Commun. math. Phys. 52, 77—101 (1977)

Communications in
Mathematical
Physics

Induced Representations, Reproducing Kernels and the Conformal Group

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Abstract. The properties of "induced" (or multiplier) representations of groups which act in Hilbert spaces with a reproducing kernel are investigated. A resumé of earlier work is followed by a discussion of criteria for the irreducibility of such representations. The notions of reproducing kernel and positive definite spherical function are found to overlap. As a result, functional equations (analogous to those of Godement for spherical functions) are found for the reproducing kernel. The abstract theory is illustrated by certain discrete series representations of the conformal group and by their "limit points". In particular the so-called ladder representations (which give rise to the conformal symmetry of zero mass particles) are analysed from this viewpoint.

Reproducing Kernels

1. Introduction

This paper is mainly concerned with developing some results on group representations in Hilbert spaces with reproducing kernels. The ideas were motivated by the papers of Ruhl [2] on the conformal group, Bargmann's basic work [3] on the commutation relations and the paper of Perelomov [4] in which he points out that the notion of "coherent state" and reproducing kernel are the same.

The theory has its origins in the papers of Krein [9] and Aronszajn [10] although the foundations for the ideas developed here were laid by Kunze [1]. Subsection 2.1 is devoted to a resume of the results of [1]. This is followed by a discussion of criteria for the irreducibility of "induced representations" in reproducing kernel Hilbert spaces. The rest of Section 2 is concerned with the relationship between spherical functions and reproducing kernels. The existence of functional equations for reproducing kernels and their relationship to the functional equations satisfied by spherical functions of height one (see Godement [5] for this terminology) is investigated.

The general theory is illustrated by using the work of Graev [6] and Ruhl [2] on the discrete series representations of the conformal group [which for our purposes will be SU(2, 2)]. (Note that it was observed by Langlands [7] that the holomorphic discrete series representations of a semisimple Lie group can be constructed in Hilbert spaces with reproducing kernels.) In particular, we consider a series of representations labelled by a positive integer n which, for $n \geqq 4$, lie in the discrete series but for $n<4$ may be regarded as "limit points". An example of physical interest is the case $n=1$ which corresponds to the zero mass, zero helicity representation of the conformal group. This leads us to consider the construction of the so-called "ladder representations" of SU(2, 2) [8] in Hilbert spaces of functions holomorphic in the field theoretic "future tube". The reproducing kernels for these spaces are exhibited and the relationship to the usual infinitesimal form of the ladder representations [8] is determined by the fact that the Minkowski space wave functions are the boundary values in a "limit in mean" sense, of the holomorphic functions. Connections are made with the classical formulae for solutions of the wave equation.

2. General Theory

2.1. The Work of Kunze

In this subsection I will review the results of [1] and introduce the basic definitions and notation.[1]

Definition 2.1.1. Let X be a locally compact Hausdorff space, V a complex Hilbert space and H a Hilbert space whose elements are functions from X to V which are continuous in the norm topology on V. Suppose there exists a function $\phi: X \times X \to B(V)$ (the bounded operators on V) such that

(i) the functions

$$y \to \phi_x{}^* v(y) \overset{\text{def}}{=} \phi(x, y)^* v\,, \qquad v \in V\,.$$

lie in H for all $x \in X$,

(ii) $\langle \phi_x{}^* v, F \rangle_H = \langle v, F(x) \rangle_V$

for all $F \in H$ and $v \in V$,
then H is called a reproducing kernel Hilbert space with kernel ϕ.

Remarks. 1. The functions $\phi_x{}^* v$ are the "coherent states".

2. We use the adjoint $\phi(x, y)^*$ of $\phi(x, y)$ in the definition to accord with the notation of the examples of Section 3.

Let G be a locally compact group with X of the form G/K, K some closed subgroup of G.

Definition 2.1.2. A strongly continuous function $\tau: X \times G \to B(V)$ is a cocycle if

(i) $\tau(x, e) = I$ (I, the identity operator and e the identity of G),

(ii) $\tau(x, g_1 g_2) = \tau(x, g_1)\tau(g_1{}^{-1} \cdot x, g_2)$.

If K fixes the point x_0 say, in X, then (i) and (ii) imply that $k \to \tau(x_0, k)$ is a representation of K. Call this representation σ.

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[1] Note that our notation and formulation differs in several important respects from that used in [1]

If H is a Hilbert space whose elements are functions from X to V such that the equation

$$(U_g F)(x) = \tau(x, g) F(g^{-1} \cdot x) \tag{2.1}$$

defines $g \to U_g$ as a strongly continuous unitary representation of G then we call U the representation induced by the cocycle τ and write $\mathrm{ind}_\tau(H)$. This definition can be seen to be equivalent to the usual one (Varadarajan [10]) in certain circumstances. To see this define $b: X \to B(V)$ by

$$b(x) = \tau(x_0, \gamma(x)^{-1})$$

where $\gamma: X \to G$ is a regular Borel cross-section [10] of $G \to G/K$ such that $\gamma(x) \cdot x_0 = x$ and $\gamma(x_0) = e$. Now define

$$(BF)(x) = b(x) F(x).$$

Then provided $BH \leqq L^2(X, V, \mu)$ where μ is an invariant measure on X, it is not difficult to show that B intertwines $\mathrm{ind}_\tau(H)$ with the representation induced by the cocycle

$$\sigma_\gamma(x, g) = \tau(x_0, \gamma(x)^{-1} g \gamma(g^{-1} \cdot x)).$$

Hence whenever $\sigma_\gamma(x, g)$ is unitary for all x, g then B sets up an equivalence of $\mathrm{ind}_\tau(H)$ with a subrepresentation of the representation of G on $L^2(X, V, \mu)$ induced by σ. In general however, this is not the case. Examples of such representations will be considered in Sections 3 and 4.

The first result of [1] is as follows.

Theorem 2.1.3 (Kunze). *If H is a Hilbert space (whose elements are functions from X to V) having reproducing kernel ϕ, then the formula (2.1) defines a strongly continuous unitary representation of G if and only if ϕ satisfies the covariance relation*

$$\phi(g^{-1} \cdot x, y)^* \tau(x, g)^* = \tau(y, g^{-1}) \phi(x, g \cdot y)^*. \tag{2.2}$$

Now, the kernel ϕ on the homogeneous space X may be "lifted" to a kernel on the group G. Following [1] define:

$$\chi(g_1, g_2)^* = \tau(x_0, g_2^{-1}) \phi(x, y)^* \tau(x_0, g_1^{-1})^* \tag{2.3}$$

where $g_1 \cdot x_0 = x$ and $g_2 \cdot x_0 = y$. Then it is straightforward to verify

(i) $\chi(g_1, g_2)^* = \chi(g_2, g_1)$, [note that by definition $\phi(x, y)^* = \phi(y, x)$],

$$\text{(ii)}\quad \chi(g g_1, g g_2)^* = \chi(g_1, g_2)^*, \tag{2.4}$$

$$\text{(iii)}\quad \chi(g_1 k_1, g_2 k_2)^* = \tau(x_0, k_2^{-1}) \chi(g_1, g_2)^* \tau(x_0, k_1^{-1})^*. \tag{2.5}$$

We can also "lift" the Hilbert space H by defining for each $F \in H$ a V-valued function on G by

$$F^\sim(g) = \tau(x_0, g^{-1}) F(g \cdot x_0).$$

The functions $F^\sim$ form a Hilbert space $H^\sim$ with inner product

$$\langle F^\sim, F'^\sim \rangle \stackrel{\text{def}}{=} \langle F, F' \rangle_H.$$

Hence,

$$\begin{aligned}\langle \chi_g{}^*v, F^\sim\rangle &= \langle \phi^*_{g\cdot x_0}\tau(x_0, g^{-1})^*v, F\rangle_H \\ &= \langle v, \tau(x_0, g^{-1})F(g\cdot x_0)\rangle_V \\ &= \langle v, F^\sim(g)\rangle_V .\end{aligned}$$

So $H^\sim$ has reproducing kernel χ and carries a representation W defined by

$$(W_g F^\sim)(g_0) = F^\sim(g^{-1}g_0) \tag{2.6}$$

which is equivalent to $\mathrm{ind}_\tau(H)$.

Let us assume that $k \to \tau(x_0, k)$ is unitary. Let H be as above. If we make the further restriction that the maps $E_x: f \to f(x)$ from H to V have dense range for all $x \in X$, then Kunze has characterized those group representations which may be constructed in a reproducing kernel Hilbert space. (Note however that there exist reproducing kernel Hilbert spaces which do not satisfy this restriction, cf. Section 4.)

Definition 2.1.4 (Kunze). Let U be a continuous unitary representation of G on a Hilbert space H_0, K a closed subgroup of G and σ a strongly continuous unitary representation of K on a Hilbert space V. If there exists a closed subspace V' of H_0 which carries a representation of K unitarily equivalent to σ and is such that the linear span of $\{U_g v | g \in G, v \in V'\}$ is dense in H_0, then we say that U is of type σ.

Given a representation U of type σ, and $A: V \to V'$ a unitary transformation setting up the equivalence of the preceding definition, define $\Phi: G \to B(V)$ by

$$\langle u, \Phi(g)v\rangle = \langle Au, U_g Av\rangle , \qquad u, v \in V . \tag{2.7}$$

Now, defining $\chi(g_1, g_2)^* = \Phi(g_1{}^{-1}g_2)^*$ Kunze shows (Theorem 5 of [1]) that there is a reproducing kernel Hilbert space $H^\sim$ (with kernel χ) consisting of continuous functions from G to V, such that χ satisfies

$$\chi(g_1k_1, g_2k_2)^* = \sigma(k_2{}^{-1})\chi(g_1, g_2)^*\sigma(k_1) . \tag{2.8}$$

Furthermore, the representation of G defined by

$$(g\cdot F)(g_0) = F(g^{-1}g_0)\,(F \in H^\sim) \tag{2.9}$$

is equivalent to U.

The above discussion establishes the main results of [1]. The remainder of this section is concerned with developing some elementary consequences of the above definitions. Note that the assumption that E_x has dense range for all $x \in X$ will not be made (this leads to a more cumbersome notation than that adopted by Kunze).

(i) Let H be as above and H_0 a subspace of H. Then H_0 has a reproducing kernel $\psi: X \times X \to B(V)$ defined by

$$\langle \psi(x, y)^*u, v\rangle_V = \langle P\phi_x{}^*u(y), v\rangle_V \tag{2.10}$$

where P is the orthogonal projection from H to H_0.

(ii) If we drop the continuity requirement from the definition of H then it can be shown that a Hilbert space of functions from X to V has a reproducing kernel if and only if every element F of the space satisfies

$$\|F(x)\|_V \leqq \omega(x)\|F\|_H \quad \text{for all} \quad x \in X,$$

where $\omega(x)>0$ depends only on x. (This requires some work to prove.)

(iii) Quite often H will be a subspace of a larger Hilbert space H_1. Then ϕ defines the orthogonal projection onto H_1 by $f \to F$ where

$$\langle v, F(x)\rangle_V = \langle \phi_x{}^* v, f\rangle_{H_1}, \qquad f \in H_1. \tag{2.11}$$

(iv) By the Cauchy-Schwarz inequality we have the useful bounding property of ϕ:

$$|\langle v, F(x)\rangle_V| \leqq \langle v, \phi(x, x)v\rangle_V^{1/2} \|F\|_H \tag{2.12}$$

for all $v \in V$.

For the remainder of the paper we will assume that $k \to \tau(x_0, k)$ is unitary, that $\phi(x_0, x_0) = I$ and that H carries the unitary representation $\mathrm{ind}_\tau(H)$ of G. With these assumptions we prove (cf. Theorem 4 of [1]):

Lemma 2.1.5. *Denote the restriction of* $\mathrm{ind}_\tau(H)$ *to* K *by* $k \to S_k$. *Then the vectors* $\{\phi_{x_0}{}^* v | v \in V\}$ *span a subspace* V_0 *of* H *carrying a representation of* K *equivalent to* σ. *Further, the projection* P_0 *onto this subspace is given by*

$$\begin{aligned} P_0(\phi_x{}^* u)(y) &= \phi(x_0, y)^* \phi(x, x_0)^* u \\ &\equiv \phi_{x_0}{}^* \phi(x, x_0)^* u(y). \end{aligned} \tag{2.13}$$

Proof. By virtue of (2.2),

$$\begin{aligned} S_k(\phi_{x_0}{}^* u)(x) &= \tau(x, k)\phi_{x_0}{}^* u(k^{-1} \cdot x) \\ &= \phi_{x_0}{}^* \tau(x_0, k^{-1})^* u(x). \end{aligned}$$

Hence $S_k \phi_{x_0}{}^* u = \phi_{x_0}{}^* \sigma(k) u$. Clearly therefore, S acting in the space V_0 gives a representation equivalent to σ (note that the map $V \to V_0$ defined by

$$v \to \phi_{x_0}{}^* v$$

is an isometry).

Consider the operator P_0 defined on the vectors $\phi_x{}^* u$ by (2.13). Observe that the space H_ϕ defined as the linear span of $\{\phi_x{}^* v | v \in V, x \in X\}$ is dense in H. So extending P_0 linearly to H_ϕ makes P_0 densely defined. Further, if

$$F = \sum\nolimits_{x,u} \lambda_x \phi_x{}^* u$$

is an element of H_ϕ, then by direct computation

$$\|P_0 F\|^2 = \langle F(x_0), F(x_0)\rangle_V.$$

But by (2.12) we have

$$\|F(x_0)\| \leqq \|F\|.$$

So P_0 is bounded on H_ϕ and hence on H. As $\phi(x_0, x_0) = I$ so $P_0^2 = P_0$. It is straightforward to check that $P_0 = P_0{}^*$ thus completing the proof.

Remark. Acting on an arbitrary $F \in H$, P_0 has the form

$$(P_0F)(x) = \phi(x_0, x)^* F(x_0).$$

2.2. Criteria for Irreducibility

Following Krein [9] we introduce the

Definition 2.2.1. Let H, ϕ be as for Lemma 2.1.5 and denote the strongly continuous V-valued functions on X by $C(X, V)$. Call ϕ (G, K, V)-zonal if the conditions:

(i) there is a function $\psi: X \times X \to B(V)$ such that both ψ and $\phi - \psi$ are reproducing kernels for Hilbert spaces of functions contained in $C(X, V)$,

(ii) ψ satisfies (2.2), force ψ to be a multiple of ϕ.

Note that whenever ψ satisfies (i) we can define

$$A(\phi_x{}^* u, \phi_y{}^* v) = \langle \psi(x, y)^* u, v \rangle_V.$$

A can be extended to a sesquilinear functional on $H_\phi \times H_\phi$ (H_ϕ as in the proof of Lemma 2.1.5). Since $\phi - \psi$ is also a reproducing kernel we have

$$A(F, F) \leqq \langle F, F \rangle_H$$

for all $F \in H_\phi$. Thus A is continuous from $H_\phi \times H_\phi$ to $\mathbb{C}$ and therefore extends uniquely to $H \times H$. By the Riesz representation theorem there is a bounded (self-adjoint) operator $A^\sim: H \to H$ such that

$$\langle A^\sim F, F' \rangle = A(F, F')$$

for all $F, F' \in H$.

Now let us record the

Lemma 2.2.2. *If $B: H \to H$ is bounded then B has a kernel $b: X \times X \to B(V)$ such that*

$$\langle u, (BF)(x) \rangle_V = \langle b_x u, F \rangle_H$$

where $b_x(y) = b(x, y)$ is defined by

$$\langle b(x, y) u, v \rangle_V = \langle B^* \phi_x{}^* u(y), v \rangle_V.$$

Further, B is in the commuting algebra of $\mathrm{ind}_\tau(H)$ *if and only if b satisfies (2.2).*

The proof is straightforward so we omit it. The point of the preceding discussion lies in

Lemma 2.2.3. *ϕ is (G, K, V)-zonal if and only if* $\mathrm{ind}_\tau(H)$ *is irreducible.*

Proof. Suppose $\mathrm{ind}_\tau(H)$ is not irreducible and P is a non-zero projection onto a proper invariant subspace. Now PH has a reproducing kernel say ψ and hence the kernel for $(I-P)H$ is $\phi - \psi$. Now applying Lemma 2.2.2 to P forces ψ to satisfy conditions (i) and (ii) of Definition 2.2.1. So ϕ is not zonal.

Conversely, let ψ satisfy conditions (i) and (ii) of the definition with ψ not a multiple of ϕ. By the remarks preceding Lemma 2.2.2, ψ defines an operator $A^\sim: H \to H$. Now by Lemma 2.2.2 $A^\sim$ is in the commuting algebra of $\mathrm{ind}_\tau(H)$ (noting that $A^{\sim}$ necessarily has as its kernel, ψ). Since ψ is not a multiple of ϕ, this forces the commuting algebra to be non-trivial. Equivalently, $\mathrm{ind}_\tau(H)$ is not irreducible.

Our object now is to determine conditions under which a given kernel is zonal. As before denote the restriction of $\mathrm{ind}_\tau(H)$ to K by $k \to S_k$.

Theorem 2.2.4. *Let H, ϕ, $\mathrm{ind}_\tau(H)$ be as above and P_0 be the projection defined in Lemma 2.1.5. Suppose that*

$$k \to \tau(x_0, k)$$

is an irreducible representation of K and there exists a subgroup G_0 of G such that the following condition holds: there is a projection P_c with $P_0 \geqq P_c$, lying in the centre of the commuting algebra of the representation $\mathrm{ind}_\tau(H)$ restricted to G_0. Then ϕ is (G, K, V)-zonal.

Remark. One way in which the condition on P_c may be realized is if there is a subspace P_cH of P_0H, such that on restricting $\mathrm{ind}_\tau(H)$ to G_0, one obtains a maximal primary (or factor) representation of G_0 (in the terminology of Mackey [11]) in P_cH.

Proof. Suppose that $\mathrm{ind}_\tau(H)$ is not irreducible and that P is the projection onto a proper invariant subspace. Define the kernel ψ for PH by (2.10). From the hypotheses on P_c we deduce that $P_cP_0 = P_0P_c$ and $P_cP = PP_c$. Further, the map $v \to \phi_{x_0}{}^*v$ is an isometry from V onto P_0H and therefore P_c defines a projection $P_c^{\sim}$ say, on V. Hence, for $u \in P_c^{\sim}V$:

$$\psi_{x_0}{}^*u = P\phi_{x_0}{}^*u = PP_c\phi_{x_0}{}^*u = P_c\psi_{x_0}{}^*u\,. \tag{2.14}$$

Further ψ satisfies (2.2) and hence the relation

$$\psi(x_0, x_0) = \tau(x_0, k^{-1})\psi(x_0, x_0)\tau(x_0, k)\,.$$

But $k \to \tau(x_0, k) = \sigma(k)$ is irreducible and so $\psi(x_0, x_0)$ is a multiple: λI of the identity. Now (2.14) implies that $\psi_{x_0}{}^*u$ lies in P_0H and hence that

$$\langle \phi_{x_0}{}^*v, \psi_{x_0}{}^*u\rangle_H = \lambda\langle v, u\rangle_V$$

for all $v \in V$. From this it follows readily that $\psi_{x_0}{}^*u = \lambda\phi_{x_0}{}^*u$ for all $u \in P_cV$. This forces $\lambda = 1$ and since σ is irreducible we must have

$$\psi(x_0, x)^* = \phi(x_0, x)^*\,.$$

Now let $g \cdot x_0 = x$ so that from (2.2)

$$\begin{aligned}\phi(x, y)^*\tau(x_0, g^{-1})^* &= \tau(y, g)\phi(x_0, g^{-1}\cdot y)^*\\ &= \tau(y, g)\psi(x_0, g^{-1}\cdot y)^*\\ &= \psi(x, y)^*\tau(x_0, g^{-1})^*\,.\end{aligned}$$

Hence $\phi = \psi$, implying that P is the identity; a contradiction. So $\mathrm{ind}_\tau(H)$ is irreducible.

Corollary 2.2.5. *If the multiplicity of the representation σ in the restriction of $\mathrm{ind}_\tau(H)$ to K is one then ϕ is (G, K, V)-zonal.*

Proof. Set P_c equal to P_0.

Remark. In this context I should mention the very important result of Kobayashi [12]. With the notation as before suppose that X has the structure of an n-dimensional complex manifold with quasi-invariant measure μ. Form the holomorphic vector bundle E over G/K with fibre V (this is the induced bundle). The representation σ acts in the fibres and on the space of holomorphic n-forms on G/K with values in E, it induces a representation U of G. Kobayashi shows that there is a reproducing kernel ϕ for the Hilbert space H of square integrable holomorphic n-forms. Furthermore whenever H is non-trivial the restriction of U to H defines a unitary representation of G so that ϕ satisfies an analogue of (2.2). Kobayashi then shows that this representation is always irreducible. In Section 3 we will see the idea behind Kobayashi's proof used in a particular example.

In the case where σ is not irreducible it is more difficult to find conditions for zonality. In this direction we have the following result. Recall the definition of $H^{\frown}$, the kernel $\chi: G\times G\to B(V)$ [Eq. (2.3)] and the representation W of G defined by (2.6).

Proposition 2.2.6. *If P_0 lies in the centre of the commuting algebra of the representation $k\to W_k$ of K then the commuting algebra of $\{W_g|g\in G\}$ is isomorphic (as a von Neumann algebra) to the algebra of operators which commute with $\{\chi(g)|g\in G\}$.*

The proof is not difficult and so we omit it. We conclude this subsection with the remark that there are close connections between the ideas discussed here and Naimark's notion of "quasihomogeneity" [13]. In fact when σ is irreducible and K is compact it is not difficult to show that $\mathrm{ind}_\tau(H)$ is quasihomogeneous in the sense of [13]. One should compare Proposition 2.2.6 with Naimark's Theorem 5.

2.3. Functional Equations

We lead into the connection between spherical functions and reproducing kernels with a brief introduction to Godement's work [5]. Suppose that K is a compact subgroup of G, that T is a representation of G on a Banach space B and that in the restriction of T to K, σ occurs with finite multiplicity. Denote by $B(\sigma)$ the closed subspace of B formed by taking the linear span of those vectors which transform under the representation $k\to T_k$ of K according to σ. If there exists a continuous projection $E(\sigma)$ of B onto $B(\sigma)$ define

$$\phi_\sigma(g)=\mathrm{Tr}(E(\sigma)T_g)\,,\qquad g\in G\,.$$

Then ϕ_σ is called a spherical function for G, K and Godement proves [5] that ϕ_σ satisfies

$$\int_K \phi_\sigma(kg\,k^{-1}g')dk=\phi_\sigma(g)\phi_\sigma(g') \tag{2.15}$$

if and only if σ occurs once only in the representation $k\to T_k$ of K. Let 1 denote the trivial representation of K and set

$$T=\mathrm{ind}_1(H)$$

where H is a reproducing kernel Hilbert space with kernel ϕ.

Suppose that 1 occurs once only in $k \to T_k$ and set

$$\chi_{x_0}(g)^* = \phi_{x_0}(g \cdot x_0)^* = \mathrm{Tr}(P_0 T_{g^{-1}}).$$

So χ_{x_0} is a spherical function and the functional Equation (2.15) becomes:

$$\int_K \phi_{x_0}(kg\, k^{-1} g' \cdot x_0) dk = \phi_{x_0}(g \cdot x_0) \phi_{x_0}(g' \cdot x_0).$$

Using (2.2) this reduces to:

$$\int_K \phi_{x_0}(g'^{-1} k g^{-1} \cdot x_0)^* dk = \int T_{k^{-1}} \phi_{g' \cdot x_0}(g^{-1} \cdot x_0)^* dk.$$

But $\int_K T_k dk$ is the projection P_1 onto the fixed point set of the representation $k \to T_k$ of K. So the functional equation is

$$(P_1 \phi_x^*)(y) = \phi_x(x_0)^* \phi_{x_0}(y)^*$$

for all $x, y \in X$ (setting $g' \cdot x_0 = x$ and $g^{-1} \cdot x_0 = y$).

Now remove the compactness assumption on K and return to the general situation with the assumptions as in the preamble to Lemma 2.1.5. We impose, for the remainder of this section, the additional restriction that σ be irreducible (and hence that the criterion for zonality given by Corollary 2.2.5 is valid). With $k \to S_k$ denoting the restriction to K of $\mathrm{ind}_\tau(H)$ write $C(S)$ for the commuting algebra of S. By Lemma 2.1.5 there is a subspace of H, say $H(\sigma)$, which carries the representation S^σ of K satisfying (see Mackey [11] for the terminology):

(i) S^σ is maximal primary.

(ii) Every subrepresentation of S^σ is quasiequivalent to σ.

[To see this one takes the central support in $C(S)$ of P_0.] As a consequence of Lemma 2.1.5 we have

Theorem 2.3.1. *With the restrictions of the previous paragraph, the projection P_σ onto $H(\sigma)$ satisfies for all $u \in V$, $x \in X$:*

$$P_\sigma \phi_x^* u = \phi_{x_0}^* \phi(x, x_0)^* u \tag{2.16}$$

if and only if the multiplicity of σ in $\mathrm{ind}_\tau(H)$ *restricted to K is one.*

In the particular case where G is unimodular and K is compact P_σ has the form

$$(P_\sigma F)(g) = \int_K \overline{C(k)}\, F(k^{-1} g) dk \tag{2.17}$$

for $F \in H$, where $C(k) = d_\sigma \mathrm{Tr}\, \sigma(k)$ with d_σ the dimension of σ (see [5]). Recalling the definition of χ [Eq. (2.3)] we have

Corollary 2.3.2. *With the assumptions above, the irreducible representation σ occurs once only in the restriction of* $\mathrm{ind}_\tau(H)$ *to K if and only if*

$$\int_K \overline{C(k)} \chi(g_1, k^{-1} g_2)^* dk = \chi_e(g_2)^* \chi_{g_1}(e)^*. \tag{2.18}$$

Proof. Apply both sides of (2.18) to some $v \in V$ and then substitute from (2.3). The result then follows directly from Theorem 2.3.1.

When $V = \mathbb{C}$ we can interpret the theorem as asserting that the function χ_e is a positive definite spherical function of height one (see [5]). For, in this case

$$\overline{c(k)} \chi_{g_1}(g_2)^* = \chi_{g_1}(g_2 k)^*$$

and so

$$\int_K \chi_e(g_1{}^{-1}k^{-1}g_2k)^* dk = \chi_e(g_2)^* \chi_{g_1}(e)^* .$$

Noting that a reproducing kernel is always a positive definite operator valued function [1], this suggests we consider the converse problem of when a positive definite spherical function is a reproducing kernel. Following Warner [14] we introduce the following assumptions and definitions.

(a) Let G be unimodular with K a large compact subgroup and let μ be a finite dimensional representation of K acting on a Banach space V.

(b) Define a μ-spherical function to be a continuous function

$$\Psi : G \to \mathrm{Hom}(V, V)$$

such that

$$\Psi(k_1 g k_2) = \mu(k_1)\Psi(g)\mu(k_2) .$$

(c) Let U be a topologically completely irreducible strongly continuous representation of G in a Banach space B such that in the restriction of U to K, a given irreducible σ occurs with multiplicity $m \neq 0$ (necessarily finite as K is large in G). Let μ be the representation $m\sigma$ (i.e. the direct sum of m copies of σ). Suppose further that there is a continuous projection P_μ onto the space of vectors in B which transform according to σ under $k \to U_k$ and define

$$\Psi^U_\sigma(g) = P_\mu U_g P_\mu .$$

Then Ψ^U_σ is a μ-spherical function. All spherical functions ψ, "of type σ and height m" (see [5]), have the form $\psi(g) = \mathrm{Tr}\, \Psi^U_\sigma(g)$ for some μ-spherical function Ψ^U_σ [14].

Now, on $V = P_\mu B$ we can define an inner product such that μ is unitary. Then ψ defined by

$$\psi(g) = \mathrm{Tr}\, \Psi^U_\sigma(g) \tag{2.19}$$

is positive definite whenever Ψ^U_σ is such that

$$g \to \langle v, \Psi^U_\sigma(g) v \rangle_V$$

is positive definite for all $v \in V$. In this case we call Ψ^U_σ a positive definite μ-spherical function. One may now use Theorem 1 of [1] to obtain a reproducing kernel Hilbert space for which

$$g, g_0 \to \Psi^U_\sigma(g^{-1}g_0)^*$$

is the reproducing kernel. Alternatively consider the following interesting argument (due essentially to K.C. Hannabuss).

Define Ψ^U_σ as above to be a positive definite μ-spherical function. For convenience write Ψ for Ψ^U_σ. Let $C_0(G, V)$ be the space of strongly continuous V-valued functions with compact support in G. Define:

$$\Psi(f)(g_0) = \int_G \Psi(g^{-1}g_0)^* f(g) dg$$

for all $f \in C_0(G, V)$. Denote by $\Psi(G, V)$ the image of $C_0(G, V)$ under the map $f \to \Psi(f)$. On $\Psi(G, V)$ there is an inner product

$$\langle \Psi(f), \Psi(f) \rangle_\Psi = \iint \langle f(g_0), \Psi(g^{-1}g_0)^* f'(g) \rangle_V dg\, dg_0 .$$

Positivity of the inner product follows from the positive definite property of Ψ. If for some f, $\langle\Psi(f), \Psi(f)\rangle_\Psi=0$ then $\langle\Psi(f'), \Psi(f)\rangle_\Psi=0$ for all $f'\in C_0(G, V)$ by the Cauchy-Schwarz inequality. Equivalently,

$$\int_G \langle f'(g), \Psi(f)(g)\rangle_V dg=0$$

for all $f'\in C_0(G, V)$ and hence $\Psi(f)=0$. So $\langle\ ,\ \rangle_\Psi$ is non-degenerate and we may complete $\Psi(G, V)$ to give a Hilbert space $H(\Psi)$ say.

Lemma 2.3.3. *$H(\Psi)$ is a reproducing kernel Hilbert space with kernel $g, g_0\to \Psi(g^{-1}g_0)^*$.*

Proof. Let δ_e be the Dirac measure at the identity and as usual write $\Psi_g{}^*v$ for the function $g_0\to\Psi(g^{-1}g_0)^*v (v\in V)$. Then by taking an approximate identity $\{f_\alpha\}_{\alpha\in A}$ for G it is not difficult to see that $\Psi(f_\alpha v)$ converges weakly in $H(\Psi)$ to $\Psi(\delta_e v)$. Hence $\Psi_g v(g_0)^*=\Psi(\delta_g v)(g_0)$ and it is straightforward to verify that $g_0, g\to\Psi(g_0{}^{-1}g)^*$ has the reproducing property for functions in $\Psi(G, V)$. An arbitrary element $F\in H(\Psi)$ may be identified with the function $F^\sim$ defined by $\langle v, F^\sim(g)\rangle_V=\langle\Psi_g{}^*v, F\rangle_\Psi$ (cf. Theorem 1 of [1]).

Returning to the main argument, observe that on $H(\Psi)$ we can define a strongly continuous unitary representation by

$$W_g\Psi(f)(g')=\int_G \Psi(g_0{}^{-1}g')^* f(g^{-1}g_0)dg_0\,.$$

As the elements of $H(\Psi)$ satisfy $F(xk)=\mu(k^{-1})F(x)$ the representation $g\to W_g$ is of the "induced" form we have been discussing to date.

Now let

(i) V_0 be the space of vectors $\{\Psi_e{}^*v|v\in V\}$.

(ii) P_0 be the projection onto V_0 given by (Lemma 2.1.5)

$$P_0\Psi_g{}^*v=\Psi_e{}^*\Psi(g, e)^*v\,.$$

Then we have for all $u, v\in V$

$$\langle\Psi_e{}^*u, P_0W_gP_0\Psi_e{}^*v\rangle=\langle u, \Psi(g, e)^*v\rangle$$

so we have, from the definition of $\Psi=\Psi_\sigma^U$,

$$\Psi_e(g)=E(\mu)U_gE(\mu)=P_0W_gP_0\,. \tag{2.20}$$

Lemma 2.3.4. *The representations U and W are Naimark equivalent (see* [14]*).*

Proof. Let B_0 denote the dense subspace of B spanned by $\{U_gv|g\in G, v\in V\}$. Let H_Ψ be the span of $\{\Psi_g{}^*v|g\in G, v\in V\}$. Following Kunze [1] define

$$SU_gv=\Psi_g{}^*v\,.$$

Extend S linearly to B_0. Then clearly S is one-one and onto H_Ψ and satisfies $SU_g=W_gS$ on B_0. To complete the proof we will show that $S^{-1}: H_\Psi\to B_0$ is closable. For this, it is sufficient to show that if $\{F_r\}\subseteq H_\Psi$ is a sequence converging to zero in $H(\Psi)$ and $S^{-1}F_r\to f\in B_0$ then $f=0$.

If $F_r(g)=(\sum_j \Psi_{g_j}{}^*v_j(g))^{(r)}$ then, as norm convergence in $H(\Psi)$ implies pointwise convergence we have

$$(\textstyle\sum_j \Psi(g_j{}^{-1}g)^*v_j)^{(r)}\to 0 \quad \text{for all} \quad g\in G\,.$$

That is, $E(\mu)U_{g^{-1}}(\sum_j U_{g_j}v_j)^{(r)}\to 0$ [using (2.20)], for all $g\in G$. Equivalently $E(\mu)U_g f=0$ for all $g\in G$. But U is topologically completely irreducible and so f must be zero.

We have now almost proved

Theorem 2.3.5. (i) *If ψ is a spherical function of type σ and height m defined via* (2.19) *in terms of a positive definite μ-spherical Ψ^U_σ (where U is topologically completely irreducible and $\mu=m\sigma$) then there is a unitary irreducible representation W of G in a Hilbert space $H(\Psi)$ with reproducing kernel*

$$g, g_0\to\Psi^U_\sigma(g^{-1}g_0)^*$$

such that U is Naimark equivalent to W. Further, with P_0 denoting the projection onto the subspace of $H(\Psi)$ spanned by $\{\Psi_e{}^*v|v\in V\}$,

$$\psi(g)=\mathrm{Tr}(P_0W_gP_0).$$

(ii) *The reproducing kernel satisfies the functional equation*

$$\int_K \overline{C(k)}\Psi_{g_0}(k^{-1}g)^*dk=\Psi_e(g)^*\Psi_{g_0}(e)^*. \tag{2.21}$$

Proof. (i) It remains only to prove that W is irreducible. This follows from the remark on page 305 of [14].

(ii) If we can show that $P_\sigma=P_0$ then the proof is complete. This follows from Proposition 4.5.1.6 of [14] (cf. also 6.1.1.8 of Volume 2 of [14].)

The theorem asserts the existence of a functional equation for positive definite μ-spherical functions of arbitrary height. Note that only in the case where the height is one does there exist a corresponding equation for the spherical function

$$\psi(g)=\mathrm{Tr}\,\Psi^U_\sigma(g).$$

Further, the reproducing kernel approach allows a reinterpretation of the functional equation as asserting (when the multiplicity of σ is one) a sufficient condition for the irreducibility of the representation determined by the kernel.

3. Application to SU(2, 2)

3.1. Notation

Throughout this section $G=\mathrm{SU}(2,2)$ and $K=\mathrm{SU}(2)\times\mathrm{SU}(2)\times U(1)$ is the maximal compact subgroup of G. The notation and preliminary results are to be found in Ruhl [2]. The usual form of SU(2,2) consists of 4×4 matrices

$$M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$$

(where A, B, C, D are 2×2 matrices over $\mathbb{C}$) such that

$$M^*E=EM^{-1}, \qquad \det M=1, \tag{3.1}$$

with

$$E=\begin{pmatrix}\sigma_0 & 0\\ 0 & -\sigma_0\end{pmatrix},$$

σ_0 being the 2×2 identity. We will consider the form of SU(2,2) obtained from that above by letting

$$U=2^{-1/2}\begin{pmatrix}\sigma_0 & -\sigma_0\\ \sigma_0 & \sigma_0\end{pmatrix}$$

and defining the map $M\to UMU^{-1}$.

This takes SU(2,2) onto the group of matrices of determinant one which preserve $\begin{pmatrix}0 & \sigma_0\\ \sigma_0 & 0\end{pmatrix}$. In this realisation G/K is naturally homeomorphic to the future tube [2]

$$T=\{x+iy|x, y\in\mathbb{R}^4, y_0>|\boldsymbol{y}|\}.$$

(Here we will use the usual notation of relativistic quantum mechanics.) SU(2,2) acts on T by

$$\begin{aligned}g\cdot W&=(RW+S)(TW+Q)^{-1}\\ &=(WT^*+Q^*)^{-1}(WR^*+S^*)\end{aligned}\tag{3.2}$$

where

$$g=\begin{pmatrix}R & iS\\ -iT & Q\end{pmatrix}=U\begin{pmatrix}A & B\\ C & D\end{pmatrix}U^{-1}$$

and $W=w_0\sigma_0+\boldsymbol{w}\cdot\boldsymbol{\sigma}(\boldsymbol{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$ being the Pauli matrices).

The representations considered by Ruhl [2] are those of the holomorphic discrete series of SU(2,2) (see Harish-Chandra [15] and Graev [6] for the general theory). These may be regarded as representations induced from the compact Cartan subgroup of SU(2,2) or equally, from the maximal compact subgroup K (cf. Langlands [7]). We will consider only a subclass of these representations.

Consider the Hilbert space K_n whose elements are $\mathbb{C}$-valued functions holomorphic in T and satisfying

$$\|F\|_n^2=\int|F(W)|^2(y^2)^{n-4}d^4W<\infty\tag{3.3}$$

where $y^2=y_0^2-\boldsymbol{y}^2$, d^4W is Lebesgue measure on $\mathbb{C}^4$ and $n\geqq 4$ is integral. Acting on this space is a unitary irreducible representation of SU(2,2) defined by

$$(U_g^nF)(W)=\det(-TW^*+Q^*)^{-n}F(g^{-1}\cdot W)\tag{3.4}$$

where

$$g=\begin{pmatrix}R & iS\\ -iT & Q\end{pmatrix}.$$

The proof of this fact may be deduced from [6] using the results of [2]. We will give a proof, which depends on the reproducing kernel for K_n, at the end of Subsection 3.2.

3.2. Limit Points

We are interested in the cases $n=1,2,3$ treated briefly by Ruhl [2]. The case $n=3$ (but not $n=1,2$) is actually covered by the general theory of Knapp and Okamoto [16] so this analysis has some interest as a worked example. We follow the treatment by Gelfand et al. [17] of SU(1,1) and introduce a new normalisation for the inner product. Define

$$R_n(y)=(y^2)^{n-4}\theta(y_0)\theta(y^2)/\Gamma(n-2)\,\Gamma(n-3)\,,$$

where $\theta(y_0)\theta(y^2)$ is the characteristic function of T. Define

$$\langle F,F\rangle_n=\int_{\mathbb{C}^4}|F(W)|^2R_n(y)d^4W\,.$$

Recalling the definition of the generalised function $(y^2)^\lambda_+$ of Gelfand et al. [18] we see that

$$R_n(y)=\theta(y_0)\,(y^2)^{n-4}_+/\Gamma(n-2)\Gamma(n-3)\,.$$

It is in fact an analytic function of the index n, the factor $\Gamma(n-2)\Gamma(n-3)$ "cancelling" the poles of the numerator.

This generalized function takes the values

$$R_3(y)=\delta_+(y^2)=\theta(y_0)\delta(y^2)$$
$$R_2(y)=(\pi/3)\,\delta(y)$$
$$R_1(y)=(\pi/4)\,\Box\delta(y)\,,$$

for $n=3,2,1$ respectively. Substitution in (3.3) gives

$$\|F\|_3{}^2=\int_{\mathbb{C}^4}|F(w)|^2\delta_+(y^2)d^4xd^4y \tag{3.5}$$

where $w=x+iy\in\mathbb{C}^4$ and

$$\|F\|_2{}^2=(\pi/3)\int_{\mathbb{M}}|F(x)|^2d^4x \tag{3.6}$$

where $\mathbb{M}$ = Minkowski space. As $\|F\|_1{}^2$ is degenerate we will reserve the $n=1$ case until the next section.

From the work of Ruhl [2] one can deduce that there are Hilbert spaces K_3, K_2 of functions holomorphic on T carring unitary representations of SU(2, 2) given by (3.4) for $n=3,2$. One may also deduce that K_3 and K_2 have norms given by (3.5) and (3.6) respectively. I will give an independent treatment of these facts which illustrates some of the reproducing kernel techniques.

Let

$$\partial T=\{x+iy|x,y\in\mathbb{R}^4,\,y^2=0,\,y_0>0\}\,.$$

Consider the Hilbert spaces $L^2(\mathbb{M})$ and $L^2(\partial T, dv)$ where dv is the measure $d^4xd^3y/2y_0$ on ∂T. Given $f\in L^2(\mathbb{M})$ and $f'\in L^2(\partial T, dv)$, define functions F and F' holomorphic in T by

$$F(w_1)=(\pi/3)\,C_2\int_{\mathbb{M}}f(x)\,((w_1-x)^2)^{-2}d^4x \tag{3.7}$$
$$F'(w_1)=C_3\int_{\mathbb{C}^4}f'(w)\,((w_1-\bar{w})^2)^{-3}d^4xd^4y\delta_+(y^2)\,. \tag{3.8}$$

where $w = x + iy$ and C_2 and C_3 are normalisation constants. These integrals exist as the functions

$$x \to ((w_1 - x)^2)^{-2}, \qquad x \in \mathbb{M}$$
$$w \to ((w_1 - \bar{w})^2)^{-3}, \qquad w \in \partial T$$

lie in $L^2(\mathbb{M})$ and $L^2(\partial T, dv)$ respectively for all $w \in T$ (see below). We define function spaces K_2 and K_3 by (3.7) and (3.8) as the holomorphic extensions into T of elements of $L^2(\mathbb{M})$ and $L^2(\partial T, dv)$. It remains to show that K_3 and K_2 are Hilbert spaces with norms given by (3.5) and (3.6). To do this we follow Ruhl [2] and use the Fourier-Laplace transform.

Let

$$V_p^+ = \{p \in \mathbb{R}^4 | p_0 > 0, p^2 > 0\} \tag{3.9}$$

and define Hilbert spaces H_λ^p from functions $h: V_p^+ \to \mathbb{C}$ satisfying

$$(h, h)_\lambda = (\pi/2) \int |h(p)|^2 \theta(p_0)\theta(p^2) d^4p (p^2)^{-\lambda} < \infty$$

where $\lambda \geqq 0$. Now define Hilbert spaces H_λ^w of functions holomorphic in T via the Fourier-Laplace transform $\mathscr{L}$:

$$(\mathscr{L}h)(w) = \int h(p) \exp(ip \cdot w) \theta(p_0) \theta(p^2) d^4p \tag{3.10}$$

where $p \cdot w = p_0 w_0 - \boldsymbol{p} \cdot \boldsymbol{w}$. Observe that the functions k_w^λ defined by

$$k_w^\lambda(p) = \exp(-ip \cdot \bar{w})(p^2)^\lambda$$

are in H_λ^p for all $w \in T$. Hence (3.10) is defined as an ordinary integral being just the inner product $(k_w^\lambda, h)_\lambda$. Now Ruhl shows that $H_{n-2}^w = K_n$ for $n \geqq 4$. Further, the reproducing kernels for these spaces are defined by

$$\begin{aligned}\phi^n(w_1, w_2)^* &= \int \exp ip \cdot (w_2 - \bar{w}_1)(p^2)^{n-2} d^4p \theta(p_0) \theta(p^2) \\ &= C_n \det\left(\frac{1}{2i}(W_2 - W_1{}^*)\right)^{-n}\end{aligned} \tag{3.11}$$

where C_n is a normalization factor (see [2]). The reproducing property follows from the relations

$$\begin{aligned}(\mathscr{L}h)(w) &= F(w) = (k_w^{n-2}, h)_{n-2} \\ &= \langle \mathscr{L}k_w^{n-2}, \mathscr{L}h \rangle_n \\ &= \langle \phi_w^{n*}, F \rangle_n, \qquad F \in K_n.\end{aligned}$$

Now, for $n = 2$ the above formulae still hold provided we define $\langle \phi_w^{2*}, F \rangle_2$ by (3.7) (for a suitably chosen constant C_2). The relationship between F and its "boundary value" f in (3.7) is determined by the fact that if $F = \mathscr{L}h$ then f is the Fourier transform of h. Further, the function F_y defined by $F_y(x) = F(w)$, $w = x + iy$ converges in the L^2 norm on $\mathbb{M}$ to f as $y \to 0$ (see [2]).

When $n = 3$ we need to do a little more work. Define a map $\mathscr{T}$ from H_1^p into $L^2(\partial T, dv)$ by setting

$$(\mathscr{T}f)(x + iy) = \int_{V_p^+} \exp ip \cdot (x + iy) f(p) d^4p, \qquad x + iy \in \partial T,$$

for those $f \in H_1^p$ which are integrable with respect to Lebesgue measure on V_p^+. $\mathscr{T}$ is norm preserving and so defines an isometry of H_1^p onto a closed subspace of $L^2(\partial T, dv)$ (it is not difficult to show that $\mathscr{T}$ cannot map onto $L^2(\partial T, dv)$). Setting $f' = \mathscr{T}h$ for $h \in H_1^p$ we see that

$$\begin{aligned}(\mathscr{L}h)(w) &= F'(w) = (k_w^1, h)_1 \\ &= \langle \mathscr{T}k_w^1, \mathscr{T}h\rangle_3 \\ &= \langle \phi_w^{1*}, f'\rangle_3 .\end{aligned}$$

That is, K_3 is just the image under $\mathscr{L}$ of H_1^p and the norm (3.5) is well defined provided the "boundary value" f' of $F' = \mathscr{L}h \in K_3$ is defined to be $\mathscr{T}h$.

We now derive proofs of unitarity and irreducibility of the representations U^n defined by (3.4). The covariance relation (2.2) follows directly from the identity

$$(g^{-1} \cdot W_1 - g^{-1} \cdot W_2{}^*) = (W_2{}^* T^* + Q^*)^{-1} (W_1 - W_2{}^*)(TW_1 + Q)^{-1}$$

where

$$g^{-1} = \begin{pmatrix} R & iS \\ -iT & Q \end{pmatrix}$$

using the definition (3.11) of ϕ^n. Unitarity now follows from Theorem 1 of [1] (i.e. Theorem 2.1.3 of Subsection 2.1). Irreducibility may be obtained in several ways. The following argument may be found in Kobayashi's paper [12] and in Bargmann [3].

Let ψ^{n*} be the kernel for some non-trivial invariant subspace of K_n, then ψ^{n*} necessarily satisfies the covariance relation

$$\psi^n(g^{-1} \cdot W_1, g^{-1} \cdot W_2)^* = \det[(W_2{}^* T^* + Q^*)(TW_1 + Q)]^n \cdot \psi^n(W_1, W_2)^* .$$

Since SU(2,2) acts transitively on T we can write

$$\psi^n(W, W) = \det[(W^* T^* + Q^*)(TW + Q)]^n \psi^n(i\sigma_0, i\sigma_0)$$

where $\begin{pmatrix} R & iS \\ -iT & Q \end{pmatrix}^{-1}$ takes $i\sigma_0$ to W. But equally $\phi^n(W, W)$ satisfies a similar relation and so

$$\psi^n(W, W) = d_n \phi^n(W, W),$$

for some $d_n \in \mathbb{R}$. Define a function χ^n by

$$\chi^n(W_1, W_2) = d_n \phi^n(W_1, W_2)^* - \psi^n(W_1, W_2)^* .$$

Make a change of variable by setting

$$\zeta = W_1{}^* + W_2 , \qquad \eta = (W_1{}^* - W_2)/i .$$

Then $\zeta, \eta \to \chi^n(\zeta, \eta)$ is holomorphic in ζ and η and is zero when $W_1 = W_2$. This is precisely when ζ, η are both real forcing $\chi^n \equiv 0$. Hence the representations U^n are irreducible.

4. The Ladder Representations of SU(2, 2)

4.1. Preliminaries

The ladder representations [8] are those which give rise to the conformal symmetry of zero mass particles. They have the interesting property that on restriction to a Poincaré subgroup of SU(2,2) they remain irreducible [8]. Our intention in this section is to give a treatment of the ladder representations which makes use of the reproducing kernel techniques. We begin with some facts about the zero mass representations of the Poincare group (see [19] for an elaboration of the results listed here).

We will take the Poincare group P to be $\mathbb{R}^4 \circledS \mathrm{SL}(2,\mathbb{C})$. Write $\mathbb{P}^4$ for momentum space and let $X_0^+ = \{p \in \mathbb{P}^4 | p_0 > 0 \text{ and } p^2 = 0\}$ be the surface of the forward light cone. The P invariant measure on X_0^+ is

$$d\mu_0(p) = \delta_+(p^2)d^4p = \theta(p_0)\delta(p^2)d^4p\,.$$

Let V_s be the $(2s+1)$-dimensional Hilbert space carrying the representation σ_s of $\mathrm{SL}(2,\mathbb{C})$ labelled $(s,0)$ (see [19] for this labelling) where the norm is chosen so that σ_s restricted to $\mathrm{SU}(2) \subseteqq \mathrm{SL}(2,\mathbb{C})$ is unitary. Let $d\mu_s$ be the measure

$$d^3\boldsymbol{p}/2|\boldsymbol{p}|^{2s+1} \equiv p_0^{-2s}\delta_+(p^2)d^4p$$

on X_0^+ and define a non-unitary representation N of P on $L^2(X_0^+, V_s, d\mu_s)$ by

$$(N_g f)(p) = \sigma_s(\Lambda)\exp(in\cdot p)f(\Lambda^{-1}\cdot p) \tag{4.1}$$

where $g = (n, \Lambda) \in P$.

I will take the Lie algebra of $\mathrm{SL}(2,\mathbb{C})$ to be spanned by $\{J_i, L_i | i = 1, 2, 3\}$ with

$$[J_i, J_j] = i\varepsilon_{ijk}J_k\,; \qquad [L_i, L_j] = -i\varepsilon_{ijk}J_k$$
$$[J_i, L_j] = i\varepsilon_{ijk}L_k\,.$$

Denote by $p\hat{}$ the element $(1,0,0,1) \in \mathbb{P}^4$ and let $\chi_s(p\hat{})$ be the unique vector of norm one in V_s such that

$$\sigma_s(\exp(i\lambda J_3))\chi_s(p\hat{}) = \exp(i\lambda s)\chi_s(p\hat{})\,. \tag{4.2}$$

If p is any point on $X_0{}^+$ let

$$l_p : p\hat{} \to (p_0, 0, 0, p_0)$$

be the $\mathrm{SL}(2,\mathbb{C})$ matrix

$$\begin{pmatrix} p_0{}^{1/2} & 0 \\ 0 & p_0{}^{-1/2} \end{pmatrix}$$

and R_p be the rotation taking $(p_0, 0, 0, p_0)$ to (p_0, p_1, p_2, p_3). Putting the above together we define

$$\chi_s(p) = \sigma_s(R_p l_p)\chi_s(p\hat{})\,.$$

Now, the zero mass (positive energy) helicity s representation of $P(s = 0, 1/2, 1, 3/2, \ldots)$ is constructed on the subspace Γ_s of $L^2(X_0{}^+, V_s, d\mu_s)$ consisting of solutions of the equation:

$$\sum_{i=1}^{3} \dot{\sigma}_s(J_i)p_i f(p) = sp_0 f(p) \tag{4.3}$$

where $\dot{\sigma}_s$ denotes the representation of the Lie algebra of $SL(2, \mathbb{C})$ corresponding to σ_s. A straightforward calculation shows that the *non*-orthogonal projection onto Γ_s is given by

$$(Pf)(p) = \sigma_s(R_p l_p) P_0 \sigma_s(l_p^{-1} R_p^{-1}) f(p),$$

where P_0 is the projection in V_s defined by

$$P_0 v = \langle \chi_s(p\hat{}), v \rangle_{V_s} \chi_s(p\hat{}), \quad v \in V_s.$$

We will write

$$E(p) = \sigma_s(R_p l_p) P_0 \sigma_s(l_p^{-1} R_p^{-1}). \tag{4.4}$$

The advantage of defining the Poincaré representations on $L^2(X_0{}^+, V_s, d\mu_s)$ is that the inner product may be transformed onto Minkowski space. To see this we introduce first the Schwarz space on $\mathbb{R}^4(\mathbb{P}^4)$ of C^∞ functions of fast decrease taking values in V_s: $\mathscr{S}(\mathbb{R}^4, V_s)$ $(\mathscr{S}(\mathbb{P}^4, V_s))$. Let $\mathscr{S}_0(\mathbb{P}^4, V_s)$ be the subspace of $\mathscr{S}(\mathbb{P}^4, V_s)$ consisting of functions of compact support not containing the origin. Write $\mathscr{F}$ for the Fourier transform from both $L^2(\mathbb{P}^4, V_s)$ to $L^2(\mathbb{R}^4, V_s)$ and from $\mathscr{S}'(\mathbb{P}^4, V_s)$ to $\mathscr{S}'(\mathbb{R}^4, V_s)$. Finally, note that every function $f \in L^2(X_0{}^+, V_s, d\mu_s)$ defines a distribution $f\delta_+$ on $\mathscr{S}(\mathbb{P}^4, V_s)$ by

$$(f\delta_+)(\phi) = \int_{X_0^+} \sum_{\nu=1}^{2s+1} f_\nu(\omega, \boldsymbol{p}) \phi_\nu(\omega, \boldsymbol{p}) d^3\boldsymbol{p}/2\omega \tag{4.5}$$

where $\omega = |\boldsymbol{p}|$.

Now, it is not hard to show that the inner product in $L^2(X_0{}^+, V_s, d\mu_s)$ transforms, for functions $\phi, \psi \in \mathscr{F}\mathscr{S}_0(\mathbb{P}^4, V_s)$ into:

$$\begin{aligned} \langle \phi, \psi \rangle_0^s = i \int_{x_0=0} \Bigg[& \frac{\partial}{\partial x_0} (\beta_s * \phi^+)(x)^* \cdot (\beta_s * \psi^-)(x) \\ & - (\beta_s * \phi^+)(x)^* \frac{\partial}{\partial x_0} (\beta_s * \psi^+)(x) \Bigg] d^3\boldsymbol{x} \end{aligned} \tag{4.6}$$

where

(i) $\phi^+ = \mathscr{F}(\phi\delta_+) = (\mathscr{F}\phi) * (\mathscr{F}\delta_+)$,

(ii) β_s is the Fourier transform of the tempered distribution

$$p_0^{-s} \equiv (\tfrac{1}{2})(p_0 + i0) + (\tfrac{1}{2})(p_0 - i0))^{-s}$$

using the notation of Gelfand et al. [18],

(iii) $\beta_s * \phi^+$ denotes convolution in the x_0 variable only. (Note that it is not trivial that $\beta_s * \phi^+$ is well defined.)

Now (4.6) is an integral over the spacelike hypersurface $x_0 = 0$. If we write $\langle\, , \rangle_\Sigma^s$ for the corresponding integral over any sufficiently smooth spacelike hypersurface Σ, an application of Green's theorem gives

$$\langle\, , \rangle_\Sigma^s = \langle\, , \rangle_0^s.$$

Essentially what we are doing here is taking a space of V_s-valued solutions of $\Box F(x)=0$ and defining an inner product by an integral over a Cauchy initial data surface. The form $\langle\,,\rangle_\Sigma^s$ is positive definite because we are restricting to positive energy solutions of the wave equation. This space of solutions $\mathscr{F}\mathscr{S}_0(\mathbb{P}^4, V_s)$ may now be completed to give a Hilbert space H_s say, whose elements may be identified with those of $\mathscr{T}L^2(X_0{}^+, V_s, d\mu_s)$ where $\mathscr{T}$ is the map

$$f\to\mathscr{F}(f\delta_+), \qquad f\in L^2(X_0{}^+, V_s, d\mu_s).$$

4.2. Some Properties of Γ_s

If $f\in\Gamma_s$ and $g=(0, R_pl_p)^{-1}\in P$ then we have, from Equation (4.3)

$$\textstyle\sum_i \dot\sigma_s(J_i)p_i\hat{}(N_g\cdot f)(p\hat{})=s(N_g\cdot f)(p\hat{}),$$

and this reduces to

$$\dot\sigma_s(J_3)(N_g\cdot f)(p\hat{})=s(N_g\cdot f)(p\hat{}).$$

Recall that $\chi_s(p\hat{})$ is the unique vector of norm one satisfying this last equation. Hence

$$(N_g\cdot f)(p\hat{})=c_g\chi_s(p\hat{})$$

where $c_g\in\mathbb{C}$ depends only on p. Thus we may define a measurable function $f^0:X_0{}^+\to\mathbb{C}$ by $f^0(p)=c_g$ where $g=(0, R_pl_p)^{-1}$. We now have

$$\begin{aligned} f(p)&=\sigma_s(R_pl_p)(N_g\cdot f)(p\hat{})\\ &=\sigma_s(R_pl_p)c_g\chi_s(p\hat{})\\ &=f^0(p)\chi_s(p).\end{aligned}$$

Our object is to show that $f^0\in\Gamma_0=L^2(X_0{}^+, d\mu_0)$.

The first step is to note that with

$$L_3=\begin{pmatrix} i/2 & 0\\ 0 & -i/2\end{pmatrix}$$

we have $l_p=\exp(-i\log(p_0)L_3)$. Secondly, recall that in the representation $(s, 0)$ [19]

$$i\dot\sigma_s(J_3)=\dot\sigma_s(L_3)$$

and therefore

$$\dot\sigma_s(l_p)\chi_s(p\hat{})=p_0^s\chi_s(p\hat{}).$$

Finally since $\sigma_s(R_p)$ is unitary

$$\begin{aligned}\|f\|^2&=\int|f^0(p)|^2\chi_s(p\hat{})^*\chi_s(p\hat{})\delta_+(p^2)d^4p\\ &=\int|f^0(p)|^2\delta_+(p^2)d^4p<\infty.\end{aligned}$$

This proves

Lemma 4.2.1. *Every function $f\in\Gamma_s$ has the form*

$$f(p)=f^0(p)\chi_s(p)$$

for some $f^0 \in \Gamma_0$. *Furthermore the inner product of two functions in* Γ_s *is given by*

$$\langle f_1, f_2 \rangle = \int_{\mathbb{P}^4} \overline{f_1^0(p)} f_2^0(p) \delta_+(p^2) d^4 p$$

so that the map $f^0 \to f$ *from* Γ_0 *onto* Γ_2 *is an isometry.*

Let $f_j (j=1,2)$ be elements of Γ_s and denote by F_j^0 the Fourier transform of $f_j^0 \delta_+$. Then the inner product has the simple form:

$$\langle \mathcal{T} f_1, \mathcal{T} f_2 \rangle_\Sigma^s = i \int_\Sigma [\partial_\mu \overline{F_1^0(x)} F_2^0(x) - \overline{F_2^0(x)} \partial_\mu F_1^0(x)] \cdot d\sigma^\mu(x) \tag{4.7}$$

$\left(\text{with } \partial_\mu = \dfrac{\partial}{\partial x^\mu} \text{ and } d\sigma^\mu \equiv \text{“area measure” in } \Sigma \right)$ whenever $f_j^0 \delta_+$ is actually the product of a function f_j^0 in $\mathcal{S}_0(\mathbb{P}^4, \mathbb{C})$ with the distribution δ_+.

We conclude this subsection by determining some dense subspaces of Γ_s and $L^2(X_0^+, V_s, d\mu_s)$.

Lemma 4.2.2. *Define for each* $w \in T$, $G_w^-(p) = \exp(-ip \cdot \bar{w})$. *The functions* $G_w^- \in \Gamma_0$ *for all* $w \in T$ *and their linear span is dense.*

Proof.

$$\|G_w^-\|^2 = \int \exp(-2p \cdot y) \delta_+(p^2) d^4 p$$

where y is the imaginary part of w. Hence

$$\|G_w^-\|^2 = \pi/2(y_0^2 - |\boldsymbol{y}|^2) < \infty \text{ since } y_0 > |\boldsymbol{y}| \text{ in } T.$$

Density follows from two known results:

(i) the functions $\{\omega \to \exp(-2\omega y_0) | y_0 > |\boldsymbol{y}|, \boldsymbol{y} \text{ fixed}\}$ span a dense subspace of $L^2(0, \infty)$;

(ii) the functions $\{\boldsymbol{p} \to \exp(i\boldsymbol{p} \cdot \boldsymbol{x}) | \boldsymbol{x} \in \mathbb{R}^3\}$ span a dense subspace of the space of functions square integrable on the sphere of radius $\omega = |\boldsymbol{p}|$ (the measure being that arising from $d^3\boldsymbol{p}/2\omega$). As a corollary we note that the functions f_w defined by $f_w(p) = G_w^-(p) \chi_s(p)$ span a dense subspace of Γ_s. A similar argument to that above gives

Lemma 4.2.3. *The functions* $\overline{h_w^\nu}: p \to p_0^{2s} \overline{G_w(p)} e_\nu$ *where* w *ranges over* T *and*

$$e_\nu (\nu = 1, \ldots, 2s+1)$$

ranges over an orthonormal basis of V_s, *span a dense subspace of* $L^2(X_0^+, V_s, d\mu_s)$.

4.3. *Hilbert Spaces of Holomorphic Functions*

This is the central topic of this section. Here we produce the Hilbert spaces in which the ladder representations act, as Hilbert spaces of holomorphic functions and exhibit the reproducing kernels.

Let $f: X_0^+ \to V_s$ be square integrable with respect to $d\mu_s$, then its Fourier-Laplace transform defined componentwise by

$$F_\nu(w) = \int \exp(ip \cdot w) f_\nu(p) \delta_+(p^2) d^4 p \tag{4.8}$$

exists as an ordinary integral by Lemma 4.2.3 and the fact that (4.8) is just

$$F_v(w) = \langle h_w{}^{\neg}, f \rangle .$$

Furthermore, $f_v\delta_+$ is a tempered distribution with support in the closure of $V_p{}^+$ [$V_p{}^+$ defined by (3.9)] implying that (4.8) defines a function holomorphic in T [2]. Denoting the Fourier-Laplace transform by $\mathscr{L}$ define

$$H_s^w = \mathscr{L}L^2(X_0{}^+, V_s, d\mu_s)$$

as the Hilbert space with inner product

$$\langle F_1, F_2 \rangle_s = \int f_1(p)^* f_2(p)\delta_+(p^2)d^4p/p_0^{2s}$$

where $F_j = \mathscr{L}(f_j\delta_+)$ $j=1,2$.

The first question we will settle is: what can be said about the Shilov boundary values of elements of H_s^w (Minkowski space is the Shilov boundary of T [21]).

We know [20] that such functions will converge in a distribution theoretic sense to their boundary values $\mathscr{T}f = \mathscr{F}(f\delta_+)$ but in fact more is true.

Theorem 4.3.1. *With $F = \mathscr{L}f$ and F_y defined by*

$$F_y(x) = F(w)\,; \qquad w = x + iy\,,$$

$$\lim_{y\to 0} \|F_y - \mathscr{T}f\|_\Sigma^s = 0\,.$$

So F converges in norm to its boundary value $\mathscr{T}f$.

Proof. Firstly, $\mathscr{T}f$ is an element of H_s and for each y in V_+ where

$$V_+ = \{y \in \mathbb{R}^4 | y_0 > |\mathbf{y}|\}\,,$$

the function F_y is in H_s. Secondly, since F_y is the image under $\mathscr{T}$ of the function

$$p \to \exp(-p\cdot y)f(p)\,,$$

we have

$$\|F_y\|_s^2 = \int_{\mathbb{P}^4} \exp(-2p\cdot y)\|f(p)\|^2{}_{V_s}\delta_+(p^2)d^4p/p_0^{2s}\,.$$

Thus

$$\|F_y - \mathscr{T}f\|_\Sigma^s = \int_{\mathbb{P}^4} \|(\exp(-p\cdot y) - 1)f(p)\|_{V_s}\delta_+(p^2)d^4p/p_0^{2s}\,.$$

But

$$|\exp(-p\cdot y) - 1|^2 \|f(p)\|^2/p_0^{2s} \leqq \|f(p)\|^2/p_0^{2s}$$

and

$$|\exp(-p\cdot y) - 1|^2 \|f(p)\|^2/p_0^{2s} \to 0 \quad \text{as} \quad y \to 0\,.$$

Therefore by the Lebesgue dominated convergence theorem,

$$\lim_{y\to 0} \|F_y - \mathscr{T}f\|_\Sigma^s = 0\,.$$

Next observe that H_s^w is a reproducing kernel Hilbert space with kernel

$$\phi^s(w_1, w_2)^* = \int \exp(ip\cdot(w_2-\bar{w}_1))p_0^{2s}\delta_+(p^2)d^4p$$
$$= \langle h_{w_2}^{-\nu}, h_{w_1}^{-\nu}\rangle \cdot I$$

where I is the identity operator on V_s. This is clear from

$$\langle \phi_{w_1}^s{}^* e_\nu, F\rangle \stackrel{\text{def}}{=} \int \exp(ip\cdot w_1) f_\nu(p)\delta_+(p^2)d^4p = F_\nu(w_1).$$

Define

$$\psi(w_1, w_2)^* = \langle h_{w_2}^-, h_{w_1}^-\rangle .$$

From the known Laplace transform of $\delta_+(p^2)$ we have

$$\psi(w_1, w_2)^* = \pi/2(w_2-\bar{w}_1)^2 ,$$

whence for $s=0$ we have recovered the reproducing kernel for the $n=1$ case of Section 3.

Proposition 4.3.2. *The space $\Gamma_s^w = \mathscr{L}\Gamma_s$ is a reproducing kernel Hilbert space with an operator-valued kernel whose components are*

$$\Phi_{\mu\nu}(w_1, w_2)^* = \langle e_\mu, \Phi(w_1, w_2)^* e_\nu\rangle_{V_s}$$
$$= \langle e_\mu, \int \exp[ip\cdot(w_2-\bar{w}_1)]\ p_0^{2s}E(p)^* e_\nu\rangle_{V_s}\delta_+(p^2)d^4p . \tag{4.9}$$

Proof. We prove firstly that $\Phi_w{}^* u \in \Gamma_s^w$ for all $u \in V_s$. Recalling the definition of $E(p)$ by (4.4) we have

$$p_0^{2s}E(p)^* u = \langle \chi_s(p), u\rangle \chi_s(p)$$

and hence $\Phi_w{}^* u$ lies in Γ_s^w provided the function

$$p \to \exp(-ip\cdot\bar{w})\ \langle \chi_s(p), u\rangle$$

is in $L^2(X_0{}^+, d\mu_0)$ (Lemma 4.2.1). But this last function has norm

$$\int \exp(-2p\cdot y)|\langle \chi_s(p), u\rangle|^2 \delta_+(p^2)d^4p$$
$$= \|u\|^2 \int \exp(-2p\cdot y)p_0^{2s}\delta_+(p^2)d^4p < \infty .$$

It remains to check the reproducing property. Given $F \in \Gamma_s^w$ we have

$$F(w) = \int \exp(ip\cdot w) f^0(p)\chi_s(p)\delta_+(p^2)d^4p$$

and therefore

$$\langle \Phi_w{}^* u, F\rangle_s = \int \exp(ip\cdot w)\ \langle E(p)^* u, \chi_s(p)\rangle f^0(p)\delta_+(p^2)d^4p$$
$$= \int \exp(ip\cdot w)\ \langle u, \chi_s(p)\rangle f^0(p)\delta_+(p^2)d^4p$$

by the definition of $\chi_s(p)$ and $E(p)$. So

$$\langle \Phi_w{}^* u, F\rangle_s = \langle u, F(w)\rangle_{V_s}$$

as required.

Using the results of Subsection 4.1 we can rewrite the above formulae in Minkowski space terms. Let $f \in L^2(X_0{}^+, V_s, d\mu_s)$ then for $F = \mathscr{L}(f\delta_+)$

$$\begin{aligned} F_v(w) &= \langle \phi_w^s{}^* e_v, F\rangle_\Sigma^s \\ &= \int \exp(ip\cdot w) f_v(p)\delta_+(p^2)d^4p \\ &= \langle G_w^-, f_v\rangle . \end{aligned}$$

Now $(\mathscr{T}G_w^-)(x)^* = \pi/2(w-x)^2$ and so for f "sufficiently well behaved",

$$\begin{aligned} F_v(w) &= \langle G_w^-, f_v\rangle = \langle \mathscr{T}G_w^-, \mathscr{T}f_v\rangle_0 \\ &= \frac{i\pi}{2}\int_{x_0=0}\left[\frac{\partial}{\partial x_0}(1/(w-x)^2)\mathscr{T}f_v(x) - \frac{1}{(w-x)^2}\frac{\partial}{\partial x_0}(\mathscr{T}f_v(x))\right]d^3\boldsymbol{x} \\ &\equiv \frac{i\pi}{2}\int_\Sigma\left[\partial_\mu(1/(w-x)^2)\mathscr{T}f_v(x) - \frac{1}{(w-x)^2}\partial_\mu(\mathscr{T}f_v(x))\right]d\sigma^\mu(x) . \end{aligned}$$

These are just the classical Kirchoff formulae (expressing the solution $\mathscr{T}f_v$ of the wave equation in terms of its initial data via the Green's function for positive energy solutions), extended to functions holomorphic in T. These formulas have a distribution theoretic meaning for arbitrary $f \in L^2(X_0{}^+, V_s d\mu)$ [22]. Thus the reproducing property of the kernel merely expresses the "reproducing property" of the Green's function (cf. [23]). Similar formulae can be derived for elements of Γ_s^w.

Finally, we note that from (2.12) there exist bounds for the elements of Γ_s^w and H_s^w. For H_s^w we compute $(F = \mathscr{L}(f\delta_+))$,

$$|F_v(w)|^2 \leqq \|h_w^-\|^2\|f\|^2 = (-1/2)^{2s}\frac{\partial^{2s}}{\partial y_0^{2s}}(1/y^2)\,\|f\|^2$$

where y is the imaginary part of w. For Γ_s^w the function $w \to \Phi(w,w)^*$ can be determined after some calculation as

$$\Phi(w,w)_{\mu\nu}^* = [\pi(2s)!\,\delta_{\mu\nu}/(2y^2)^{2s+1}]\,(\sigma_s)_{\mu\nu}\begin{pmatrix} y_0+|\boldsymbol{y}| & 0 \\ 0 & y_0-|\boldsymbol{y}|\end{pmatrix}$$

where $(\sigma_s)_{\mu\nu}$ denotes the $\mu\nu^{\text{th}}$ matrix element. This gives a bound by substitution in

$$|F_v(w)| \leqq \Phi(w,w)_{\nu\nu}^{1/2}\|F\| .$$

Remark. Let D be the domain of $(-\nabla^2)^{1/2}$ in $L^2(\mathbb{R}^3)$. Then the elements of Γ_0^w are the holomorphic extensions into T of tempered distributions on Minkowski space defined via the Kirchoff formula from initial data in D.

4.4. The Ladder Representations

Consider the following action of SU(2,2) on V_s-valued functions holomorphic in T:

$$(g\cdot F)(W) = \det(TW+Q)^{-1-2s}\sigma_s(WT^*+Q^*)F(g^{-1}\cdot W) \tag{4.10}$$

where

(i) $W = w_0\sigma_0 + \boldsymbol{w}\cdot\boldsymbol{\sigma}$ and $g^{-1} = \begin{pmatrix} R & iS \\ -iT & Q \end{pmatrix}$,

(ii) $g^{-1}\cdot W = (RW+S)(TW+Q)^{-1}$,

(iii) σ_s is extended in the obvious way to $\mathrm{GL}(2,\mathbb{C})$.

A distinguished Poincaré subgroup of SU(2,2) is given by matrices with $T=0$, i.e. matrices of the form

$$g = \begin{pmatrix} a & iVa^{*-1} \\ 0 & a^{*-1} \end{pmatrix}; \quad g^{-1} = \begin{pmatrix} a^{-1} & -ia^{-1}V \\ 0 & a^* \end{pmatrix}$$

where $a \in \mathrm{SL}(2,\mathbb{C})$ and $V = v_0\sigma_0 + \boldsymbol{v}\cdot\boldsymbol{\sigma}$, $v_\mu \in \mathbb{R}$. Restricting (4.10) to Poincaré elements we have

$$(g\cdot F)(W) = \sigma_s(a)F(a^{-1}(W-V)a^{*-1}).$$

This is just the representation N defined by (4.1) transferred via the Laplace transform onto functions in T. Restricting therefore to Γ_s^w we obtain the (positive energy) zero mass helicity s representation of the Poincaré group.

Now it is not hard to compute the expressions for the generators of SU(2,2) acting on the functions holomorphic in T. Setting the imaginary part of w equal to zero in these expressions yields the formal expressions of Mack and Todorov [8] for the infinitesmal form of the ladder representations acting on functions on Minkowski space. It would be nice to have a global proof of the unitarity of these representations but unfortunately only the $s=0$ case is easily treated. The proof is as follows.

The reproducing kernel is

$$\phi^0(W_1, W_2)^* = \tfrac{\pi}{2}\det\left(\frac{1}{2i}(W_2 - W_1^*)\right)^{-1}$$

and the covariance relation for

$$g^{-1} = \begin{pmatrix} R & iS \\ -iT & Q \end{pmatrix}$$

is just

$$\phi^0(g^{-1}W_1, g^{-1}\cdot W_2)^* = \det[(W_1^*T^* + Q^*)(TW_2+Q)]\cdot\phi^0(W_1, W_2)^*.$$

But this is no more than the covariance relation (2.2). So by Theorem 1 of [1] we have unitarity of the representation (4.10) for $s=0$.

Note that this proof fails for the higher spin cases simply because it is not clear that these are of the "induced form" that we have been discussing (cf. Gross [24]). One can prove however that $(g\cdot F)(W)=0$ for all $g\in G$ if and only if

$$\textstyle\sum_i \sigma_s(J_i)\dfrac{\partial}{\partial w_i}F(W) = s\dfrac{\partial}{\partial w_0}F(W).$$

So H_s^w is not invariant under the SU(2,2) action (cf. [25]).

Acknowledgement. Most of the work described here was done at the Mathematical Institute, Oxford under the supervision of Dr. K. C. Hannabuss whose assistance and encouragement I am pleased to acknowledge. I would also like to thank Mr. M. A. Gerzon and Dr. E. B. Davies for some helpful discussions. The paper was completed with the financial support of a Rothman's Fellowship.

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Communicated by H. Araki

Received October 11, 1975