamiltonian* $H$ *is bounded below by* $b$: $\sigma_e(H)\subset[b,\infty)$ *for* $b<\infty$; *if* $b=\infty$, $H$ *has a pure point spectrum of finite multiplicity.*

If $H$ commutes with a unitary representation of a compact symmetry group acting on $\mathscr{H}$ one can consider the restriction of $H$ to an invariant subspace $\mathscr{H}_G$ (e.g. $\mathscr{H}_G$ carries an irreducible representation of the group). Let $b(\mathscr{H}_G)$ be defined as $b$, but the infimum is taken only over $\Psi\in\mathscr{D}(H)\cap P_p(d)\mathscr{H}_G$, then one gets

**Theorem 2.** *The essential spectrum of* $H|\mathscr{H}_G$ *is bounded below by* $b(\mathscr{H}_G)$.

We will discuss the potentials for which our assumptions hold in Section III. There we will also express $b$ in terms of the Hamiltonians of subsystems, if the potential vanishes at infinity.

## II. Proof of the Theorems

First we introduce smooth multiplication operators $E_p^d$ which map onto states with far separated subsystems. Let $\varphi \in \mathscr{D}(\mathbb{R})$, $0 \leqq \varphi \leqq 1$ obey $\varphi(\lambda)=1$ if $|\lambda| \leqq 1$, $\varphi(\lambda)=0$ if $\lambda \geqq 2$. For any equivalence class of partitions $[q]$, labelled by $p$, we define

$$\chi_p(\xi) = \bigvee_{[q]} \{\prod_{i\in \Delta_q,\, j \in \Delta'_q} (1-\varphi(|\boldsymbol{x}_i - \boldsymbol{x}_j|))\}$$

where $f \vee g := f + g - f \cdot g$. These $\chi_p(\xi)$ are smooth analogues of the projections $P_p(d)$.

$$\begin{aligned} &E_1 := \chi_1\,; \quad E_p := \chi_p \prod_{l=1}^{p-1} (1-\chi_l) \quad \text{for} \quad 2 \leqq p \leqq K\,; \\ &E_0 := \prod_{p=1}^{K} (1-\chi_p)\,; \\ &E_k^d(\xi) := E_k(\xi/d) \quad \text{for} \quad k=0,1,\ldots,K. \end{aligned} \tag{3}$$

The multiplication operators $E_k^d$ have the following simple properties:

$$\begin{aligned} &\textstyle\sum_{k=0}^{K} E_k^d = \mathbb{1}, \quad \|E_k^d\| = 1 \quad \text{for all} \quad d>0, \\ &P_p(d) E_p^d = E_p^d, \quad P_p(2d) E_p^d = P_p(2d), \\ &E_0^d F_R = E_0^d \quad \text{if} \quad R \geqq 2(N-1)d. \end{aligned} \tag{4}$$

Due to assumption ii) the following expansion is possible for any normalized $\Psi \in \mathscr{D}$:

$$\begin{aligned} \langle \Psi, (H-a\mathbb{1})\Psi\rangle &= \textstyle\sum_{k,l=0}^{K} \langle E_k^d \Psi, (H-a\mathbb{1}) E_l^d \Psi\rangle \\ &= \textstyle\sum_l \langle E_l^d \Psi, (H-a\mathbb{1}) E_l^d \Psi\rangle \\ &\quad + \textstyle\sum_{k<l} \langle \Psi, [E_k^d, [H, E_l^d]]\Psi\rangle \\ &\quad + \textstyle\sum_{k<l} 2\,\mathrm{Re}\langle E_l^d E_k^d \Psi, (H-a\mathbb{1})\Psi\rangle. \end{aligned} \tag{5}$$

On one hand we can estimate with $b_p(d)$ and $b$ as defined in (2):

$$\begin{aligned} &\textstyle\sum_{p=1}^{K} \langle E_p^d \Psi, (H-a\mathbb{1}) E_p^d \Psi\rangle \\ &\geqq \textstyle\sum_{p=1}^{K} (b_p(d)-a) \|E_p^d \Psi\|^2 \\ &\geqq (b-a) \textstyle\sum_p \|E_p^d \Psi\|^2 \cdot 1/2 \\ &\geqq (b-a) \|\textstyle\sum_{p=1}^{K} E_p^d \Psi\|^2 \cdot 1/2K \\ &> 2^{-N}(b-a)\|\textstyle\sum_{p=1}^{K} E_p^d \Psi\|^2 \geqq 2^{-N}(b-a)(\|\Psi\| - \|E_0^d\Psi\|)^2 \\ &\forall \Psi \in \mathscr{D}, \quad \forall a < b < \infty, \quad \forall d \geqq d_0(a)\,; \end{aligned} \tag{6}$$

or in case $b=\infty$ the given quantity is bounded below by $M(\|\Psi\| - \|E_0^d \Psi\|)^2$ for any $M<\infty$, $\forall \Psi \in \mathscr{D}$, $\forall d \geqq d_0(M)$.

On the other hand the same quantity is bounded above using (5) and (1):

$$\begin{aligned} &\textstyle\sum_{p=1}^{K} \langle E_p^d \Psi, (H-a\mathbb{1}) E_p^d \Psi\rangle \\ &\leqq \langle E_0^d \Psi, (-H+a\mathbb{1}) E_0^d \Psi\rangle \\ &\quad + \textstyle\sum_{k<l} h_d(E_k, E_l) \|(\mathbb{1}+|H|)\Psi\|^2 \\ &\quad + [K(K+1)+1] \|(H-a\mathbb{1})\Psi\|. \end{aligned} \tag{7}$$

Now we set $a=\inf\sigma_e(H)$. If we derive a contradiction for $a<b$, Theorem 1 is proved.

Choosing $d$ and $\Psi$ suitably we will show that the r.h.s. of (7) is arbitrarily small. By an extension of Weyl's criterion (Theorem 1.2 in [5]) there exists an orthonormal sequence $\Psi_n\in\mathscr{D}$ such that $\|(H-a\mathbb{1})\Psi_n\|\to 0$ as $n\to\infty$. Let $n_0$ be so big that $\forall n \geqq n_0:(K+1)^2\|(H-a\mathbb{1})\Psi_n\|<2^{-N}(b-a)\cdot 1/6$ (or $M/6$). Fix $d'\geqq d_0$ such that

$$\textstyle\sum_{k<l} h_{d'}(E_k,E_l)\sup\limits_n \|(\mathbb{1}+|\mathrm{H}|)\Psi_n\|^2<2^{-N}(b-a)\cdot 1/6\ (\text{or } M/6).$$

So far we have used only assumption (i) and (ii). The remaining term in (7) will be estimated using the compactness criterion.

$$\begin{aligned}&\langle E_0^{d'}\Psi_n,(-H+a\mathbb{1})E_0^{d'}\Psi_n\rangle\\ &\leqq|-c+a|\cdot\|E_0^{d'}\Psi_n\|^2\leqq|-c+a|\cdot\|F_R\Psi_n\|^2\\ &=|-c+a|\cdot\|F_R[i\mathbb{1}+H]^{-1}[i\mathbb{1}+H]\Psi_n\|^2\end{aligned}\tag{8}$$

by (4) for $R=2Nd'$, $c$ is the lower bound of $H$.

The uniformly bounded sequence $\Phi_n=[i\mathbb{1}+H]\Psi_n$ converges strongly to $(i+a)\Psi_n$ which converges weakly to 0. So $\Phi_n$ converges weakly to 0. The compactness criterion states that $F_R[i\mathbb{1}+H]^{-1}$ is a compact operator which implies that $F_R\Psi_n$ converges strongly to 0. We can choose $n\geqq n_0$ such that (8) is bounded by $(b-a)2^{-N}\cdot 1/6$ (or $M/6$) and that $\|E_0^{d'}\Psi_n\|\leqq 1/4$. Then $(1-\|E_0^{d'}\Psi_n\|)^2>1/2$. Thus $a<b$ leads to a contradiction of (6) and (7) and Theorem 1 is proved.

The functions $E_k^d$ are symmetric under permutations of identical particles, so they map $\mathscr{H}$ into itself. Furthermore, they are invariant under rotations and reflections of the relative coordinates. If there happens to be another compact symmetry group on $\mathscr{H}$ one should take the mean over the group of the transformed $E$'s. Then the proof of Theorem 2 goes on the restricted Hilbert space exactly as before.

## III. Applications

Hunziker proved Theorem 1 for potentials $V\in L^2+L_\varepsilon^\infty$ [4]. Later it was extended by several authors to larger classes of potentials, spin and symmetry groups were included. (See e.g. the introduction of [5] and the references given there.) The most general result for $H=H_0+V$ defined as operator sum is given by Jörgens and Weidmann [5]. If $H=H_0+V$ is defined as quadratic form Simon [7] proved it for $V\in R+L_\varepsilon^\infty$. We will treat now the two cases separately.

### *a) Operator Sum $H_0+V$*

We will show that our assumptions are weaker than those made by Jörgens and Weidmann (p. 50—54 of [5]). The Hamiltonian is supposed to be bounded below and essentially selfadjoint on $C_0^\infty(\mathbb{R}^{3N})$. In the case where exterior forces are absent this is equivalent to $H$ being essentially selfadjoint on $\mathscr{D}=C_0^\infty(\mathbb{R}^{3N-3})\subset\mathscr{D}(H_0)\cap\mathscr{D}(V)$. $C_0^\infty$ is left invariant by multiplication with smooth functions $f(\xi)$, $g(\xi)$, therefore $f(\xi)\psi(\xi)\in\mathcal{D}\subset\mathcal{Q}(H)$. Furthermore $f(\xi)\psi(\xi)\in\mathcal{D}(H_0)\cap\mathcal{D}(V)$ which allows to compute for $\Psi\in\mathcal{D}$:

$$\begin{aligned}&|\langle\Psi,[f(\xi/d),[H,g(\xi/d)]]\Psi\rangle|\\&\leqq|\langle\ldots H_0\ldots\rangle|+|\langle\ldots V\ldots\rangle|.\end{aligned}\tag{9}$$

The double commutator in the first term is bounded by

$$\begin{aligned}&\textstyle\sum_{i=1}^{N}(2m_i)^{-1}|(\mathrm{grad}_i f(\xi/d)\cdot\mathrm{grad}_i g(\xi/d))|\\&=d^{-2}\textstyle\sum(2m_i)^{-1}|(\mathrm{grad}_i f(\xi)\cdot\mathrm{grad}_i g(\xi))|\leqq\mathrm{const}\,d^{-2}.\end{aligned}$$

If $V$ is a local interaction (i.e. a multiplication operator in $x$-space) the second term of (9) vanishes.

For non local $V$ one can use that the potential is assumed to be ($H_0$- and therefore) $H$-small at infinity. A simple calculation together with Corollary 6.7 of [5] shows the desired decrease of (1). Therefore our assumptions (i) and (ii) are fulfilled.

The compactness property follows from the well known compactness of $F_R(\mathbb{1}+H_0)^{-1}$ and the boundedness of $(\mathbb{1}+H_0)(i\mathbb{1}+H)^{-1}$ which is assumed in [5]. (For a detailed discussion of situations where the compactness criterion holds see Section III of [1].) So all our assumptions are fulfilled for the previously treated operator sum Hamiltonians.

### b) Form Sum $H_0+V$

Simon [7] treats local pair potentials $V\in R+L_\varepsilon^\infty$ which define semibounded selfadjoint Hamiltonians $H$ with $\mathcal{Q}(H)=\mathcal{Q}(H_0)=\mathcal{Q}(H_0)\cap\mathcal{Q}(|V|)$. Since $\mathcal{Q}(H_0)$ is left invariant under multiplication with smooth functions one has for all $\Psi\in\mathcal{D}=\mathcal{D}(H)\subset\mathcal{Q}(H)$: $f\Psi\in\mathcal{Q}(H)=\mathcal{Q}(H_0)\cap\mathcal{Q}(|V|)$. The commutator estimate with $H_0$ is exactly as in the operator sum case and the potential-term vanishes. The compactness criterion is fulfilled because $H_0$ is form bounded by $H$ (see Proposition 4 of [1]). Thus all our assumptions are fulfilled.

### c) Calculation of $b$

In most of the physically interesting situations two far separated subsystems do not interact. This is expressed mathematically by the $H$-smallness at infinity for the potentials (see Section III of [5] for a discussion of this notion). We expand the potential into a sum of terms corresponding to 2-, 3-, ..., $N$-particle interactions. For a given partition $p$ we denote by $I_p$ the sum over those potential terms which couple the two subsystems and $H_p=H-I_p$ is the Hamiltonian where only the particles within the clusters interact. Now in the operator sum case Jörgens and Weidmann assume that (if all particles are different)

$$\|I_p\Psi\|\leqq V_p(d)\|(\mathbb{1}+H_0)\Psi\|\leqq\mathrm{const}\,V_p(d)\|(\mathbb{1}+|H|)\Psi\|$$

for all $\Psi\in C_0^\infty\subset\mathcal{D}(H)\subset\mathcal{D}(V)\subset\mathcal{D}(I_p)$ where $\psi(\xi)=0$ whenever for some $i\in\Delta_p, j\in\Delta_p'$: $|\boldsymbol{x}_i-\boldsymbol{x}_j|<d$, and

$$V_p(d)\to 0\quad\text{as}\quad d\to\infty\,.\tag{10}$$

Thus $\inf \sigma(H_p) \leqq \text{const}\, V_p(d) + (1 + \text{const}\, V_p(d)) b_p(d)$ and

$$\inf \sigma(H_p) \leqq \lim_{d \to \infty} b_p(d). \tag{11}$$

Actually (11) is an equality, because $H_p$ is invariant under translations of one subsystem alone.

For two body potentials of the Rollnik class one can easily show the analogous form bounds leading to the same results. Thus for asymptotically vanishing potentials we obtain the commonly used formulation of Hunziker's theorem

$$\inf \sigma_e(H) \geqq \min_p \inf \sigma(H_p).$$

A proof of the opposite inequality for asymptotically vanishing potentials can be found e.g. in [5, 7].

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