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The Wightman Axioms for the Weakly Coupled Yukawa Model in Two Dimensions

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Abstract. We prove the convergence of a cluster expansion for the weakly coupled Yukawa model in two dimensions.

I. Introduction and Results

The purpose of this paper is to prove the convergence of a cluster expansion [8, 3] for the Yukawa model in two dimensions[1]. We use here the model as defined by Seiler [13] and McBryan [10], and we shall use the presentation of Seiler and Simon [14].

The Yukawa model has been also studied by Glimm [4], Glimm and Jaffe [5] and [6], Schrader [12], Brydges and Federbush [2] and Brydges [1].

In this introduction we define the problem and state the main results, in the second chapter we define and give the properties of our main tool: a set of decoupling functions allowing to do the cluster expansion—see also [9]—, in the last chapter we prove the convergence of the cluster expansion.

Let us give some definitions, see [14].

The partition function in a volume Λ is:

$$Z_\Lambda = \int d\mu \, \det{}_{\mathrm{ren}}(1+K_\Lambda)\,. \tag{I.1}$$

The unnormalized Schwinger functions in a volume Λ are:

$$\begin{aligned} &S_\Lambda(f_1, \dots, f_n; g_1, \dots, g_N; h_1, \dots, h_N) \\ &= \int d\mu \left\{ \det{}_{ik} S_F\left((P^2+m^2)^{-1/4} g_i, \frac{\not{P}+m}{(P^2+m^2)^{3/4}} h_k \right) \right\} \prod_{l=1}^{n} \varphi(f_l) \det{}_{\mathrm{ren}}(1+K_\Lambda) \end{aligned}$$

where:

$$S_F(g', h') = \left(g', \frac{1}{1+K_\Lambda} h' \right)_{L^2}$$

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1 A. Cooper and L. Rosen have shown also the same result [17]

and iP is the gradient operator K_Λ is the operator in L^2 of kernel:

$$K_\Lambda(x,y) = \left\{\int \lambda \left[(P^2+m^2)^{-1/4} \frac{\not{P}+m}{(P^2+m^2)^{1/2}}\right](x,z)\Gamma\phi(z)\Lambda(z)[P^2+m^2]^{-1/4}(z,y)\,dz\right\}.$$

λ is the coupling constant and is real, m is the mass of the fermion, $\Gamma=i\gamma_5$ in the pseudoscalar case and $\Gamma=1$ in the scalar case, $\Lambda(x)$ is the characteristic function of the volume Λ.

Also:

$$\det_{\text{ren}}(1+K_\Lambda)=\det_3(1+K_\Lambda)B$$

with

$$\det_p(1+K):=\det(1+K)\exp\left(\sum_{n=1}^{p-1}(-1)^n\frac{1}{n}\operatorname{Tr}K^n\right)$$

and

$$B:=\exp\left\{-\frac{M^2}{2}\int :\phi^2:(x)\Lambda(x)\,dx-\frac{1}{2}\operatorname{Tr}_{\text{reg}}:K_\Lambda^2:\right\}.$$

$\operatorname{Tr}_{\text{reg}}:K_\Lambda^2:$ is defined as in [14] and we note:

$$\operatorname{Tr}_{\text{reg}}:K_\Lambda^2:=\int dx\,dy\,:\phi(x)\,b_{\text{reg}}(x,y)\,\phi(y):.$$

Also $d\mu$ is the Gaussian measure of mean zero and covariance

$$C_{\bar m}(x,y)=\frac{1}{(2\pi)^2}\int\frac{e^{ip(x-y)}}{p^2+\bar m^2}\,d^2p.$$

Finally f_l, g_j, h_k are functions in suitable spaces (defined later), and for convenience we suppose that their supports are localized in unit squares of a lattice cover of R^2:

$$R^2=\bigcup_{\alpha\in z^2}\Delta_\alpha,\qquad \Delta_\alpha \text{ is the unit square centered at } \alpha.$$

Let:

$$\hat g_i^\alpha(x)=[(P^2+m^2)^{-1/4}g_i](x)\chi_{\Delta_\alpha}(x)$$

$$\hat h_k^\alpha(x)=\left[\left(\frac{\not{P}+m}{(P^2+m^2)^{3/4}}\right)h_k\right](x)\chi_{\Delta_\alpha}(x)$$

where χ_Δ is the characteristic function of the unit square Δ. Then:

$$S_\Lambda=\sum_{\alpha_i,\beta_k}\tilde S_\Lambda(f_1,\dots,f_n;\hat g_1^{\alpha_1},\dots,\hat g_N^{\alpha_N};\hat h_1^{\beta_1},\dots,\hat h_N^{\beta_N})$$

with

$$\tilde S_\Lambda(f_1,\dots,f_n;\hat g_1,\dots,\hat g_N;\hat h_1,\dots,\hat h_N)$$
$$:=\int d\mu\,\det_{ik}S_F(\hat g_i,\hat h_k)\prod_{l=1}^n\phi(f_l)\det_{\text{ren}}(1+K_\Lambda).\qquad\text{(I.2)}$$

We construct the cluster expansion for $\tilde{S}_\Lambda$ (all the functions, f, $\hat{g}$, $\hat{h}$ having their support in unit squares); S_Λ is obtained by resummation.

Remark. In the sequel we shall take the boson mass $\bar{m}$ to be equal to m, because in the analysis of the convergence of the cluster expansion the two masses play the same role.

Let us call Z^{2*} the sides of the squares in the net defined by the lattice cover of R^2. To each $b \in Z^{2*}$ is associated a variable s_b, $0 \leqq s_b \leqq 1$. For each choice of $\{s_b\}_{b\in Z^{2*}}$ we define s-dependant quantities:

$$C(s)(x,y) = \sum_{\alpha,\beta,\gamma\in Z^2} H(s;\Delta_\alpha,\Delta_\beta,\Delta_\gamma)M(\Delta_\alpha,\Delta_\beta,\Delta_\gamma)C_m(x,y)\chi_{\Delta_\alpha}(x)\chi_{\Delta_\beta}(y)$$

$$K(s)_\Lambda(x,y) = \sum_{\alpha,\beta,\gamma\in Z^2} H(s;\Delta_\alpha,\Delta_\beta,\Delta_\gamma)K_{\Delta_\gamma\cap\Lambda}(x,y)\chi_{\Delta_\alpha}(x)\chi_{\Delta_\beta}(y)\,.$$

The definition of $b_{\text{reg}}(s)(x,y)$ follows in a natural way from the definition of $K(s)$.

Then if in formula (I.1) and (I.2) we replace all the quantities by the corresponding s-dependant quantities this defines $\tilde{S}_{(s)}$ and $Z_{(s)}$.

In Chapter II we define $H(s;\Delta_\alpha,\Delta_\beta,\Delta_\gamma)$ and $E(\Delta_\alpha,\Delta_\beta,\Delta_\gamma)$ and prove:

Lemma I.1. $0 \leqq H(s;\Delta_\alpha,\Delta_\beta,\Delta_\gamma) \leqq 1$, $0 \leqq M(\Delta_\alpha,\Delta_\beta,\Delta_\gamma) \leqq 1$,

$$\sum_\gamma M(\Delta_\alpha,\Delta_\beta,\Delta_\gamma) = 1\,.$$

If $s_b = 1\,\forall b$ *then* $H(s;\Delta_\alpha,\Delta_\beta,\Delta_\gamma) \equiv 1$.

Let $D \subset Z^{2*}$ *defined by* $D = \{b \in Z^{2*} | s_b = 0\}$ *and let* $R^2 \setminus D = X_1 U \ldots U X_p$ *be the decomposition of* $R^2 \setminus D$ *as a disjoint union of connected components then:*

$$H(s;\Delta_\alpha,\Delta_\beta,\Delta_\gamma) = \sum_k H(s;\Delta_\alpha,\Delta_\beta,\Delta_\gamma)|_{X_k}$$

where

$$H(s;\Delta_\alpha,\Delta_\beta,\Delta_\gamma)|_{X_k} = \begin{cases} H(s;\Delta_\alpha,\Delta_\beta,\Delta_\gamma) & \text{if} \quad \Delta_\alpha \subset X_k,\, \Delta_\beta \subset X_k,\, \Delta_\gamma \subset X_k \\ 0 \quad \text{otherwise}\,. \end{cases}$$

It is then clear that $C(s)(x,y) = \bigoplus_k C(s)(x,y)|_{X_k}$ and also that $K(s)_\Lambda = \sum_k K(s)|_{X_k}$ [where $K(s)|_{X_k}$ means the restriction of $K(s)$ to $L^2(X_k)$].

Then

$$\frac{1}{1+K(s)} = \prod_k \frac{1}{1+K(s)|_{X_k}} \quad \text{and} \quad \det(1+H(s)) = \prod_k \det(1+K(s)|_{X_k})$$

and this is obviously true also for $\det_{\text{ren}}(1+K(s))$ because $\operatorname{Tr} K^q$ decomposes itself in a sum $\sum_k \operatorname{Tr}(K|_{X_k})^q$.

Also

$$\det_{ik} S_F(\hat{g}_i,\hat{h}_k) = \pm \prod_k \det_{jl} S_F(\hat{g}_j,\hat{h}_l)$$

where for k given the determinant is formed with the function:

$$\operatorname{supp}\hat{g}_j \subset X_k, \quad \operatorname{supp}\hat{h}_l \subset X_k\,.$$

Then with obvious notation we have:

$$Z=\prod_k Z_{X_k} \qquad \tilde{S}=\prod_{X_k} \tilde{S}|_{X_k}\,.$$

As a consequence, according to the general scheme of the cluster expansion (see [8] and [3]), one has a convergent cluster expansion if one can prove that for some values of the parameters λ and $m(m\geqq 1)$:

A_1) Δ being a unit square. $Z_{(s),\Delta}>0$ for $s_b=0$ $b\in\partial\Delta$.

A_2) Let $\Gamma\subset Z^{2*}$ and ∂^Γ note $\prod_{b\in\Gamma}\frac{d}{ds_b}$. Let X be one of the connected components of $R^2\setminus(Z^{2*}\setminus\Gamma)$ then:

$$|\partial^\Gamma \tilde{S}_{(s),\Delta}(f_1, f_n; \hat{g}_1, \dots, \hat{g}_N; \hat{h}_1, \dots, \hat{h}_N)|$$
$$\leqq \prod_{i=1}^n \|f_i\|_{-1} \prod_{j=1}^N \|\hat{g}_j\|_{L_2}\|\hat{h}_j\|_{L_2} O(1)^N O(1)^n \prod_\Delta (n(\Delta)!)^{1/2} e^{-Q_1|\Gamma|}$$

where $|\Gamma|$ is the number of bonds in Γ, $\|f\|^2_{-1}=\int\frac{|\tilde{f}(p)|^2}{(p^2+1)}dp$, $n(\Delta)$ is the number of f_i with support in Δ and Q_1 is some positive constant large enough. Also for a matrix $|A|=\sup_{i,j}|A_{ij}|$.

Indeed, giving us $\lambda_0>0$, there exists $m(\lambda_0)$ such that for $|\lambda|\leqq\lambda_0$, λ real and $m\geqq m(\lambda_0)A_1$ and A_2 are satisfied[2], and even by taking m large enough one can take Q_1 as large as we want.

We now want to show that one can bound the norms of the $\hat{g}$'s functions by norms of the initial functions g.

First suppose that $\operatorname{supp} g\subset\Delta$, and that $\Delta\cap\Delta_\alpha=\emptyset$. Let η_Δ be a C_0^∞ function such that $\eta_\Delta(x)=1$ if $x\in\Delta$ and such that $\operatorname{dist}(\operatorname{supp}\eta_\Delta, \Delta_\alpha)>\frac{1}{2}$ (if $\Delta\cap\Delta_\alpha=\emptyset$). We have

$$\|\hat{g}^\alpha\|^2_{L^2}=\int dp\,dr\,dt\,dq\,\frac{\tilde{g}(p)\tilde{\eta}_\Delta(-p+r)\tilde{X}_{\Delta_\alpha}(-r+t)\tilde{\eta}_\Delta(-t+q)\tilde{\bar{g}}(-q)}{(r^2+m^2)^{1/4}(t^2+m^2)^{1/4}}$$

defining $D:=(P^2+m^2)$, then:

$$\begin{aligned}\|\hat{g}^\alpha\|^2_{L^2}&=\|g\eta_\Delta D^{-1/4}\chi_{\Delta_\alpha}D^{-1/4}\eta_\Delta g\|_1\\&=\|gD^{-1/4}D^{1/4}\eta_\Delta D^{-1/4}\chi_{\Delta_\alpha}D^{-1/4}\eta_\Delta D^{1/4}D^{-1/4}g\|_1\\&\leqq\|g\|^2_{-1/2}\|D^{1/2}\eta_\Delta D^{-1/4}\chi_{\Delta_\alpha}D^{-1/4}\eta_\Delta D^{1/2}\eta_\Delta D^{-1/4}\chi_{\Delta_\alpha}D^{-1/4}\eta_\Delta\|_1\end{aligned}$$

where

$$\|g\|_{-1/2}=\left(\int|\tilde{g}(p)|^2(p^2+1)^{-1/2}dp\right)^{1/2}\geqq\|D^{-1/4}g\|_{L^2}\,.$$

Using then the Theorem 2.2 of Seiler and Simon [14]:

$$\begin{aligned}\|\hat{g}^\alpha\|^2_{L^2}&\leqq\|g\|^2_{-1/2}e^{-2Q_7 d(\Delta,\Delta_\alpha)}\|D^{1/2}\eta_\Delta D^{-1/4}\chi_{\Delta_\alpha}D^{-1/8}\|^2_2\\&\leqq O(l)\,e^{-Q_7 d(\Delta,\Delta_\alpha)}\|g\|^2_{-1/2}\end{aligned}$$

where Q_7 is as big as we want if m is big enough, and $d(\Delta,\Delta_\alpha)=\sup(1,\operatorname{dist}(\Delta,\Delta_\alpha))$.

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2 Remark that the theory depends only of the ratio λ/m

Then if $\Delta \cap \Delta_\alpha \neq \emptyset$, we use

$$\|\hat{g}^\alpha\|^2{}_L{}^2 = \int |(D^{-1/4} g)(x)|^2 \chi_{\Delta_\alpha}(x)\, dx \leqq \|g\|^2_{-1/2} .$$

We have thus obtained that:

$$\|\hat{g}^\alpha\|_{L^2} \leqq \|g\|^2_{-1/2} O(1) e^{-Q_7 d(\Delta, \Delta_\alpha)} .$$

Using then the fact that $\sum_{\Delta_\alpha} e^{-d(\Delta,\Delta_\alpha)} \leqq O(1)$ we obtain for the original Schwinger functions bounds in $\|g\|_{-1/2}$ and $\|h\|_{-1/2}$, and the exponential decrease between the support of the $\hat{g}$ and $\hat{h}$ give exponential decreases between the support of the g and h using the exponential decrease:

$$e^{-(Q_7 - 1) d(\Delta, \Delta_\alpha)} .$$

Then as a consequence of A_1 and A_2 we get

Theorem 1. *Let $\lambda_0 > 0$ be given.*

Uniformly in s, and λ, $|\lambda| < \lambda_0$, these exists m large enough such that:

$$\lim_{\Lambda \to \infty} \frac{S_{(s),\Lambda}(f; g; h)}{Z_{(s),\Lambda}}$$

exists and is bounded by

$$O(1)^n O(1)^N \prod_{i=1}^{n} \|f_i\|_{-1} \prod_{k=1}^{N} \|g_k\|_{-1/2} \|h_k\|_{-1/2} \prod_{\Delta} (n(\Delta)!)^{1/2} .$$

Moreover, there is an exponential clustering which is as big as we want if m is taken large enough.

Finally under the same conditions as for Theorem 1, and S_Λ being defined as in (I.1), one has:

Theorem 2. *The infinite volume limits $\lim_{\Lambda \to \infty} Z_\Lambda^{-1} S_\Lambda(f; g; h)$ exists and satisfy all the Osterwalder-Schrader axioms, including an exponential clustering.*

As an obvious consequence:

Corollary 1. *There exists a 2 dimensional Yukawa relativistic theory satisfying the Wightman axioms and possessing a mass gap.*

Theorem 2 follows from Theorem 1. Indeed we proceed as in [9]. We define new Schwinger functions $S_{\Lambda;Y}$, for Y a big square union of lattice squares and containing Λ, by:

$$S_{\Lambda;Y} := S_{(s),\Lambda} \quad \text{for} \quad s_b = 1 \quad \text{if} \quad b \in B_Y \quad \text{and} \quad s_b = 0 \quad \text{otherwise} .$$

B_Y is the set of lattice lines strictly contained in Y. Then by the equivalent of Proposition IV.1.3 of [9]:

$$\lim_{Y \to \infty} S_{\Lambda;Y} = S_\Lambda .$$

On the other hand from Theorem 1, it follows that

$$\lim_{\Lambda\to\infty}\lim_{Y\to\infty}\frac{S_{\Lambda;Y}}{Z_{\Lambda;Y}}$$

exists; this proves the existence part of Theorem 2. Now since this limit is also the limit of the theory defined by Seiler [13] and McBryan [10], it is obvious that all the Osterwalder Schrader axioms are satisfied.

It remains thus to prove A_1 and A_2. The proof of A_1 results from Seiler [13]. In fact $C_{(s)}(x,y)|_\Delta$ and $K_{(s)}(x,y)|_\Delta$ are proportional to $C(x,y)$ and $K(x,y)$, $x,y\in\Delta$. Thus the proof of $Z_{(s),\Delta}>0$ reduces itself to the Seiler's proof of $Z\neq 0$ for the volume $\Lambda=\Delta$, i.e. to have $\|K|_\Delta\|_4^4<1$. This condition is obtained by taking λ/m small enough.

The remaining of the article is devoted to the proof of A_2.

II. The Combinatoric of the Cluster Expansion

We give now the explicit form of H which is a function of a parameter m_1, and of M which is a function of another parameter m_2:

Definition. Let $m_1>0$ and $m_2>0$:

$$H_{m_1}(s;\Delta_1,\Delta_2,\Delta_3)=\sum_{\gamma\subset Z^{2^*}}\prod_{b\in\gamma}s_b\prod_{b\notin\gamma}(1-s_b)\frac{\bar{C}^\gamma_{m_1}(\Delta_1;\Delta_3)}{\bar{C}_{m_1}(\Delta_1;\Delta_3)}\frac{\bar{C}^\gamma_{m_1}(\Delta_3;\Delta_2)}{\bar{C}_{m_1}(\Delta_3;\Delta_2)} \tag{II.1}$$

and

$$M_{m_2}(\Delta_1,\Delta_2,\Delta_3)=E_{m_2}(\Delta_1,\Delta_2,\Delta_3)\left[\sum_\alpha E_{m_2}(\Delta_1,\Delta_2,\Delta_\alpha)\right]^{-1}$$

$$E_{m_2}(\Delta_1,\Delta_2,\Delta_3)=e^{-m_2[d(\Delta_1,\Delta_3)+d(\Delta_3,\Delta_2)]}$$

$$C^\gamma_{m_1}(x,y)=\int e^{-m_1^2T}\int\prod_{b\in\gamma^c}J^T_b(z)\,dz^T_{xy}\,dT$$

where dz^T_{xy} is the Wiener density for the paths in R^2 and

$$J^T_b(z)=\begin{cases}0 & \text{if}\quad z(\tau)\in b\quad 0\leqq\tau\leqq T\\ 1 & \text{otherwise}\end{cases}$$

$$\bar{C}^\gamma_{m_1}(\Delta_\alpha,\Delta_\beta)=\int C^\gamma_{m_1}(x,y)\chi_{\Delta_\alpha}(x)\chi_{\Delta_\beta}(y)\,dx\,dy,\qquad \bar{C}_{m_1}=\bar{C}^{Z^{2^*}}_{m_1}$$

Proof of Lemma I.1. It is obvious. In particular H being a convex sum of non negative quantities smaller than one, we have $0\leqq H\leqq 1$. If $s\equiv 1$ then in formula (II.1) the only contribution to the sum over γ is for $\gamma=Z^{2^*}$ and so $H\equiv 1$.

Finally remark that if $\bar{C}^\gamma_{m_1}(.\,;.)=\sum_k\bar{C}^\gamma_{m_1}(.\,;.)|_{X_k}$ then $\bar{C}^\gamma_{m_1}(\Delta_1;\Delta_3)\bar{C}^\gamma_{m_1}(\Delta_3;\Delta_2)$ is equal to zero if Δ_1, Δ_2, and Δ_3 don't belong to the same X_k. This finishes the proof.

We now reduce A_2 to a proposition whose proof is given in Chapter III. This is obtained through a lemma showing that $H_{m_1}E_{m_2}$ has essentially the same combinatoric properties as Dirichlet covariances, see [8]. Thus let us consider $\partial^\Gamma\tilde{S}_{(s)}$.

The d/ds derivations acting on $C(s)$ or $K(s)$ are localized:

$$\frac{d}{ds}C(s)=\sum_{\Delta,\Delta'}\sum_{\alpha}\frac{d}{ds}H(s;\Delta,\Delta',\Delta_\alpha)M(\Delta,\Delta',\Delta_\alpha)\chi_\Delta C\chi_{\Delta'}$$

with obvious notations

$$\frac{d}{ds}K(s)=\sum_{\Delta,\Delta',\Delta''}\frac{d}{ds}H(s;\Delta,\Delta',\Delta'')\chi_\Delta K_{\Delta''}\chi_{\Delta'}.$$

We then can write (see [8]):

$$\partial^\Gamma \tilde{S}_{(s)}=\sum_{\substack{\pi\in\mathscr{P}(\Gamma)\\ \pi=\{\gamma_1,\ldots,\gamma_L\}}}\sum_{\substack{\Delta_1,\Delta'_1,\Delta''_1\\ \vdots\\ \Delta_L,\Delta'_L,\Delta''_L}}\partial^{\gamma_j}H(s;\Delta_j,\Delta'_j,\Delta''_j)\sum_R\int R\,d\mu \tag{II.2}$$

where R contains L, K, or C. M localized in $(\Delta_i,\Delta'_i,\Delta''_i)$ and $\mathscr{P}(\Gamma)$ is the set of all the partitions of Γ.

Now the following lemma summarizes the cluster expansion estimates:

Lemma II.1. *Let b_j be an arbitrary element of γ_j, for $j=1,\ldots,L$ and let Q_4 and Q_3 be any positive constants, then there exist a positive constant Q_2 and m_1 and m_2, $m_2>m_1$ such that:*

$$\sum_{\substack{\pi\in\mathscr{P}(\Gamma)\\ \pi=\{\gamma_1,\ldots,\gamma_L\}}}\sum_{\substack{\Delta_1,\Delta'_1,\Delta''_1\\ \vdots\\ \Delta_L,\Delta'_L,\Delta''_L}}\prod_{j=1}^{L}\{\partial^{\gamma_j}H(s;\Delta_j,\Delta'_j,\Delta''_j)E(\Delta_j,\Delta'_j,\Delta''_j)e^{-Q_2}e^{Q_3[d(b_j,\Delta_j)+d(b_j,\Delta'_j)]}\}$$
$$\leqq e^{-Q_4|\Gamma|}. \tag{II.3}$$

Proof. Let $\gamma\subset\gamma_0$, we define:

$$\partial^\gamma\bar{C}^{\gamma_0}_{m_1}(\Delta,\Delta')=\iint e^{-m_1^2T}\int\prod_{b\in\gamma}(1-J_b^T(z))\prod_{\gamma_0^c}J_b^T(z)\,dz^T_{xy}\chi_\Delta(x)\chi_{\Delta'}(y)\,dx\,dy.$$

Then:

$$\partial^\gamma H(s;\Delta,\Delta',\Delta'')=\sum_{\substack{\tilde\gamma\subset Z^{2*}\\ \gamma\subset\tilde\gamma}}\prod_{b\in\tilde\gamma\setminus\gamma}s_b\prod_{b\notin\tilde\gamma}(1-s_b)\sum_{\substack{\gamma_1\cup\gamma_2=\gamma\\ \gamma_1\cap\gamma_2=\emptyset}}\frac{\partial^{\gamma_1}\bar{C}^{\tilde\gamma\setminus\gamma_2}_{m_1}(\Delta;\Delta'')}{\bar{C}^{\tilde\gamma}_{m_1}(\Delta;\Delta'')}\frac{\partial^{\gamma_2}\bar{C}_{m_1}(\Delta';\Delta'')}{\bar{C}^{\tilde\gamma}_{m_1}(\Delta';\Delta'')}.$$

Using

$$e^{-m_2d(\Delta,\Delta'')}[\bar{C}_{m_1}(\Delta;\Delta'')]^{-1}\leqq O(1)e^{-(m_2-m_1-1)d(\Delta,\Delta'')}$$

$$\sum_{\Delta''}\exp\{-d(\Delta,\Delta'')-d(\Delta',\Delta'')\}\leqq O(1)$$

and $\partial^\gamma\bar{C}^{\tilde\gamma}\leqq\partial^\gamma\bar{C}$ we obtain:

$$\sum_{\Delta''}\partial^\gamma H(s;\Delta,\Delta',\Delta'')E(\Delta,\Delta',\Delta'')\leqq O(1)\sum_{\gamma_1\cup\gamma_2=\gamma}\sup_{\Delta''}\partial^{\gamma_1}\bar{C}_{m_1}(\Delta;\Delta'')$$
$$\cdot e^{-(m_2-m_1-2)[d(\Delta,\Delta'')+d(\Delta',\Delta'')]}\partial^{\gamma_2}\bar{C}_{m_1}(\Delta';\Delta'').$$

In formula (II.3), for any $\pi\in\mathscr{P}(\Gamma)$, $\pi=\{\gamma_1,\ldots,\gamma_L\}$ let b_i be any element of γ_i then:

$$\sum_{\{\Delta_i,\Delta'_i\}}\prod_{j=1}^{L}e^{-[d(b_j,\Delta_j)+d(b_j,\Delta'_j)]}\leqq O(1)^L.$$

To finish the proof it is then sufficient to prove

$$\sum_{\substack{\pi\in\mathscr{P}(\Gamma)\\ \pi=\{\bar\gamma_1,\dots,\bar\gamma_L\}}}\ \sum_{\substack{\gamma_i\cup\gamma_i'=\bar\gamma_i\\ \gamma_i\cap\gamma_i'=\emptyset}}\ \sup_{\Delta_i,\Delta_i',\Delta_{\alpha_i}}\ \prod_{i=1}^{L}\{\partial^{\gamma_i}\bar C_{m_1}(\Delta_i,\Delta_{\alpha_i})\,\partial^{\gamma_i'}\bar C_{m_1}(\Delta_i',\Delta_{\alpha_i})$$

$$\cdot e^{-(m_2-m_1-2)[d(\Delta_i,\Delta_{\alpha_i})+d(\Delta_i',\Delta_{\alpha_i})]}e^{(Q_3+1)[d(b_i,\Delta_i)+d(b_i,\Delta_i')]}O(1)e^{-Q_2}\}\leqq e^{-Q_0|\Gamma|}$$

where b_i is any element of $\bar\gamma_i$. Suppose that $b_i\in\gamma_i$, we have

$$d(b_i,\Delta_i')\leqq d(\Delta_i,\Delta_i')+d(b_i,\Delta_i)+2\leqq d(\Delta_i,\Delta_{\alpha_i})+d(\Delta_{\alpha_i},\Delta_i')+d(b_i,\Delta_i)+4\,.$$

If $\gamma_i=\emptyset$ (or $\gamma_i'=\emptyset$) then we use: $\bar C_{m_1}(\Delta_i.\ \Delta_{\alpha_i})\leqq O(1)$ and we can choose the parameters such that:

$$O(1)e^{-Q_2/2}e^{2(Q_3+1)}e^{-(m_2-m_1-Q_3-3)d(\Delta_i,\Delta_{\alpha_i})}\leqq 1$$

so that in the remaining of the proof we can forget about the empty γ_i. We symmetrize: let now b_i (resp. b_i') be an arbitrary element of γ_i (resp. γ_i') then for $\gamma_i\neq\emptyset$ we define:

$$f(\gamma_i)=\sup_{\tilde\Delta_i,\tilde{\tilde\Delta}_i}\partial^{\gamma_i}\bar C_{m_1}(\tilde{\tilde\Delta}_i,\tilde\Delta_i)e^{-(m_2-m_1-Q_3-3)d(\tilde\Delta_i,\tilde{\tilde\Delta}_i)}e^{2(Q_3+1)d(b_i,\Delta_i)}$$

$$\cdot e^{2(Q_3+1)}O(1)e^{-Q_2/2}e^{d(b_i,\tilde\Delta_i)+d(\tilde\Delta_i,\tilde{\tilde\Delta}_i)}$$

and for $\gamma_i=\emptyset, f(\emptyset)=1$.

To prove (II.3) it is sufficient to prove that:

$$\sum_{\substack{\pi\in\mathscr{P}(\Gamma)\\ \pi=\{\bar\gamma_i,\dots,\bar\gamma_L)}}\ \sum_{\substack{\gamma_i\cup\gamma_i'=\bar\gamma_i\\ \gamma_i\cap\gamma_i'=\emptyset}}\ \prod_i O(1)e^{-d(b_i,b_i')}f(\gamma_i)f(\gamma_i')\leqq e^{-Q_4|\Gamma|}\,. \tag{II.4}$$

We assert then that the l.h.s. in (II.4) is smaller than [3]:

$$\sum_{\substack{\pi\in\mathscr{P}(\Gamma)\\ \pi=(\tilde\gamma_i\dots\tilde\gamma_p)}} O(1)^p\,2^p\prod_{i=1}^{p}.f(\tilde\gamma_i)$$

the proof that this is smaller than $e^{-Q_4|\Gamma|}$ for a correct choice of Q_2, $m_1(m_2>m_1)$ is given in the references [8] and [15] and is one of the main combinatoric tool of the cluster expansion. This finishes the proof of the lemma.

With this lemma the proof of A_2 is reduced to the proof of:

Proposition II.1. *For given Q_2, λ_0, and m_2 (large) there exists Q_3 and Q_6 (independant of the parameters) such that for m large enough and $|\lambda|<\lambda_0$* (see formula II.2):

$$\sup_L\ \sup_{\substack{b_j\in\Gamma\\ j=1,\dots,L}}\ \sup_{\substack{\Delta_1,\Delta_1',\Delta_1''\\ \vdots\\ \Delta_L,\Delta_L',\Delta_L''}}\ \prod_{i=1}^{L}\{e^{-Q_3[d(\Delta_j,b_j)+d(b_j,\Delta_j')]}e^{Q_2}\}$$

$$\prod_{\text{derived }k} e^{m_2[d(\Delta_j,\Delta_j'')+d(\Delta_j',\Delta_j'')]}\prod_{\text{derived }C} e^{2m_2d(\Delta_j,\Delta_j')}\sum_R\left|\int R\,d\mu\right|$$

$$\leqq O(1)^n O(1)^N\prod_\Delta (n(\Delta)!)^{1/2}e^{Q_6|\Gamma|}\prod_{i=1}\|f_i\|_{-1}\prod_{j=1}^{N}\|\hat g_j\|_{L^2}\|\hat h_j\|_{L_2}\,. \tag{II.5}$$

Remark. We have used that:

$$E^{-1}(\Delta_j,\Delta_j',\Delta_j'')=\exp\{m_2[d(\Delta_j,\Delta_j'')+d(\Delta_j',\Delta_j'')]\}$$

$$(E^{-1}M)(\Delta_j,\Delta_j',\Delta_j'')\leqq\exp\{2m_2\,d(\Delta_j,\Delta_j')\}$$

[3] It has been pointed out by Lon Rosen that this inequality in a preliminary version and the corresponding formula in [9] are incorrect

and R is defined by what we obtain after performing the derivations up to the factors $\partial^\gamma H$ and M.

Also in the formula above $O(1)$ in $O(1)^n$ depends on Q_2.

Then A_2 follows from the fact that one can take Q_4 as large as we want by choosing m_2 (and thus m) large enough and define $Q_1 = Q_4 - Q_6$.

III. The Cluster Expansion: Proof of Proposition II.1

First let us see what we obtain when we do a derivation d/ds_b, $b \in \Gamma$ on a Schwinger function $\tilde{S}(\omega)$

$$\tilde{S}(\omega) = \iint \det_{jk} S_F(x_j, y_k; \phi) \prod \phi(t_i) \det_{\mathrm{ren}}(1+K)$$
$$\omega(x_1 \ldots; y_1, \ldots; t_1, \ldots)\, d\mu \prod_j dx_j dy_j \prod_i dt_i \qquad \text{(III.1)}$$

where $S_F(x, y; \phi)$ stands for the kernel of $(1+K)^{-1}$, and ω is some function with each argument localized in some unit square of the lattice cover. Also for simplicity since we look at some algebraic aspects we omit any reference to λ, Γ of Λ.

Acting on an expression of the form $\int R\, d\mu$ where R and $d\mu$ depend on s, the derivation d/ds produces two categories of terms:

$$\frac{d}{ds} \int R\, d\mu = \int \frac{dR}{ds} d\mu + \frac{1}{2} \int \frac{dC}{ds}(x, y) \left(\frac{\partial^2}{\partial\phi(x)\, \partial\phi(y)} R \right) dx\, dy\, d\mu .$$

R being of the form of the integrand in (III.1) one sees that one has essentially to know the effect of d/ds or $\partial/\partial\phi$ on $S_F(x, y; \phi)$ or on $\det_{\mathrm{ren}}(1+K)$.

One gets:

$$\begin{aligned}
&\frac{d}{ds} \det_{\mathrm{ren}}(1+K) \\
&= \det_{\mathrm{ren}}(1+K) \Bigg\{ \sum_{\alpha,\beta} dx\, dy\, \chi_{\Delta_\alpha}(x) \chi_{\Delta_\beta}(y) \\
&\quad \cdot \Bigg[\sum_{\gamma,\delta} \int dw\, dz\, \chi_{\Delta_\gamma}(z) \chi_{\Delta_\delta}(w) S_F(y, z; \phi) K(z, w) K(w, x) \frac{dK}{ds}(x, y) \\
&\quad - \frac{1}{2} :\phi(x) \frac{d}{ds} b_{\mathrm{reg}}(x, y) \phi(y): \Bigg] \Bigg\} \\
&= :\det_{\mathrm{ren}}(1+K) \{ \mathrm{Tr}\, S_F A_2 + A_3 \}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{ds} S_F(x, y; \phi) = \sum_{\alpha,\beta} \Bigg\{ &- \int dz \frac{dK}{ds}(x, z) S_F(z, y; \phi) \chi_{\Delta_\alpha}(x) \chi_{\Delta_\beta}(z) \\
&+ \int dz\, dw\, \chi_{\Delta_\alpha}(z) \chi_{\Delta_\beta}(w) K(x, z) \frac{dK}{ds}(z, w) S_F(w, y; \phi) \\
&- \sum_{\gamma,\delta} \int dz\, dw\, dv\, du\, \chi_{\Delta_\alpha}(v) \chi_{\Delta_\beta}(u) \chi_{\Delta_\gamma}(z) \chi_{\Delta_\delta}(w) \\
&\cdot S_F(x, z; \phi) K(z, w) K(w, v) \frac{dK}{ds}(v, u) S_F(u, y; \phi) \Bigg\} \\
= :&A_1 S_F - S_F A_2 S_F .
\end{aligned}$$

The formula of derivations for $\partial/\partial\phi(z)$ are the same with $\frac{K}{ds}$ replaced by

$$\frac{\partial K(x,y)}{\partial\phi(z)} = \sum_{\Delta,\Delta',\Delta''} \lambda H(s;\Delta,\Delta',\Delta'')\left[(P^2+m^2)^{-1/4}\frac{\not{P}+m}{(P^2+m^2)^{1/2}}\right](x,z)$$
$$\chi_{\Delta''}(z)\,(P^2+m^2)^{-1/4}(z,y)\,\chi_\Delta(x)\,\chi_{\Delta'}(y)\,.$$

We note $\partial/\partial\phi\ \det_{\mathrm{ren}}(1+K) = \det_{\mathrm{ren}}(1+K)\ \{\mathrm{Tr}\, S_F A'_2 + A'_3\}$

$$\frac{\partial}{\partial\phi} S_F = A'_1 S_F - S_F A'_2 S_F\,.$$

By their definition the A_i or A'_i are completely localized expressions. In each of these terms between any two localization squares there is always a chain of "propagators":

$$(P^2+m^2)^{-1/4}(x,y) \quad \text{or} \quad \left[(P^2+m^2)^{-1/4}\frac{\not{P}+m}{(P^2+m^2)^{1/2}}\right](x,y)\,.$$

Each boson propagator, each boson field and each propagator is localized.

The A_i and A'_i are polynomials in the boson field of degree 3 at most, and between any two such fields there is always a chain of "propagators". Each A_i or A'_i has at most $3K$. Finally, acting on S_F a derivation generates an expression or order 2 in S_F, and acting on $\det_{\mathrm{ren}}(1+K)$ an expression of degree 1 in S_F.

Before estimating the number of terms produced by derivation we have first to reorder them in view of preserving the antisymmetric structure since it is essential for the volume dependance estimate ([13, 10]). Therefore after each derivation we perform the following operation:

– when the derivation acts on $\det_{i\,k} S_F \det_{\mathrm{ren}}(1+K)$ put together the terms which increase the degree in S_F. They form a new determinant of one order higher (this can be checked easily).

– other terms are left unchanged.

Now as a first step in proving Proposition II.1 we bound the number of terms produced by the derivations. To do this we use the technique of the combinatoric factors (see [7]).

To take account of the sum over the localization squares we need exponential localization factors of the type $e^{O(1)d(\Delta,\Delta')}$ or $e^{O(1)d(b,\Delta)}$ where Δ and Δ' are two localization squares in an A_i or A'_i generated by a derivation relatively to s_b.

Two localizations squares in an A_i or A'_i are linked by a chain of "propagators".

There is at most 6 "propagators" by chain. We thus distribute the localization factors to the "propagators" using:

$$d(\Delta,\Delta') \leqq d(\Delta,\Delta_1) + \ldots + d(\Delta_i,\Delta') + 10 \qquad (i \leqq 5)$$
$$d(b,\Delta) \leqq d(b,\Delta_1) + \ldots + d(\Delta_i,\Delta) + 12 \qquad (i \leqq 6)\,.$$

Let L be the number of derived K or C. We take account of these factors by the following combinatoric factors:

$$O(1)^L \prod_{\text{derivations}} e^{O(1)[d(b,\Delta_1)+d(b,\Delta_2)]} \prod_{\text{"propagators"}} e^{O(1)d(\Delta,\Delta')}$$

where product over the derivations means product over the derived K or C, Δ_1, and Δ_2 being the localizations squares of $K(x, y)$ or $C(x, y)$ and b is one of the bond relatively to which K or C is derived.

We will not list these factors in the following.

Because of Lemma II.1 we don't need to take account of how many times, and relatively to what set of bonds a K or C is derived i.e. we count only the derivations acting on a non derived K or C. In a given term we attribute a factor 2 to each K or C to decide whether it is derived or not. This gives a $2^{O(1)L}$ since the number of K or C is bounded by $O(1)L$.

Also for each derived A_i, A_i' we fix each localization square Δ using a combinatoric factor $O(1)e^{d(b,\Delta)}$.

We are thus ready to compute the number of terms generated by the derivations. Giving a factor 2 to each derivation we separate its effect according to the following cases.

α) $\frac{d}{ds}$ acts on everything except the measure $d\mu$.

β) $\frac{d}{ds}$ acts on $d\mu$.

We first compute the combinatoric factors for the case α. With a factor $O(1)$ given to the derivation, we divide case α in several subcases:

α_1) $\frac{d}{ds}$ derives S_F and we select $A_1 S_F$.

α_2) We consider the sum of terms of higher degree in S_F which form a determinant of higher order.

α_3) $\frac{d}{ds}$ derives $\det_{\text{ren}}(1+K)$ and we select A_3.

α_4) $\frac{d}{ds}$ derives a K or C created by a previous derivation (i.e. a fermion propagator in A_i or A_i').

Let us consider the combinatoric for each case separately (excluding the localization factors).

Case α_4: the combinatoric factor is 1 since the derived propagators have already been chosen.

Case α_3: the combinatoric factor is 1 since there is only one term.
Case α_2: the combinatoric factor is 1 since there is only one term.
Case α_1: it is the case: $S_F(x, y; \phi) \to (A_1 S_F)(x, y)$.

Let i in $\det_{ik} S_F(x_i, y_k; \phi)$ labels the columns. The term given by the determinant in case α_1 is a sum of determinant each with a column $A_1 S_F$. The combinatoric factors we are looking for control the number of columns (initial or produced by derivations). With a factor 2 we distinguish 2 subcases:

a) the smearing function for the variable x is a A_i or A_i' created by a previous derivation.

b) the smearing function for the variable x is an original $\hat{g}_i$ of formula (I.2).

We consider first the case a). The function A_i (or A_i') has been produced by a localized derivation $\frac{d}{ds}$ and thus contains:

$$\chi_{\Delta_1}(u)\frac{dK}{ds}(u,v)\chi_{\Delta_2}(v) \quad \text{or has been produced by } \chi_{\Delta_2}(z)\frac{\delta}{\partial\phi(z)}\,.$$

The square Δ_2 is chosen with a localization factor $\exp\{O(1)d(b,\Delta_2)\}$.

Let $R_{a,c}(\Delta)$ be the number of times that in a) $\Delta_2=\Delta$. In a given term at the end of the expansion, let $n_c(\Delta)$ be the sum of the number of times that in $\frac{dK}{ds}(u,v)$, v is localized in Δ and of the number of times that in $\frac{dC}{ds}(u,v)$, u or v is localized in Δ. Then the number of A_i or A_i' with $\Delta_2=\Delta$ is bounded by $n_c(\Delta)$. Doing this choice $R_{a,c}$ times we obtain as combinatoric factor:

$$\prod_{\Delta}\{n_c(\Delta)\}^{R_{a,c}(\Delta)} \leqq \prod_{\Delta} n_c(\Delta)^{n_c(\Delta)} \prod_{\Delta} R_{a,c}(\Delta)^{R_{a,c}(\Delta)}\,.$$

We then "attribute" to each derivation in case a) with $\Delta_2=\Delta$, a localization factor $e^{2d(b,\Delta)}$, this attribution allows us to use the following lemma:

Lemma III.1.

$$\prod_{\Delta}\left\{n_c(\Delta)^{n_c(\Delta)} \prod_{b \text{ such that } \Delta_2=\Delta} e^{-d(b,\Delta)}\right\} \leqq O(1)O(1)^L$$

$$\prod_{\Delta}\left\{R_{a,c}(\Delta)^{R_{a,c}(\Delta)} \prod_{b \text{ such that } \Delta_2=\Delta} e^{-d(b,\Delta)}\right\} \leqq O(1)O(1)^L\,.$$

Proof. The first inequality is just Lemma (10.2) of [8]. The second inequality follows also from this lemma if one remarks that: $R_{a,c}(\Delta)\leqq 2n_c(\Delta)$.

The overall combinatoric factor for case a) is therefore $O(1)O(1)^L$.

Consider now case b). Let $N_c(\Delta_0)$ be the number of functions $\hat{g}_i$, $i=1,\dots,N$ which have support in Δ_0 (it is also in the determinant the number of columns with functions $\hat{g}_i$ localized in Δ_0. We choose with a localization factor $e^{O(1)d(b,\Delta_0)}$ the square Δ_0 support of the function $\hat{g}_i$.

Let now $R_{b,c}(\Delta)$ be the number of times that in case b) $\Delta_0=\Delta$. We thus get a combinatoric factor:

$$\prod_{\Delta} N_c(\Delta)^{R_{b,c}(\Delta)}\,.$$

Attributing [as in case a)] to each derivation a factor $e^{d(b,\Delta)}$, we have at our disposal a factor $e^{-d(b,\Delta)}$ that we used in the following lemma.

Lemma III.2.

$$\prod_{\Delta} N_c(\Delta)^{R_{b,c}(\Delta)} \prod_{b \text{ such that } \Delta_0=\Delta} e^{-d(b,\Delta)} \leqq O(1)O(1)^N\,.$$

Proof. One has (see [8], Lemma 10.2):

$$\prod_{b \text{ such that } \Delta_0=\Delta} e^{-d(b,\Delta)} \leqq O(1)^{-O(1)R_{b,c}(\Delta)^{3/2}}$$

so that:

$$N_c(\Delta)^{R_{b,c}(\Delta)} \prod_b e^{-d(b,\Delta)} \leqq e^{R_{b,c}(\Delta) \ln N_c(\Delta) - O(1) R_{b,c}(\Delta)^{3/2}}$$

$$\leqq O(1) e^{O(1)(\ln N_c(\Delta))^3} \leqq O(1) e^{O(1) N_c(\Delta)}$$

but now $\prod_\Delta \exp\{O(1) N_c(\Delta)\} = O(1)^N$, this proves the lemma.

The total combinatoric factor for case α_1 is thus $O(1) O(1)^L O(1)^N$.

We now compute the combinatoric factors for case β. Since in this case each d/ds derivation generates two $\partial/\partial\phi$ derivations we compute the combinatoric factor of $\partial/\partial\phi$ derivations.

With a factor $O(1)$ by $\partial/\partial\phi$ derivation we divide the effect of $\partial/\partial\phi$ in several cases in analogy to case α:

α_1') $\partial/\partial\phi$ derives S_F and we select $A_1' S_F$.

α_2') We consider the sum of therms of higher degree in S_F which form a determinant of higher order.

α_3') $\partial/\partial\phi$ derives $\det_{\text{ren}}(1+K)$ and we select A_3'.

α_4') $\partial/\partial\phi$ derives fields ϕ created by previous derivations (i.e. fields in A_i or A_i').

α_5') $\partial/\partial\phi$ derives $\prod_{i=1}^n \phi(f_i)$ (of formula I.2).

The derivation $\partial/\partial\phi$ is localized in some square Δ_0 (already chosen). Let us now consider the combinatoric factor for each case. Cases α_1', α_2', and α_3' are as above.

Case α_4': the field $\phi(z)$, $z \in \Delta_0$, which is derived is in some A_1 or A_i' produced by $\chi_{\Delta_1}(u) \chi_{\Delta_2}(v) \dfrac{d}{ds}$ or by $\chi_{\Delta_2}(t) \dfrac{\partial}{\partial\phi(t)}$.

We choose Δ_2 with a factor $e^{O(1) d(\Delta_0, \Delta_2)}$. Now all A_i or A_i' generated by derivations localized in Δ_2 have at most $3n_c(\Delta_2)$ fields, since there is at most 3 fields in each A_i or A_i'. The combinatoric factor is then:

$$\prod_\Delta (3n_c(\Delta))^{3n_c(\Delta)}.$$

We deal with this factor as above, see Lemma III.1, this gives a bound: $O(1) O(1)^L$.

Case α_5': let $R(\Delta_0)$ be the number of times that derivations $\partial/\partial\phi$, acting on $\prod_{i=1}^n \phi(f_i)$, are localized in Δ_0. Each time the number of choices is $n(\Delta_0)$ (remember it is the number of f_i with support in Δ_0). Thus the total combinatoric factor is

$$\prod_\Delta n(\Delta)^{R(\Delta)}.$$

Attributing a factor $e^{d(b, \Delta_0)}$ to each derivation relative to b, localized in Δ_0, we have by Lemma III.2:

$$\prod_\Delta \Big\{ n(\Delta)^{R(\Delta)} \prod_{b \text{ deriving in } \Delta} e^{-d(b,\Delta)} \Big\} \leqq O(1) O(1)^n.$$

Finally we have got that for the cluster expansion, the combinatoric factors are (see Proposition II.1):

$$O(1)\,O(1)^L O(1)^n O(1)^N \prod_{\text{derivations}} e^{O(1)[d(b_i,\Delta_i)+d(b_i,\Delta_i')]} \prod_{\text{“propagators”}} e^{O(1)d}$$

where d stands for $d(\Delta, \Delta')$, Δ and Δ' being the localization squares of the "propagator". Define $\bar{\lambda}_0 = \sup\{1, \lambda_0\}$. Now the following lemma is sufficient to prove Proposition II.1:

Lemma III.3. *Fix m_1 and m_2 as for Lemma II.1 and let R be an element in the expansion of Proposition II.1, then there exist Q_3 large (depending on m_2 and m_1), m large enough and $Q_5 > 0$ such that:*

$$\sup_{b_i, i=1,\dots,L}\ \sup_{\substack{\Delta_1,\Delta_1',\Delta_1'' \\ \vdots \\ \Delta_L,\Delta_L',\Delta_L''}}\ \sup_R \left\{ \prod_{i=1}^{L} e^{-(Q_3 - O(1))[d(b_i,\Delta_i)+d(b_i,\Delta_i')]} \right.$$

$$\left. \cdot \prod_{\substack{\text{derived} \\ \text{boson propagators}}} e^{O(1)d} \prod_{\text{“propagators”}} e^{O(1)d}\, O(1)^L \left| \int R\, d\mu \right| \right\}$$

$$\leqq [\bar{\lambda}_0^3 e^{-Q_5}]^L [e^{Q_5} O(1)]^n O(1)^N e^{Q_6|\Gamma|} \prod_{\Delta} (n(\Delta)!)^{1/2} \prod_{i=1}^{n} \|f_i\|_{-1} \prod_{j=1}^{N} \|\hat{g}_j\|_{L^2} \|\hat{h}_j\|_{L^2}. \tag{III.2}$$

Q_5 can be taken as large as we want if m is taken sufficiently large and is independant of Γ. The $O(1)$ factors in the right hand side and Q_6 are independant of m_1, m_2, m, and Γ.

This lemma includes the combinatoric factors of $\sum_R$ in Proposition II.1, thus taking $Q_2 + 3\log\bar{\lambda}_0 \leqq Q_5$ and the supremum over L, one has proved this proposition.

Let us now prove Lemma III.3. The integrand R has the general form:

$$R = \int \det_{ik} S_F(x_i, y_k; \phi)\, w(x_1, \dots; y_1, \dots)\, \det_{\text{ren}}(1+K)\, dx_1 \dots dy_1 \dots$$

where $w(x_1, \dots; y_1, \dots)$ is a product or integral of product of $\hat{g}_i$, $\hat{h}_k$, C, A_i, A_i' and $\phi(f)$.

To bound $|\int R\, d\mu|$ we use:

Proposition III.1. *Let $1 \leqq i, k \leqq N + 2L$, and M (in the definition of* $\det_{\text{ren}}(1+K)$ *be large enough depending on λ_0 then there exists $Q_6 > 0$:*

$$\left|\int \det_{ik} S_F(x_i, y_k; \phi)\, w(x_1, \dots; y_1, \dots)\, \det_{\text{ren}}(1+K)\, dx_1 \dots dy_1 \dots d\mu\right|$$

$$\leqq \left[\int \|w\|_{L^2}^4 d\mu\right]^{1/4} O(1)\, e^{Q_6|\Delta|}\, O(1)^{N+2L}.$$

Proof. Applying twice Schwarz inequality the proposition follows from the work of Seiler and Simon [14]: their proof applies here since they have also localized each $K(x, y)$ in unit squares, the fact of multiplying it by $0 \leqq H(\Delta, \Delta, \Delta') \leqq 1 (x \in \Delta, y \in \Delta')$ leaves the proof available up to obvious modifications.

Note: Our preprint version [16] was self contained and in particular included a third proof of the linear lower bound, which differs from those of McBryan [10] and [11] and of Seiler and Simon [14].

It seems also to us that the nice proof of McBryan can also be extended to the s-dependant models described here.

Under the conditions of A_2, $|\Lambda|=|X|\leqq|\Gamma|+1$ ($|X|=$ surface of X). The next step is to bound $\int \|w\|_{L^2}^4 d\mu$.

First we estimate the effect of the functional integration $d\mu$ on $\|w\|^4$ by computing the combinatoric factors corresponding to the contractions between the fields ϕ.

We characterize the fields ϕ by squares:

1) For a field $\phi(f)$ of formula (I.1), the "characterizing square" is Δ if support of f_i is in Δ.

2) For a field belonging to some A_i (resp. A_i') the characterizing square is Δ if A_i (resp. A_i') has been generated by $\chi_\Delta(x)\dfrac{d}{ds}K(x,\cdot)$, $\left(\text{resp. by } \chi_\Delta(x)\dfrac{d}{ds}C(x,\cdot)\right.$ or by $\left.\chi_\Delta(y)\dfrac{d}{ds}C(\cdot,y)\right)$.

The number of fields characterized by Δ is less than $4n(\Delta)+12n_c(\Delta)$ (since there is at most 3 fields ϕ by A_i or A_i').

Attributing a factor $e^{O(1)d(\Delta,\Delta')}$ to each contraction between a field characterized by Δ with a field characterized by Δ', we get as a combinatoric factor for the contractions:

$$\prod_\Delta O(1)^{4n(\Delta)+12n_c(\Delta)}|[4n(\Delta)+12n_c(\Delta)]!|^{1/2}$$
$$\leqq O(1)^n O(1)^L \prod_\Delta (4n(\Delta)!)^{1/2} \prod_\Delta (12n_c(\Delta)!)^{1/2}.$$

Attributing factors $e^{O(1)d(b,\Delta)}$ to the derivations we treat the last term as before.

Finally the (localization) factors $e^{O(1)d(\Delta,\Delta')}$ can be decomposed in a product of localization factors by boson propagator and by "propagator".

The total combinatoric factor for the contractions is thus for $|\int \|w\|_{L^2}^4 d\mu|^{1/4}$:

$$O(1)^n O(1)^L \prod_\Delta (n(\Delta)!)^{1/2} \prod_{\text{derivations}} e^{O(1)[d(\Delta_i,b_i)+d(b_i,\Delta_i')]} \prod_{\text{boson propagators}} e^{O(1)d}.$$

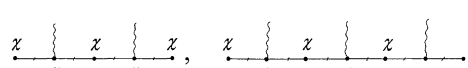
Let us consider a fully contracted term: we call such a term a big graph. A big graph is decomposed in small graphs, and each small graph will be estimated by its Hilbert-Schmidt norm, see [7]. A big graph is formed with:

vertices: the $\hat{g}_i$, $\hat{h}_k$, and f_i functions and also the functions χ_Δ,

boson propagators noted ~~ and "propagators" noted —/—.

The small graphs are:

$\hat{g}.$, $\hat{h}.$, f~~~,



and also: $\phi\, b_{\text{reg}}\, \phi$:

We first apply the H.S. norm to the vertices in ĝ and ĥ, this gives a bound:

$$\prod_{j=1}^{N} (\|\hat{g}\|_{L^2}\|\hat{h}\|_{L^2})^4 \qquad |\text{big graph}|^{1/2}.$$

To bound this new big graph, we first use the technique of [7, 9] and we extract from each "propagator" and each boson propagator localized in Δ and Δ' a factor $e^{-O(1)d(\Delta,\Delta')}$ where $O(1)$ is taken as large as we want by taking m large enough.

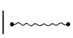
We then use the following bounds:

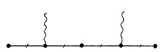
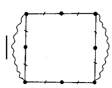
$$|\tilde{\chi}_\Delta(p)| \leqq O(1)F(p), \quad \frac{1}{(p^2+m^2)} \leqq O(1)F(p)^{1-\varepsilon/2}m^{-\varepsilon} \leqq e^{-Q_8}F(p)^{1-\varepsilon/2}$$

[for any $0<\varepsilon<1/3$ if m is taken large enough depending on Q_8], and

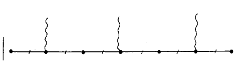
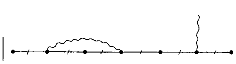
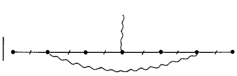
$$\frac{1}{(p^2+m^2)^{1/4}} \leqq O(1), \quad \left|(p^2+m^2)^{-1/4}\frac{\not{p}+m}{(p^2+m^2)^{1/2}}\right| \leqq O(1)F(p)^{1/4-\varepsilon/4}e^{-Q_8/2}.$$

Finally if a boson propagator is attached to a f_i-function of formula (I.2) we use $\frac{1}{p^2+m^2} \leqq \frac{1}{p^2+1}e^{Q_8}e^{-Q_8}$. The bounds for the small graphs are:

$\|$$\|_{\mathrm{H.S.}} = |$$|^{1/2} \leqq \|f\|_{-1}e^{Q_8}e^{-Q_8}$

$\|$$\|_{\mathrm{H.S.}} = |$$|^{1/2} \leqq O(1)(e^{-Q_8})^3\bar{\lambda}_0^2$.

For the small graph with 3 boson lines, because the boson line can possibly contract between themselves we are obliged to consider:

$\|$$\|_{\mathrm{H.S.}}$, $\|$$\|_{\mathrm{H.S.}}$, $\|$$\|_{\mathrm{H.S.}}$.

These 3 norms are bounded by $O(1)(e^{-Q_8})^{3+1+\frac{1}{2}}\bar{\lambda}_0^3$.

Finally if Δ and Δ' are neighbours or identic it is proved in [14] see also [16] that:

$$\left\|\int \chi_\Delta :\phi\, b_{\mathrm{reg}}\phi: \chi_{\Delta'}\right\|_{\mathrm{H.S.}} \leqq O(1)m^{-3\varepsilon}\bar{\lambda}_0^2 \leqq O(1)(e^{-Q_8})^3\bar{\lambda}_0^2.$$

If Δ and Δ' have no intersection then it is trivial that:

$$|b_{\mathrm{reg}}(x,y)\chi_\Delta(x)\chi_{\Delta'}(y)| \leqq O(1)e^{-md(\Delta,\Delta')/2}m^{-2\varepsilon}\bar{\lambda}_0^2.$$

From that we get:

$$\|\chi_\Delta :\phi\, b_{\mathrm{reg}}\phi: \chi_{\Delta'}\|_{\mathrm{H.S.}} \leqq O(1)(e^{-Q_8})^3\bar{\lambda}_0^2 e^{-md(\Delta,\Delta')/2}.$$

We have then obtained for each term a bound consisting of:

Exponentially decreasing factors for all the localizations, and the decrease is as strong as we want provided that we take m large enough.

A product of norm of small graphs, and for each term the number of small graphs is smaller than $O(1)L$ non counting the small graphs associated with $\hat{g}$, $\hat{h}$ and f.

So that each contracted terms is bounded by:

$$O(1)^L|e^{-Q_8}|^L\bar{\lambda}_0^{3L}|e^{Q_8}|^n\prod_{i=1}^n\|f_i\|_{-1}\prod_{j=1}^N\|\hat{g}_j\|_{L^2}\|\hat{h}_j\|_{L^2}\prod e^{-O(1)d}.$$

Thus collecting the various bound and taking $O(1)e^{-Q_8}=e^{-Q_5}$ we have:

$$O(1)^L|\int R\,d\mu| \leqq O(1)^N(O(1)e^{Q_5})^n \prod_{\Delta}(n(\Delta)!)^{1/2}(e^{-Q_5})^L \bar{\lambda}_0^{3L} e^{Q_6|\Gamma|}$$

$$\prod_{\text{derivations}} e^{O(1)[d(\Delta_j,b_j)+d(\Delta_j',b_j)]} \prod_{i=1}^{n} \|f_i\|_{-1} \prod_{j=1}^{N} \|\hat{g}_j\|_{L^2} \|\hat{h}_j\|_{L^2} \prod_{\text{"propagators"}} e^{-O(1)d}.$$

This finishes the proof og Lemma III.3.

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