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On the ζ-Function of a One-dimensional Classical System of Hard-Rods

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Abstract. The ζ-function of a one-dimensional classical hard-rod system with exponential pair interaction is defined as the generating function for the partition function of the system with periodic boundary conditions. It is shown, here, that the ζ-function for this system is simply related to the traces of the restrictions of the Ruelle's transfer matrix, and related operators to a suitable function space. This ζ-function does not, in general, extend to a meromorphic function.

Introduction

The new interest in classical one dimensional models of statistical mechanics has its origin in the work of Sinai [1] who found an interesting connection of these models with certain measure theoretic problems in the theory of dynamical systems. By constructing symbolic dynamics [2] for Anosov diffeomorphisms and flows on a compact manifold with the help of Markov partitions [3] he was able to apply the methods developed in the study of one dimensional models and to get interesting new results. A special role in the study of dynamical systems is played by the ζ-function of such a system introduced by Artin and Mazur [4]

$$\zeta(z)=\exp\left(\sum_{n=1}^{\infty} z^n N_n/n\right)$$

where N_n is the number of fixed points of the mapping f^n, where $f:M\to M$ is a diffeomorphism on some compact manifold M. They could show that the function $\zeta(z)$ has a non-vanishing radius of convergence for almost all diffeomorphisms f. To study the possible relevance of this ζ-function for statistical mechanics, Ruelle [5] introduced generalized ζ-functions in the following way:

Let M be some topological space and $f:M\to M$ a mapping. Let $A:M\to\mathbb{C}$ be a complex valued function on M. Then consider the formal expression

$$\zeta(z,e^A)=\exp\left[\sum_{n=1}^{\infty}\frac{z^n}{n}\left\{\sum_{x\in\mathrm{Fix}\,f^n}\left(\exp\sum_{k=0}^{n-1}A(f^k x)\right)\right\}\right]. \tag{1}$$

Properties of this function were studied in [5] and [6] and it was shown that this function extends in certain cases to a meromorphic function in the whole z plane.

Looking at the expression $\sum_{x\in \mathrm{Fix}\, f^n} \exp\left(\sum_{k=0}^{n-1} A(f^k x)\right)$ in the special case where f is the shift operator τ on the configuration space K of a one dimensional classical lattice gas system, then this is nothing else but the partition function Z_n of this system with periodic boundary conditions and interaction function A [7]. In this case the function ζ in (1) can be written

$$\zeta(z, A) = \exp \sum_{n=1}^{\infty} z^n Z_n(A)/n\,, \tag{2}$$

and ζ is just the generating function for Z_n.

By applying the transfer matrix method, one of us [8] studied the above function for a one dimensional classical lattice gas system with exponential-polynomial pair interactions and showed that in this case ζ is holomorphic in a neighbourhood of $z=0$, a fact which is closely related to the existence of the thermodynamic limit. Furthermore we showed that the function ζ extends to a meromorphic function in the whole z plane.

In this paper we study the ζ-function of a one dimensional classical hard core system with exponential pair interaction. We also apply the transfer matrix method here and show the following:

The partition function Z_n of a system of hard rods of length a with periodic boundary conditions and exponential pair interaction $\Phi(y,x)=c\lambda^{|y-x|}$, can be written as

$$Z_n = (1-\lambda^{na})\,\mathrm{tr}\,\mathscr{L}_0(\mathscr{L}_0+\mathscr{L}_1)^{n-1}$$

where $\mathscr{L}=\mathscr{L}_0+\mathscr{L}_1$ is the transfer matrix of the system. The operator $\mathscr{L}_0\mathscr{L}^n$ is a trace class operator for all $n\geqq 0$ in the Banach space $B=C(I)\hat{\otimes}_\pi A_\infty(D_R)$ on which $\mathscr{L}$ acts. In the next chapter we determine the trace of the operators $\mathscr{L}_0\mathscr{L}^n$ and show the connection with the partition function Z_n. In a final chapter we discuss some properties of the ζ function of the hard core system.

I. The Transfer Matrix $\mathscr{L}$

We use the terminology which was introduced in the paper on classical hard core systems by Gallavotti and Miracle-Sole [9]. Let K be the set of all allowed configurations X of the system, where X can be described by a sequence $X=(x_1, x_2, \ldots)$, where $x_i \in \mathbb{R}_+ = \{x\in\mathbb{R}: x\geqq 0\}$ describes for instance the left corner of a rod of length a and $|x_i - x_j|\geqq a$ for $i\neq j$. We restrict ourselves to the case where the rods interact via an exponentially decreasing pair potential

$$\Phi_k(X) = \begin{cases} 0 & \text{if} \quad k\neq 2 \\ c\lambda^{(x_2-x_1)} & \text{if} \quad k=2 \end{cases} \tag{3}$$

for $X=(x_1, \ldots, x_k)\in K$, $0<\lambda<1$ and c some constant. The transfer matrix $\mathscr{L}$ [9, 10] is defined as a linear operator on the Banach space $C(K)$ of all continuous functions on the compact space K as follows:

$$(\mathscr{L}f)(X) := \int_{Y\subset[0,a)} e^{-U(Y|\tau X)} f(Y\cup\tau X)\,dY \tag{4}$$

where $f \in C(K)$ and τ is the shift operator acting on K by $\tau X = X + a$. The interaction energy $U(Y|W)$ for $Y, W \in K$ is defined as

$$U(Y|W) = \sum_{\substack{\Phi \neq S \subset Y \\ T \subset W}} \Phi_2(S \cup T).$$

Using (3) we get the expression

$$(\mathscr{L}f)(x_1, x_2, \ldots) = f(x_1 + a, x_2 + a, \ldots) + \int_0^{x_1 \vee a} f(y, x_1 + a, x_2 + a, \ldots) \exp\left(-c \sum_i \lambda^{(x_i + a - y)}\right) dy \tag{5}$$

with $X = (x_1, x_2, \ldots)$ and $x_1 \vee a = \min(x_1, a)$.

It is known that $\mathscr{L}$ is continuous but not compact on $C(K)$. The problem is to find an operator $\mathscr{L}^{\sim}$ on a space B in which it has "good" properties such as for instance a trace. In particular we want the functions $f \equiv 1$ and the principal eigenvector h of $\mathscr{L}$ belong to B. Now h can be written as

$$h(x_1, x_2, \ldots) = \int_{Y \subset (-\infty, x_1 - a) \cap \mathbb{R}_-} d\mu(Y) \exp\left(-c \sum_{i,j} \lambda^{(-y_j + x_i)}\right)$$

where $Y = (y_1, y_2, \ldots)$ and where $d\mu(Y)$ denotes the Gibbs measure on the negative real axis, we see that h depends analytically on $\sum_i \lambda^{x_i}$ and is a continuous function of the coordinate x_1, as long as $x_1 \leqq a$, whereas for $x_1 > a$, it does not depend on x_1 except through $\sum_i \lambda^{x_i}$. One is therefore led to a space of functions which depend continuously on a variable $x = x_1$ and analytically on a variable $z = \sum_i \lambda^{x_i}$.

The action of $\mathscr{L}$ on such functions can then be written as

$$(\mathscr{L}f)(x, z) = f(a, \lambda^a z) + \int_0^x f(y, \lambda^y + \lambda^a z) \exp(-c\lambda^{a-y} z)\, dy. \tag{6}$$

Here we have used the fact that the function f does not depend on x for $x > a$ and we therefore can restrict ourselves to functions which are defined and are continuous in the interval $I = [0, a]$.

Next we want to construct a Banach space B on which the mapping $\mathscr{L}$ as defined in (6) is a well defined operator. Let $I = [0, a]$ and $D_R := \{z \in \mathbb{C} : |z| < R\}$. We denote by $C(I)$ the Banach space of all continuous functions on I with the sup norm. Let further $A_\infty(D_R)$ be the Banach space of all holomorphic functions on the open disc D_R, with the usual sup norm. Then we consider the projective topological tensor product [11] $C(I) \hat{\otimes}_\pi A_\infty(D_R)$ together with the π-norm introduced first by Schatten [12] (see also Appendix A). In [11] the following fundamental Theorem is proved:

Theorem 1. *Let E, F, G be Banach spaces and $T: E \times F \to G$ a bilinear continuous mapping of the direct product $E \times F$ into G. Then there exists a unique linear, continuous mapping $T^{\sim}: E \hat{\otimes}_\pi F \to G$ such that $T^{\sim} u = T(e, f)$ if $u = e \otimes f$ and $\|T^{\sim}\| = \|T\|$.*

From this we get immediately

Lemma 1. *Let* $R > \frac{1}{1-\lambda^a}$. *Then the operator* $\mathscr{L}$ *as defined in* (6) *is a linear, continuous operator in the Banach space* $B = C(I)\hat{\otimes}_\pi A_\infty(D_R)$.

Proof. For $\varphi \in C(I)$, $\psi \in A_\infty(D_R)$ define the operators $T_i : C(I) \times A_\infty(D_R) \to B$ as follows:

$$[T_1(\varphi, \psi)](x, z) := \varphi(a)\psi(\lambda^a z)$$

$$[T_2(\varphi, \psi)](x, z) := \int_0^x \varphi(y)dy\psi(z)$$

$$[T_3(\varphi, \psi)](x, z) := \varphi(x)\psi(z)\exp(-c\lambda^{a-x}z)$$

$$[T_4(\varphi, \psi)](x, z) := \varphi(x)\psi(\lambda^x + \lambda^a z).$$

Theorem 1 tells us that all T_i, $i = 1, \ldots, 4$ define unique mappings

$$T^{\sim}_i : C(I)\hat{\otimes}_\pi A_\infty(D_R) \to B.$$

The operator $\mathscr{L}$ is then easily seen to be given by

$$\mathscr{L} = T^{\sim}_1 + T^{\sim}_2 T^{\sim}_3 T^{\sim}_4 \quad \text{which we will write as} \quad \mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1$$

with

$$\mathscr{L}_0 = T^{\sim}_1 \quad \text{and} \quad \mathscr{L}_1 = T^{\sim}_2 T^{\sim}_3 T^{\sim}_4,$$

where

$$\|\mathscr{L}_0\| \leqq 1 \quad \text{and} \quad \|\mathscr{L}_1\| \leqq a \exp|c|R.$$

Let us next study the operators $\mathscr{L}_0$ and $\mathscr{L}_1$ more carefully.

Lemma 2. *The operator* $\mathscr{L}_0 : B \to B$ *is nuclear of order* 0.

Proof. Let $u_1 : C(I) \to C(I)$ be defined by $(u_1\varphi)(x) = \varphi(a)$ and $u_2 : A_\infty(D_R) \to A_\infty(D_R)$ by $(u_2\psi)(z) = \psi(\lambda^a z)$. Then the operator $\mathscr{L}_0$ is given by $\mathscr{L}_0 = u_1 \otimes u_2$, the tensor product of the two mappings u_1 and u_2. It follows from [6] that u_2 is nuclear of order 0. Because u_1 is a finite rank operator it is also nuclear of order 0. But then it follows [13] that the tensor product $u_1 \otimes u_2$ is also a nuclear operator of order 0 on B and has therefore a unique trace.

Because the operator $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1$ is bounded and the set of nuclear operators of order 0 is a two-sided ideal in the algebra of bounded operators on any Banach space we get from Lemma 2 as an immediate consequence that for every $n \geqq 0$ the operator $\mathscr{L}_0\mathscr{L}^n$ is nuclear of order 0. Therefore the operators $\mathscr{L}_0\mathscr{L}^n$ all have a well defined trace which is given by the sum over the eigenvalues counted according to their algebraic multiplicity [6].

For the operator $\mathscr{L}_1$ we have

Lemma 3. *The operator* $\mathscr{L}_1 : B \to B$ *is quasi-nilpotent.*

Proof. From Lemma 1 we know the action of $\mathscr{L}_1$ on any element $\varphi \otimes \psi \in B$:

$$(\mathscr{L}_1 \varphi \otimes \psi)(x, z) = \int_0^x \varphi(y)\psi(\lambda^y + \lambda^a z)\exp(-c\lambda^{a-y}z)dy$$

and therefore $|(\mathscr{L}_1\varphi\otimes\psi)(x,z)|\leqq x\|\varphi\|_{C(I)}\|\psi\|_{A_\infty}M$, where

$$M=\sup_{x\in I}\sup_{z\in\bar{D}_R}|\exp(-c\lambda^{(a-x)}z)|\,.$$

By induction we then get

$$|(\mathscr{L}_1^k\varphi\otimes\psi)(x,z)|\leqq\frac{x^k}{k!}M^k\|\varphi\|_{C(I)}\|\psi\|_{A_\infty}$$

and therefore $\|\mathscr{L}_1^k\|\leqq C^k/k!$ with $C=aM$. But then the spectrum of $\mathscr{L}_1$ can only contain the point $\varrho=0$ and $\mathscr{L}_1$ is therefore quasi-nilpotent.

Next we are going to determine the trace of the operator $\mathscr{L}_0\mathscr{L}^n$. To do this we make use of the results we have obtained above. One need not know if the operator $\mathscr{L}_1$ itself has a trace on the Banach space B. By using the theory of p-summing operators one can indeed find a Hilbert space H on which $\mathscr{L}_1$ is 2-summing, which implies that for all $n\geqq 2$ the operator $\mathscr{L}_1^n$ has a well-defined trace. Because we do not need this for the subsequent discussion, we do not treat this further.

II. The Trace of the Operator $\mathscr{L}_0\mathscr{L}^n$

Using the decomposition

$$\mathscr{L}=\mathscr{L}_0+\mathscr{L}_1 \tag{7}$$

we get for $n\geqq 1$

$$\mathscr{L}_0\mathscr{L}^n=\sum_{i_1=0}^{1}\dots\sum_{i_n=0}^{1}\mathscr{L}_0\mathscr{L}_{i_1}\dots\mathscr{L}_{i_n} \tag{8}$$

For the term $\mathscr{L}_0^{n+1}$ in the expansion (8) we get using the representation $\mathscr{L}_0=u_1\otimes u_2$ of Lemma 2: $\mathscr{L}_0^{n+1}=u_1^{n+1}\otimes u_2^{n+1}$ and therefore [15] $\operatorname{tr}\mathscr{L}_0^{n+1}=(\operatorname{tr}u_1^{n+1})(\operatorname{tr}u_2^{n+1})$. Because $\operatorname{tr}u_1^{n+1}=\operatorname{tr}u_1=1$ and $\operatorname{tr}u_2^{n+1}$ is given according to a general formula in [6] and [10] by $\operatorname{tr}u_2^{n+1}=(1-\lambda^{(n+1)a})^{-1}$ we have $\operatorname{tr}\mathscr{L}_0^{n+1}=(1-\lambda^{(n+1)a})^{-1}$ (see also Appendix B). Now the general term in expansion (8) can be written as

$$T_{\boldsymbol{\alpha},\boldsymbol{\beta}}=\mathscr{L}_0^{\alpha_1}\mathscr{L}_1^{\beta_1}\dots\mathscr{L}_0^{\alpha_\varrho}\mathscr{L}_1^{\beta_\varrho} \tag{9}$$

where $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_\varrho)$, $\boldsymbol{\beta}=(\beta_1,\dots,\beta_\varrho)$, $\alpha_i,\beta_i\in\mathbb{Z}_+=\{0,1,2,\dots\}$ such that $|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|=\sum_{i=1}^{\varrho}\alpha_i+\sum_{i=1}^{\varrho}\beta_i=n+1$. Let $|\boldsymbol{\alpha}|=j+1$ with $j\geqq 0$ and define the numbers $i_k=n+1-\sum_{l=k}^{\varrho}\beta_l$ for $k=1,\dots,\varrho$. Let $\boldsymbol{y}=(y_{j+1},\dots,y_n)\in I^{n-j}$ and define a $(n-j)$ component vector $\boldsymbol{\xi}:=(\xi_{j+1},\dots,\xi_n)$ as follows: $\xi_{i_k}=a\ \forall k=1,\dots,\varrho$ and $\xi_l=y_{l-1}$ for all other $j+1<l\leqq n$. Because $i_1=j+1$ we get $\xi_{j+1}=a$. With these definitions we can write the operator $T_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ acting on an element $f=\varphi\otimes\psi\in B$ as follows:

$$(T_{\boldsymbol{\alpha},\boldsymbol{\beta}}f)(x,z)=\int^{\boldsymbol{\xi}}d\boldsymbol{y}\,\varphi(y_n)\psi(\chi(\boldsymbol{y})+\lambda^{(n+1)a}z)\exp(-c\tau(\boldsymbol{y};z)) \tag{10}$$

where

$$\int^{\xi} d\boldsymbol{y} = \int_0^{\xi_{j+1}} dy_{j+1} \int_0^{\xi_{j+2}} dy_{j+2} \ldots \int_0^{\xi_n} dy_n\,.$$

The functions χ and τ will be determined in the subsequent discussion, at the moment we only need the following properties which can be immediately verified from the definition of the operators $\mathscr{L}_0$ and $\mathscr{L}_1$: χ and τ are C^∞ in $\boldsymbol{y}$ and for all $\boldsymbol{y} \in I^{n-j}$ the mapping $z \to \chi(\boldsymbol{y}) + \lambda^{(n+1)a} z$ is a holomorphic mapping of clos D_R into D_R. The function $\tau(\boldsymbol{y}; z)$ furthermore is holomorphic in the whole z-plane. With these remarks we can prove:

Theorem 2. *Let $T_{\alpha,\beta}: B \to B$ as defined in* (9). *Then*

$$\operatorname{tr} T_{\alpha,\beta} = [1 - \lambda^{a(n+1)}]^{-1} \int^{\xi} d\boldsymbol{y} \exp(-c\tau(\boldsymbol{y}; (1-\lambda^{a(n+1)})^{-1} \chi(\boldsymbol{y}))\,.$$

Proof. Because the mapping $z \to \chi(\boldsymbol{y}) + \lambda^{a(n+1)} z$ is holomorphic for all $\boldsymbol{y} \in I^{n-j}$ and $\psi \in A_\infty(D_R)$ we can write the action of $T_{\alpha,\beta}$ on $\varphi \otimes \psi$ as

$$\begin{aligned}(T_{\alpha,\beta} f)(x, z) = \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} \sum_{s=0}^{\infty} \sum_{p=s}^{\infty} \gamma^k z^{k+s} \binom{m}{k} \binom{p}{s} a_m (-c)^p (p!)^{-1} \int^{\xi} d\boldsymbol{y}\, \chi^{m-k}(\boldsymbol{y}) \\ \cdot \tau_2^s(\boldsymbol{y}) \tau_1^{p-s}(\boldsymbol{y}) \varphi(y_n)\,,\end{aligned} \tag{11}$$

where $\tau(\boldsymbol{y}, z) = \tau_1(\boldsymbol{y}) + z\tau_2(\boldsymbol{y})$ and $\psi(z) = \sum_{m=0}^{\infty} a_m z^m$, $\gamma = \lambda^{a(n+1)}$. If we define $\psi_{kmsp}(z) := z^{k+s} \in A_\infty(D_R)$

$$\varphi'_{kmsp}(\varphi) := \int^{\xi} d\boldsymbol{y}\, \chi^{m-k}(\boldsymbol{y})\, \tau_2^s(\boldsymbol{y}) \tau_1^{p-s}(\boldsymbol{y}) \varphi(y_n) \tag{12}$$

and $\psi'_{kmsp}(\psi) = a_m \equiv a_m(\psi)$ we can write the operator $T_{\alpha,\beta}$ acting on $\varphi \otimes \psi$ as

$$\begin{aligned}&(T_{\alpha,\beta} \varphi \otimes \psi) \\ &= \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} \sum_{s=0}^{\infty} \sum_{p=s}^{\infty} \binom{p}{s} \gamma^k \binom{m}{k} (-c)^p (p!)^{-1} (\varphi'_{kmsp} \otimes \psi'_{kmsp}) \otimes (1 \otimes \psi_{kmsp})(\varphi \otimes \psi)\,.\end{aligned} \tag{13}$$

Because $\varphi'_{kmsp} \otimes \psi'_{kmsp} \in C(I)' \otimes A_\infty(D_R)'$ (where "′" denotes the dual) and $1 \otimes \psi_{kmsp} \in B$ we can deduce from Theorem 1 that there exists a unique element $f'_{kmsp} \in B'$ with $\|f'_{kmsp}\| = \|\varphi'_{kmsp} \otimes \psi'_{kmsp}\|$ such that

$$f'_{kmsp}(\varphi \otimes \psi) = \varphi'_{kmsp}(\varphi) \psi'_{kmsp}(\psi)\,.$$

Therefore the operator $T_{\alpha,\beta}$ has the following representation

$$T_{\alpha,\beta} = \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} \sum_{s=0}^{\infty} \sum_{p=s}^{\infty} \binom{p}{s} \gamma^k \binom{m}{k} (-c)^p (p!)^{-1} f'_{kmsp} \otimes f_{kmsp} \tag{14}$$

where $f_{kmsp}(x, z) = 1 \otimes \psi_{kmsp}(z)$.

Because the trace of $T_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is then given by

$$\operatorname{tr} T_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} \sum_{s=0}^{\infty} \sum_{p=s}^{\infty} \binom{p}{s} \gamma^k \binom{m}{k} (-c)^p (p!)^{-1} f'_{kmsp}(f_{kmsp}),$$

we get

$$\operatorname{tr} T_{\boldsymbol{\alpha},\boldsymbol{\beta}} = (1-\gamma)^{-1} \int^{\xi} d\boldsymbol{y} \exp[-c\tau(\boldsymbol{y};(1-\gamma)^{-1}\chi(\boldsymbol{y}))].$$

Let us next study the functions $\chi(\boldsymbol{y})$ and $\tau(\boldsymbol{y};z)$. Because these functions depend on the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ we denote them more correctly by $\chi_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $\tau_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Let $M_\alpha = |\boldsymbol{\alpha}|$ and $M_\beta = |\boldsymbol{\beta}|$, $M_{\alpha,\beta} = M_\alpha + M_\beta$. Let $y_{\boldsymbol{\beta}} := (y_1, \ldots y_{M_\beta})$ and $\xi_{\boldsymbol{\beta}} = (\xi_1, \ldots, \xi_{M_\beta})$ be two vectors from I^{M_β}. The components ξ_i, $i = 1, \ldots, M_\beta$ are defined as follows:

$$\xi_i = a \quad \text{iff} \quad \exists k_i,\ 1 \leqq k_i \leqq \varrho : i = \sum_{j=k_i}^{\varrho} \beta_j \quad \text{and} \quad \alpha_{k_i} \neq 0,$$

$$\xi_i = y_{i+1} \quad \text{for all other} \quad i \neq M_\beta$$

$$\xi_{M_\beta} = x \in I \quad \text{if} \quad \alpha_1 = 0.$$

The operator $T_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ acting on $\varphi \otimes \psi$ is then given by

$$[T_{\boldsymbol{\alpha},\boldsymbol{\beta}} \varphi \otimes \psi](x,z) = \int^{\xi_{\boldsymbol{\beta}}} dy_{\boldsymbol{\beta}} \varphi(y_1) \psi(\chi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(y_{\boldsymbol{\beta}}) + \lambda^{aM_{\alpha,\beta}} z) \exp(-c\tau_{\boldsymbol{\alpha},\boldsymbol{\beta}}(y_{\boldsymbol{\beta}};z)) \tag{15}$$

where

$$\int^{\xi_{\boldsymbol{\beta}}} dy_{\boldsymbol{\beta}} = \int_0^{\xi_{M_\beta}} dy_{M_\beta} \ldots \int_0^{\xi_1} dy_1.$$

Consider the two transformations $R_1, R_2 : \mathbb{Z}_+^\varrho \to \mathbb{Z}_+^{\varrho+1}$ defined by

$$\begin{aligned} R_1\boldsymbol{\alpha} &= (1,\boldsymbol{\alpha}), \\ R_2\boldsymbol{\alpha} &= (0,\boldsymbol{\alpha}), \end{aligned} \qquad \boldsymbol{\alpha} \in \mathbb{Z}_+^\varrho. \tag{16}$$

We want to determine the action of these two transformations on the functions $\chi_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $\tau_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. A rather trivial calculation gives

$$\begin{aligned} \chi_{R_1\boldsymbol{\alpha},R_2\boldsymbol{\beta}}(y_{R_2\boldsymbol{\beta}}) &= \chi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(y_{\boldsymbol{\beta}}) \\ \tau_{R_1\boldsymbol{\alpha},R_2\boldsymbol{\beta}}(y_{R_2\boldsymbol{\beta}};z) &= \tau_{\boldsymbol{\alpha},\boldsymbol{\beta}}(y_{\boldsymbol{\beta}};\lambda^a z) \end{aligned} \tag{17}$$

respectively

$$\begin{aligned} \chi_{R_2\boldsymbol{\alpha},R_1\boldsymbol{\beta}}(y_{R_1\boldsymbol{\beta}}) &= \chi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(y_{\boldsymbol{\beta}}) + \lambda^{(aM_{\alpha,\beta} + y_{M_\beta+1})} \\ \tau_{R_2\boldsymbol{\alpha},R_1\boldsymbol{\beta}}(y_{R_1\boldsymbol{\beta}};z) &= \tau_{\boldsymbol{\alpha},\boldsymbol{\beta}}(y_{\boldsymbol{\beta}};\lambda^{y_{M_\beta+1}} + \lambda^a z) + \lambda^{(a-y_{M_\beta+1})} z \end{aligned} \tag{18}$$

where $y_{R_1\boldsymbol{\beta}} = (y_1, \ldots, y_{M_\beta}, y_{M_\beta+1})$.

For the special case $\boldsymbol{\alpha} = (0)$, $\boldsymbol{\beta} = (1)$ we get from the definition of the operator $\mathscr{L}_1$:

$$\chi_{0,1}(y) = \lambda^y, \qquad \tau_{0,1}(y;z) = \lambda^{a-y} z. \tag{19}$$

Let $T_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ be as defined in (15). If $\alpha_k \neq 0$, $\beta_k \neq 0$ we define the operator

$$T_{k,\boldsymbol{\alpha},\boldsymbol{\beta}} = \mathscr{L}_0^{\alpha_1} \mathscr{L}_1^{\beta_1} \ldots \mathscr{L}_0^{\alpha_k - 1} \mathscr{L}_1 \mathscr{L}_0 \mathscr{L}_1^{\beta_k - 1} \mathscr{L}_0^{\alpha_{k+1}} \ldots \mathscr{L}_1^{\beta_\varrho}.$$

Let $\Pi_k = \sum_{i=k}^{\varrho} \beta_i$. If $\alpha_1 \geqq 1$, we have from the trace formula of Theorem 2:

$$\operatorname{tr} T_{\alpha,\beta} = \int^{\xi_\beta} dy_\beta \omega_{\alpha,\beta}(y_\beta)$$

where

$$\omega_{\alpha,\beta}(y_\beta) = (1-\lambda^{aM_{\alpha,\beta}})^{-1} \exp[-c\tau_{\alpha,\beta}(y_\beta; (1-\lambda^{aM_{\alpha,\beta}})^{-1}\chi_{\alpha,\beta}(y_\beta))]\,. \tag{20}$$

Using the relations (17) and (18) we can easily show the following.

Lemma 4. *Let $T_{\alpha,\beta}$, $T_{k,\alpha,\beta}$ and Π_k be as defined above. Let $y'_\beta = (y'_1, \ldots, y'_{M_\beta})$ with $y'_i = y_i \forall i \neq \Pi_k$, $y'_{\Pi_k} = y_{\Pi_k} + a$. Then*

$$\operatorname{tr} T_{k,\alpha,\beta} = \int^{\xi'_\beta} dy_\beta \omega_{\alpha,\beta}(y'_\beta)\,,$$

where ξ'_β is determined as follows:

for

$\alpha_k \geqq 2, \beta_k \geqq 2 : \xi'_i = \xi_i \forall i \neq \Pi_k - 1\,, \xi'_{\Pi_k - 1} = a\,;$

$\alpha_k = 1, \beta_k \geqq 2 : \xi'_i = \xi_i \forall i \neq (\Pi_k, \Pi_k - 1)\,, \xi'_{\Pi_k} = y_{\Pi_k + 1}\,, \xi'_{\Pi_k - 1} = a\,;$

$\alpha_k = 1, \beta_k = 1 : \xi'_i = \xi_i \forall i \neq \Pi_k\,, \xi'_{\Pi_k} = y_{\Pi_k + 1}\,;$

$\alpha_k \geqq 2, \beta_k = 1 : \xi'_i = \xi_i \forall i\,.$

Lemma 4 allows us to determine the trace of the operator $T_{\alpha,\beta}$ for fixed M_α and M_β. Consider the operator $T_{j+1,n-j} = \mathscr{L}_0^{j+1}\mathscr{L}_1^{n-j}$. From the recursion formulas (17) and (18) we get for $0 \leqq j \leqq n-2$, $n \geqq 2$, if we introduce the vector $\boldsymbol{y} = (y_{j+1}, \ldots, y_n) \in I^{n-j}$:

$$\chi_{j+1,n-j}(\boldsymbol{y}) = \sum_{k=0}^{n-j-1} \lambda^{(ka + y_{n-k})} \tag{21}$$

$$\tau_{j+1,n-j}(\boldsymbol{y}; z) = \sum_{\sigma=0}^{n-j-2} \sum_{k_\sigma=1}^{n-j-1-\sigma} [\lambda^{(k_\sigma a - y_{n-\sigma} + y_{n-\sigma-k_\sigma})} + z \sum_{k=0}^{n-j-1} \lambda^{(n+1-k)a - y_{n-k}}]\,. \tag{22}$$

Using the trace formula we get

$$\operatorname{tr} T_{j+1,n-j} = (1-\lambda^{a(n+1)})^{-1} \int^{\boldsymbol{\xi}} d\boldsymbol{y} \exp(-\mathrm{c}f_j(\boldsymbol{y})) \tag{23}$$

with $f_j(\boldsymbol{y})$ given by

$$f_j(\boldsymbol{y}) = [1-\lambda^{a(n+1)}]^{-1} \cdot \left[(n-j)\lambda^{(n+1)a} + \sum_{i=j+1}^{n-1} \sum_{k=i+1}^{n} (\lambda^{((k-i)a - y_k + y_i)} + \lambda^{((n+1+i-k)a + y_k - y_i)})\right] \tag{24}$$

and $\boldsymbol{\xi} = (a, y_{j+1}, \ldots, y_{n-1})$.

From this we can then deduce the trace of the operator $\mathscr{L}_0\mathscr{L}^n$ for $n\geqq 1$: Let $\boldsymbol{\alpha}:=(\alpha_1,\ldots,\alpha_\varrho)$, $\boldsymbol{\beta}':=(\beta'_1,\ldots,\beta'_\varrho)\in\mathbb{Z}^{*\varrho}$, where $\mathbb{Z}^*=\{1,2,3,\ldots\}$. Let furthermore be $\gamma,\delta\in\mathbb{Z}_+$. Then

$$\mathscr{L}_0\mathscr{L}^n=\mathscr{L}_0^{n+1}+\sum_{k=0}^{n-1}\mathscr{L}_0^{1+k}\mathscr{L}_1\mathscr{L}_0^{n-1-k}$$
$$+\sum_{j=0}^{n-2}\sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta}',\gamma,\delta:\,|\boldsymbol{\alpha}|+\gamma+\delta=j+1\\ |\boldsymbol{\beta}'|=n-j-1;\,\boldsymbol{\alpha},\boldsymbol{\beta}'\in\mathbb{Z}^{*\varrho};\,\gamma,\delta\in\mathbb{Z}_+}} T_{\boldsymbol{\alpha},\boldsymbol{\beta}'}\mathscr{L}_0^\gamma\mathscr{L}_1\mathscr{L}_0^\delta \tag{25}$$

where the third term only appears for $n\geqq 2$. So let us assume $n\geqq 2$. If we define the vector $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_\varrho)$ such that $\beta_i=\beta'_i$ for $i\neq\varrho$ and $\beta_\varrho=\beta'_\varrho+1$ we get $\beta_\varrho\geqq 2$. Let us recall that the numbers i_k have been defined by $i_k:=n+1-\sum_{l=k}^{\varrho}\beta_l$ for $k=1,\ldots,\varrho$. Because all $\beta_i\geqq 1$ we get

$$i_1=j+1<i_2<\ldots<i_\varrho=n+1-\beta_\varrho\leqq n-1\,. \tag{26}$$

Denote by $M_{j,n}$ the following set of integers

$$M_{j,n}:=\{j+1,j+2,\ldots,n-1\}\,, \tag{27}$$

and by $X:=\{i_k,k=1,\ldots,\varrho\}$. Then we have $|X|=\operatorname{card}X=\varrho$. It is also straightforward to show that

$$\varrho\leqq\min(j+1,n-j-1)\,. \tag{28}$$

Therefore $X\subset M_{j,n}$ and $|X|$ obeys the relation (28).

On the other hand given a subset $X\subset M_{j,n}$ with $X=\{i_1=j+1<i_2<\ldots<i_{|X|}\}$ and $|X|\leqq\min(j+1,n-j-1)$ there exists a unique vector $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_{|X|})\in\mathbb{Z}^{*|X|}$ with $\beta_{|X|}\geqq 2$ and $\sum_{i=1}^{|X|}\beta_i=n-j$ such that $i_{|X|}=n+1-\beta_{|X|}$ and $i_k=n+1-\sum_{l=k}^{|X|}\beta_l$. One only has to define $\beta_{|X|}:=n+1-i_{|X|}$ and $\beta_k=i_{k+1}-i_k$ for $1\leqq k\leqq|X|-1$. Therefore we can write the third term in (25) as follows

$$\sum_{j=0}^{n-2}\sum_{\substack{X\subset M_{j,n}\\ X=\{i_1=j+1<i_2\ldots<i_{|X|}\}\\ |X|\leqq\min(j+1,n-j-1)}}\sum_{\substack{\boldsymbol{\alpha},\sigma,\gamma\\ |\boldsymbol{\alpha}|+\sigma+\gamma=j+1}}$$
$$\cdot\mathscr{L}_0^{\alpha_1}\mathscr{L}_1^{(i_2-i_1)}\mathscr{L}_0^{\alpha_2}\mathscr{L}_1^{(i_3-i_2)}\ldots\mathscr{L}_0^{\alpha_{|X|}}\mathscr{L}_1^{n-i_{|X|}}\mathscr{L}_0^\gamma\mathscr{L}_1\mathscr{L}_0^\sigma\,. \tag{29}$$

Let us next write the vector $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_{|X|})$ as

$$\alpha_1=j+2-|X|-\sigma_1\,,\qquad \alpha_k=1+\sigma_{k-1}-\sigma_k\quad\text{for}\quad k=2,\ldots,|X|\,. \tag{30}$$

One can then show that $j+1-|X|\geqq\sigma_1\geqq\sigma_2\geqq\ldots\geqq\sigma_{|X|}\geqq 0$. From this it follows that the mapping

$$\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_{|X|})\to\boldsymbol{\sigma}=(\sigma_1,\sigma_2,\ldots,\sigma_{|X|}) \tag{31}$$

is $1-1$ and the inverse mapping of (30) is given by

$$\sigma_i=j+i+1-|X|-\sum_{k=1}^{i}\alpha_k\,.$$

With this we can write the expression (29) as

$$\sum_{j=0}^{n-2}\sum_{X\subset M_{j,n}}\sum_{\sigma_1=0}^{j+1-|X|}\cdots\sum_{\sigma_{|X|}=0}^{\sigma_{|X|-1}}\sum_{\varrho=0}^{\sigma_{|X|}}\mathscr{L}_0^{(j+2-|X|-\sigma_1)}\mathscr{L}_1^{(i_2-i_1)}\mathscr{L}_0^{(1+\sigma_1-\sigma_2)}\cdots$$
$$\cdot\mathscr{L}_0^{(1+\sigma_{|X|-1}-\sigma_{|X|})}\mathscr{L}_1^{(n-i_{|X|})}\mathscr{L}_0^{\varrho}\mathscr{L}_1\mathscr{L}_0^{(\sigma_{|X|}-\varrho)}. \tag{32}$$

It is clear that the operator $\mathscr{L}^{\sim}=\mathscr{L}_0^{\gamma}\mathscr{L}_1^{(i_2-i_1)}\mathscr{L}_0\mathscr{L}_1^{(i_3-i_2)}\ldots\mathscr{L}_0\mathscr{L}_1^{(n-i_{|X|}+1)}$ with $\gamma=j+1-(|X|-1)$ can be obtained from the operator $T_{j+1,n-j}$ simply by shifting $|X|-k$ times the operator $\mathscr{L}_0$ through the operator $\mathscr{L}_1^{(i_{k+1}-i_k)}$, $k=1,\ldots,|X|-1$. The operator $\mathscr{L}_0^{(j+2-|X|-\sigma_1)}\mathscr{L}_1^{(i_2-i_1)}\mathscr{L}_0^{(1+\sigma_1-\sigma_2)}\ldots\mathscr{L}_1^{(n-i_{|X|})}\mathscr{L}_0^{\varrho}\mathscr{L}_1\mathscr{L}_0^{(\sigma_{|X|}-\varrho)}$ can then be obtained from the operator $\mathscr{L}^{\sim}$ again by shifting operators $\mathscr{L}_0$ around. Using Lemma 4 we get then finally in the case $\varrho=0$:

$$\operatorname{tr}\mathscr{L}_0^{(j+2-|X|-\sigma_1)}\mathscr{L}_1^{(i_2-i_1)}\mathscr{L}_0^{(1+\sigma_1-\sigma_2)}\ldots\mathscr{L}_0^{\varrho}\mathscr{L}_1\mathscr{L}_0^{(\sigma_{|X|}-\varrho)}$$
$$=\int^{\boldsymbol{\xi}'}d\boldsymbol{y}\,\omega_{j+1,n-j}(y_{j+1}+(|X|-1+\sigma_1)a,\ldots,y_{i_k}+(|X|-k+\sigma_k)a,\ldots,y_{i_{|X|}}$$
$$+\sigma_{|X|}a,\ldots,y_{n-1}+\sigma_{|X|}a,y_n+\sigma_{|X|}a))$$

where $\boldsymbol{\xi}'=(\xi'_{j+1},\ldots,\xi'_n)$ and $\xi'_k=a\forall k\in X$, $\xi'_k=y_{k-1}\forall k\notin X$. In the case $\varrho\geqq 1$ we get

$$\operatorname{tr}(\ldots)=\int^{\boldsymbol{\xi}''}d\boldsymbol{y}\,\omega_{j+1,n-j}(y_{j+1}+(|X|-1+\sigma_1a),\ldots,y_{i_k}$$
$$+(|X|-k+\sigma_k)a,\ldots,y_{i_{|X|}}+\sigma_{|X|}a,\ldots,y_{n-1}+\sigma_{|X|}a,y_n+(\sigma_{|X|}-\varrho)a),$$

where $\xi''_i=\xi_i$ for $i\neq n$ and $\xi''_n=a$ and the function $\omega_{j+1,n-j}$ as in (20). After performing the summation over ϱ we arrive at

$$\operatorname{tr}\mathscr{L}_0\mathscr{L}^n=\operatorname{tr}\mathscr{L}_0^{n+1}+\sum_{k=0}^{n-1}\operatorname{tr}\mathscr{L}_0^{(1+k)}\mathscr{L}_1\mathscr{L}_0^{(n-1-k)}$$
$$+\sum_{j=0}^{n-2}\sum_{X\subset M_{J,n}}\sum_{\sigma_1=0}^{j+1-|X|}\cdots\sum_{\sigma_{|X|}=0}^{\sigma_{|X|-1}}\int^{\boldsymbol{\xi}}d\boldsymbol{y}\,\omega_{j+1,n-j}$$
$$\cdot(y_{j+1}+(|X|-1+\sigma_1)a,\ldots,y_{i_{k-1}}+(|X|-(k-1)+\sigma_{k-1})a,y_{i_k}$$
$$+(|X|-k+\sigma_k)a,\ldots,y_{i_{|X|}}+\sigma_{|X|}a,\ldots,y_{n-1}+\sigma_{|X|}a,y_n), \tag{33}$$

where $\boldsymbol{\xi}=(\xi_{j+1},\ldots,\xi_n)$ is given by $\xi_l=a\forall l\in X$, $\xi_l=y_{l-1}$

$$l\notin X,\ l\neq n,\ \xi_n=y_{n-1}+\sigma_{|X|}a.$$

The traces of the first two terms in (33) can be easily determined and we get

$$\operatorname{tr}\mathscr{L}_0\mathscr{L}^n=\frac{1}{[1-\lambda^{a(n+1)}]}\left[1+na\exp(-\lambda^{(n+1)a}/(1-\lambda)^{(n+1)a})\right.$$
$$\left.+\sum_{j=0}^{n-2}\sum_{X\subset M_{J,n}}\sum_{\sigma_1=0}^{j+1-|X|}\cdots\sum_{\sigma_{|X|}=0}^{\sigma_{|X|-1}}\int^{\boldsymbol{\xi}}d\boldsymbol{y}\,\omega^{\sim}_{j+1,n-j}(\boldsymbol{y},\boldsymbol{\sigma})\right] \tag{34}$$

where $\omega^{\sim}_{j+1,n-j}(\boldsymbol{y},\boldsymbol{\sigma})$ can be derived from the integrand in (33) and the third term again only appears for $n\geqq 2$.

Next we want to compare this trace with the partition function Z_{n+1} for a hard core system with periodic boundary conditions with exponential pair interaction $\lambda^{(y_i - y_{i-1})}$. It is convenient to introduce the coordinates y_i in the following way:

Consider the case where there are $(n-j)$ rods distributed on the interval $[0, (n+1)a]$ with periodic repetition outside this interval where $0 \leqq j \leqq n-2$. We denote the coordinate of the left corner of the i'th rod by $y_{n-(i-1)} + (i-1)a$. Then the interaction energy of this configuration is given by:

$$W_j(y) = \frac{c}{(1-\lambda^{a(n+1)})}\left[\sum_{n-j \geqq i > k \geqq 1} \lambda^{(y_{n-(i-1)} + (i-1)a - (y_{n-(k-1)} + (k-1)a))} + \sum_{n-j \geqq k \geqq i \geqq 1} \lambda^{(y_{n-(i-1)} + (i+n)a)' - (y_{n-(k-1)} + (k-1)a)}\right].$$

Some algebraic calculation shows that $W_j(y)$ can also be written as

$$W_j(y) = \frac{c}{1-\lambda^{a(n+1)}}\left[(n-j)\lambda^{(n+1)a} + \sum_{i=j+1}^{n-1} \sum_{k=i+1}^{n} \cdot \lambda^{(y_i + (k-i)a - y_k)} + \lambda^{(y_k - y_i + (i-k+n+1)a)}\right].$$

Comparing with (24) we see that

$$cf_j(y) = W_j(y). \tag{35}$$

If one includes the contributions coming from the configurations with 0 and 1 rod on the interval, the partition function Z_{n+1} is then given by

$$Z_{n+1} = 1 + \sum_{j=0}^{n-2} \int_0^{(j+1)a} dy_{j+1} \int_0^{y_{j+1}} dy_{j+2} \dots \int_0^{y_{n-1}} dy_n \exp[-cW_j(y)] + na \exp - [c\lambda^{(n+1)a}/(1-\lambda^{(n+1)a})] \tag{36}$$

By induction on n and $j = 0, 1 \dots n-2$, one can prove the following representation of the integral in (36).

Lemma 5. *Let* $M_{j,n} = \{j+1, \dots, n-1\}$ *and let* $X \subset M_{j,n}$ *such that*

$$X = \{i_1 = j+1 < i_2 < \dots < i_{|X|}\} \quad \textit{with} \quad |X| \leqq \min(j+1, n-j-1).$$

Let $\mathbf{y} = (y_{j+1}, \dots y_n)$ *and* $\boldsymbol{\xi}' = (\xi'_{j+1}, \dots, \xi'_n)$ *with* $\xi'_{j+1} = (j+1)a$, $\xi'_k = y_{k-1}$ *for* $k \neq j+1$. *Then*

$$\int^{\boldsymbol{\xi}'} d\mathbf{y}\,\omega(\mathbf{y}) = \sum_{X \subset M_{j,n}} \sum_{\sigma_1=0}^{j+1-|X|} \sum_{\sigma_2=0}^{\sigma_1} \dots \sum_{\sigma_{|X|}=0}^{\sigma_{|X|-1}} \int^{\boldsymbol{\xi}} d\mathbf{y}\,\omega^{\sim}(\mathbf{y}, \boldsymbol{\sigma})$$

for any $\omega \in C^\infty(\mathbb{R}^{n-j})$, *where* $\omega^{\sim}(\mathbf{y}, \boldsymbol{\sigma})$ *is given in terms of the function* ω *analogous to the definition in the expression* (34) *and the vector* $\boldsymbol{\xi}$ *is given as in expression* (33). *This gives finally*

Theorem 3. *Let Z_n be the partition function for a classical hard core system with hard core length a and exponential pair interaction $c\lambda^{(y_i - y_{i-1})}$ with periodic boundary conditions. Then for $n \geqq 1$*

$$Z_n = (1 - \lambda^{na}) \operatorname{tr} \mathscr{L}_0 \mathscr{L}^{n-1}$$

where the operators $\mathscr{L}_0$ and $\mathscr{L}_1$ are defined in Lemma 1.

III. The ζ-Function for a Hard Core System

Let us now look at the formal series

$$\zeta(z) = \exp \sum_{n=1}^{\infty} z^n Z_n / n$$

where Z_n is the partition function of a hard core system with periodic boundary conditions. Inserting the expression of Theorem 3 we get

$$\zeta(z) = \left[\exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} (1 - \lambda^{na}) \operatorname{tr} \mathscr{L}_0 \mathscr{L}^{n-1}\right)\right].$$

Because $|\operatorname{tr} \mathscr{L}_0 \mathscr{L}^{n-1}| \leqq \|\mathscr{L}\|^n \frac{\|\mathscr{L}_0\|_1}{\|\mathscr{L}\|}$, where $\|\mathscr{L}\|_1$ denotes the trace norm of the trace class operator $\mathscr{L}_0$ we get that $\zeta(z)$ is a holomorphic function in a neighbourhood of $z = 0$. Let us next discuss the question if $\zeta(z)$ extends to a meromorphic function in the whole z plane. Consider the following family of operators

$$T(\mu) := \mathscr{L} + \mu \mathscr{L}_0 .$$

Because, as we remarked already, the operator $\mathscr{L}$ can be shown to be a Hilbert Schmidt operator on the Hilbert space $H^1(I; A_2(D_R))$ of all H^1 mappings of the interval I into the Hilbert space $A_2(D_R)$ of all square integrable, holomorphic functions on D_R, where $H^1(I)$ is the well known Sobolev space $W_1^2(I)$ [17], the operator $T(\mu)^n$ is for every $n \geqq 2$ a holomorphic family of trace class operators on this Hilbert space. For such families the following formula holds [18]:

$$\frac{d}{d\mu} \operatorname{tr} T(\mu)^n = n \operatorname{tr}(T(\mu)^{n-1} \mathscr{L}_0) \quad \text{for all} \quad \mu \in \mathbb{C} .$$

At $\mu = 0$ this gives

$$\left.\frac{d}{d\mu} \operatorname{tr} T(\mu)^n\right|_{\mu=0} = n \operatorname{tr} \mathscr{L}_0 \mathscr{L}^{n-1} . \tag{37}$$

The Theorem of Lidskij [14] tells us on the other hand that for $n \geqq 2$ $\operatorname{tr} T(\mu)^n = \sum_{\{k\}} \lambda_k(\mu)^n$, where $\{\lambda_k(\mu)\}$ is the set of eigenvalues of the operator $T(\mu)$.

For the rest of the discussion let us restrict ourselves to the case where the interaction constant c vanishes, that means we consider the operator $\mathscr{L}: B \to B$

$$\mathscr{L} f(x, z) = f(a, \lambda^a z) + \int_0^x f(y, \lambda^y + \lambda^a z) dy .$$

The operator $T(\mu)$ for this case then reads

$$T(\mu)f(x,z)=(1+\mu)f(a,\lambda^a z)+\int_0^x f(y,\lambda^y+\lambda^a z)dy\,. \tag{38}$$

The spectrum of this operator can be determined as follows. First notice that if $f(x,z)$ is an eigenfunction with eigenvalue ϱ then the function $\frac{d}{dz}f(x,z)$ is also an eigenfunction with eigenvalue ϱ/λ^a. There we made use of the fact that any eigenfunction is holomorphic in z in a whole neighbourhood of clos D_R which follows by analytic continuation from the eigenvalue equation. Because $T(\mu)$ is compact there must therefore exist an eigenfunction $f_0(x,z)$ such that $\frac{d}{dz}f_0(x,z)=0$, that means $f_0(x,z)=f(x)$. For this function the eigenvalue problem reads as follows:

$$(1+\mu)f(a)+\int_0^x f(y)dy=\varrho(\mu)f(x)\,. \tag{39}$$

The solution to this equation in $C(I)$ is easily found to be

$$f(x)=\exp(\alpha x)\,, \tag{40}$$

where α is a solution of the equation

$$(1+\mu)\exp\alpha a=\alpha^{-1}\,, \tag{41}$$

with eigenvalue $\varrho(\mu)=\alpha(\mu)^{-1}$.

One can verify without difficulties the following properties of the numbers $\alpha(\mu)$ satisfying (41):

a) There exists a real solution α_0 iff μ is real. For $-1<\mu<\infty$ α_0 is positive.

b) There exist two sequences of solutions α_n and α_n^*, $n=1,2,\dots$ with $\mathrm{Im}\,\alpha_n>0$ and $\mathrm{Im}\,\alpha_n^*<0$ such that $|\alpha_n|\geqq n\pi/2a, |\alpha_n^*|\geqq n\pi/2a$ for all μ with $|\mu|\leqq\delta$, where δ is some small enough number.

The spectrum of the operator $T(\mu)$ is therefore given by the set

$$\sigma(T(\mu))=\{\lambda^{am}\varrho(\mu):m=0,1,\dots;\varrho(\mu)=(1+\mu)\exp(a/\varrho(\mu))\}\,. \tag{42}$$

From this consideration it follows that all the eigenvalues $\lambda(\mu)$ of the operator $T(\mu)$ are holomorphic in the disc $|\mu|<1$ and that for all $n\geqq 2$ the sum $\sum_{\{k\}}\lambda_k^n(\mu)$ converges uniformly in some small disc $|\mu|\leqq\delta$.

Therefore we get

$$\frac{d}{d\mu}\sum_{\{k\}}\lambda_k^n(\mu)=n\sum_{\{k\}}\lambda_k^{n-1}(\mu)\lambda_k'(\mu)$$

and at $\mu=0$

$$\mathrm{tr}\,\mathscr{L}_0\mathscr{L}^{n-1}=\sum_{\{k\}}\lambda_k^n\delta_k\,,\quad \delta_k=\lambda_k^{-1}\lambda_k'\,. \tag{43}$$

Inserting (43) into the definition of $\zeta(z)$ we get

$$\zeta(z)=\exp[z]\,Q(\lambda^a z)/Q(z)\,, \tag{44}$$

with

$$Q(z)=\prod_{\{k\}}(1-z\lambda_k)^{\delta_k}\exp z(\delta_k\lambda_k)\,.$$

Next we make use of the spectrum $\sigma(T(0))$ and get finally

$$\zeta(z)=\exp\left[z\left(1-\sum_{\{k\}}\varrho'_k\right)\right]\prod_{\{k\}}(1-z\varrho_k)^{-\frac{\varrho'_k}{\varrho_k}}, \tag{45}$$

where $\{\varrho_k\}$ are the zeros of the function $z\exp(-a/z)-1$ and $\varrho'_k=\varrho_k^2/(a+\varrho_k)$. Because $\varrho'_k/\varrho_k=\varrho_k/(a+\varrho_k)$, it is clear that the function $\zeta(z)$ is not meromorphic in the z plane.

By using expression (36) for the partition functions Z_n one can also perform the summation in the ζ-function and gets after some trivial algebra the following expression:

$$\zeta(z)=\exp\int_0^z \exp(az')/(1-z'\exp az')dz'\,. \tag{46}$$

Comparing with (45) one gets therefore the following interesting representation

$$\exp\int_0^z \exp(az')/(1-z'\exp az')dz'=\exp\left[z\left(1-\sum_{\{k\}}\varrho'_k\right)\right]\prod_{\{k\}}(1-z\varrho_k)^{-\frac{\varrho'_k}{\varrho_k}}.$$

Unfortunately we are not able to prove the representation (44) also for the interacting case $c\neq 0$ but it is our conjecture that it is true also in this case. It is interesting to note at this place that Kac et al. [19] treated this hard core system with the same interaction in an interesting paper in 1963. They indeed reduced the problem to the discussion of a certain integral equation of Hilbert-Schmidt type. It would be interesting to see the exact relation between this operator and our operator $\mathscr{L}$.

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Appendix A

For the readers convenience we repeat here the definition of the projective topological tensor product of two Banach spaces and of the π norm. Consider two Banach spaces E and F with their norms $\|\ \|_E$ and $\|\ \|_F$ respectively. Let be $E\otimes F$ the tensor product of the two spaces. Then one defines the following norm on $E\otimes F$:

$$\|x\|_\pi:=\inf\sum_i \|e_i\|_E\,\|f_i\|_F$$

where the infimum is taken over all possible representations of $x\in E\otimes F$ as $x=\sum_{\{i\}} e_i\otimes f_i$ with $e_i\in E$ and $f_i\in F$. The completion of the space $E\otimes F$ with respect to the norm $\| \ \|_\pi$ is denoted by $E\hat{\otimes}_\pi F$ and called the projective topological tensor product of the two spaces E and F. The elements of this space are also called Fredholm kernels.

Appendix B. Ruelles Trace Formula [6]

Let be $D \subset \mathbb{C}^n$ a bounded connected open subset and ψ a holomorphic mapping from a neighbourhood of clos D to D. Further let be φ an element of $A_\infty(D)$. Define the following linear operator $\mathscr{L}: A_\infty(D) \to A_\infty(D)$

$$\mathscr{L} f(z) := \varphi(z) f(\psi(z)).$$

Then $\mathscr{L}$ is a nuclear operator of order 0 and Trace $\mathscr{L} = \varphi(z^*) \det(1 - \psi'(z^*))^{-1}$, where z^* is the unique fixed point of the mapping ψ and $\psi'(z^*)$ is the derivative of ψ at z^*.

Note that this formula extends in a certain way the Lefschetz trace formula [20].

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