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# A New Method for Constructing Factorisable Representations for Current Groups and Current Algebras

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**Abstract.** Let $C_e^\infty(R^n, G)$ denote the group of infinitely differentiable maps from $n$-dimensional Euclidean space into a simply connected and connected Lie group, which have compact support. This paper introduces a class of factorisable unitary representations of $C_e^\infty(R^n, G)$ with the property that the unitary operator $U_f$ corresponding to a function $f$ in $C_e^\infty(R^n, G)$ depends not only on $f$, but also on the derivatives of $f$ up to a certain order. In particular these representations can not be extended to the group of all continuous functions from $R^n$ to $G$ with compact support.

## § 1. Introduction

Let $G$ be a simply connected and connected Lie group and let $\mathscr{G}$ be its Lie algebra. Let $\exp:\mathscr{G}\to G$ denote the exponential map. We denote by $C_e^\infty(R, G)$ the class of all $C^\infty$ maps from $R$ into $G$ with compact support. A map $\varphi: R\to G$ is said to have compact support if takes the value $e$, i.e., the identity element of $G$ outside a compact set. Let $C_0^\infty(R, \mathscr{G})$ denote the class of all infinitely differentiable maps from $R$ into the vector space $\mathscr{G}$ with compact support. For any $f\in C_0^\infty(R, \mathscr{G})$, we define $\operatorname{Exp} f\in C_e^\infty(R, G)$ by writing $(\operatorname{Exp} f)(x)=\exp f(x)$, for all $x\in R$. $C_e^\infty(R, G)$ is a group (under pointwise multiplication) and $C_0^\infty(R, \mathscr{G})$ is a Lie algebra (under pointwise addition, scalar multiplication and Lie brackets). These may respectively be called as current group and current algebra over $R$. We give $C_0^\infty(R, \mathscr{G})$ the usual Schwarz topology. A homomorphism $\varphi\to U_\varphi$ of the group $C_e^\infty(R, G)$ into the group of unitary operators on a Hilbert space $H$ is said to be a *unitary representation* or simply a representation if $U_{\operatorname{Exp} f_n}$ converges weakly to $U_{\operatorname{Exp} f}$ whenever $f_n\to f$ as $n\to\infty$ in the topology of $C_0^\infty(R, \mathscr{G})$.

For any compact set $K\subset R$, let $C(K, G)\subset C_0^\infty(R, G)$ be the subgroup of all those maps with support contained in $K$. If $K_1$, $K_2$ are two disjoint compact subsets of $R$, $C(K_1\cup K_2, G)$ can be identified in a natural manner with the cartesian product $C(K_1, G)\times C(K_2, G)$. Indeed, for any $\varphi\in C(K_1\cup K_2, G)$, define

$$\begin{aligned}\varphi_i(x)&=\varphi(x) \quad \text{if} \quad x\in K_i\\ &=e \quad \text{if} \quad x\notin K_i, \quad i=1,2.\end{aligned}$$

Then $\varphi = \varphi_1 \varphi_2$. The map $\varphi \to (\varphi_1, \varphi_2)$ gives the required identification. For any representation $U$ of $C_e^\infty(R, G)$, we define the representation $U^K$ of the subgroup $C(K, G)$ by

$$U_\varphi^K = U_\varphi, \varphi \in C(K, G).$$

We say that a representation $U$ of $C_e^\infty(R, G)$ is *factorisable* if, for any two disjoint compact sets $K_1, K_2 \subset R$, the representation $U^{K_1 \cup K_2}$ is unitarily equivalent to the tensor product $U^{K_1} \otimes U^{K_2}$. This unitary equivalence will of course depend on $K_1$ and $K_2$. Examples of such factorisable representations based on the unitary representations of $G$ and their first cohomologies were first constructed by Streater [6] and Araki [1]. Further development of these ideas may be found in the works of Parthasarathy and Schmidt [4, 3], Vershik, Gelfand and Graev [7], and Guichardet [2]. However, most of these examples have the degenerate property that they factorise completely. These representations extend to borel maps from $R$ into $G$ and the factorisability property extends to pairs of disjoint borel sets. This is mainly because the representations constructed in these papers do not involve the derivatives of smooth maps in a certain sense. One may compare this with the following situation in the classical theory of distributions. To evaluate the Dirac $\delta$ at a testing function $\varphi$ one need not know the derivations of $\varphi$. However to evaluate the distributions $\delta', \delta'', \ldots$ one requires a knowledge of $\varphi', \varphi'', \ldots$. The main aim of this paper is to construct factorisable representations $U$ which for their evaluation at $\operatorname{Exp} f, f \in C_0^\infty(R, \mathscr{G})$ requires a knowledge of $f, f', f'', \ldots$. A beginning in this direction was already made by Schmidt [5] in the case when $G$ is the Heisenberg group, whose representations lead to canonical commutation relations.

## § 2. The Leibnitz Extension of a Lie Algebra

In order to outline the method of constructing factorisable representations we need to construct an extension of the Lie algebra $\mathscr{G}$. To this end consider the space $\mathscr{G}_n$ which is the $n+1$-fold Cartesian product of $\mathscr{G}$. Any element $\boldsymbol{X}$ of $\mathscr{G}_n$ can be written as

$$\boldsymbol{X} = (X_0, X_1, \ldots, X_n), X_i \in \mathscr{G} \quad \text{for each } i.$$

Between two elements $\boldsymbol{X}$ and $\boldsymbol{X}'$ in $\mathscr{G}_n$ define the bracket operation by

$$[\boldsymbol{X}, \boldsymbol{X}'] = \boldsymbol{X}'',$$

where

$$X_0'' = [X_0, X_0'],$$
$$X_j'' = \sum_{r=0}^{j} \binom{j}{r} [X_r, X_{j-r}']. \tag{2.1}$$

An easy computation shows that for $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathscr{G}_n$,

$$[[\boldsymbol{X}, \boldsymbol{Y}]\boldsymbol{Z}] = \boldsymbol{T}$$

where

$$T_r = \sum_{k_1+k_2+k_3=r} (r!/k_1!k_2!k_3!)[[X_{k_1}, Y_{k_2}], Z_{k_3}].$$

This shows that

$$[[\boldsymbol{X}, \boldsymbol{Y}], \boldsymbol{Z}] + [[\boldsymbol{Y}, \boldsymbol{Z}], \boldsymbol{X}] + [[\boldsymbol{Z}, \boldsymbol{X}], \boldsymbol{Y}] = 0.$$

In other words $\mathscr{G}_n$ becomes a Lie algebra. We shall call $\mathscr{G}_n$ the $n^{\text{th}}$ *Leibnitz extension* of the Lie algebra $\mathscr{G}$. The mapping $X \to (X, 0, 0, \ldots, 0)$ is an isomorphism of $\mathscr{G}$ into $\mathscr{G}_n$. All elements of the form $(0, X_1, X_2, \ldots, X_n)$, $X_1 \in \mathscr{G}$, $i = 1, 2 \ldots n$ constitute a nilpotent Lie subalgebra $\mathscr{h}^{(n)}$ of $\mathscr{G}_n$. Further

$$[(X, 0, 0, \ldots, 0), (0, X_1, X_2, \ldots, X_n)] = (0, [X, X_1], [X, X_2], \ldots, [X, X_n]).$$

Thus $\mathscr{G}$ acts as a Lie algebra of derivations of the nilpotent Lie algebra $\mathscr{h}^{(n)}$. In other words $\mathscr{G}_n$ is a semi-direct sum of $\mathscr{G}$ and $\mathscr{h}^{(n)}$.

*Remark 2.1.* Since any Lie algebra $\mathscr{G}$ can be represented as a Lie algebra of matrices, we shall assume that $\mathscr{G}$ is a Lie algebra of real matrices in all our computations hereafter. Let the order of the matrices in $\mathscr{G}$ be $k \times k$.

**Lemma 2.2.** *The map*

$$A : (0, X_1, X_2, \ldots, X_n) \to A(X_1, X_2, \ldots, X_n),\ X_i \in \mathscr{G},\ i = 1, 2 \ldots n$$

*where*

$$A(X_1, X_2, \ldots, X_n) = \begin{pmatrix} 0 & X_1/1! & X_2/2! & \ldots & & X_n/n! \\ 0 & 0 & X_1/1! & X_2/2! & \ldots & X_{n-1}/n-1! \\ 0 & 0 & 0 & X_1/1! & \ldots & X_{n-2}/n-2! \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \ldots & 0 \end{pmatrix}$$

*is an isomorphism of the Lie algebra $\mathscr{h}^{(n)}$ into the Lie algebra of all matrices of order $k(n+1) \times k(n+1)$.*

*Proof.* This follows from a routine computation and is left to the reader.

**Lemma 2.3.** *Let $A$ be the map defined in the preceding lemma. Then the matrix* $\exp A(X_1, X_2, \ldots, X_n)$ *is of the form*

$$\begin{pmatrix} I & A_1 & A_2 & \ldots & \ldots & A_n \\ 0 & I & A_1 & A_2 & \ldots & A_{n-1} \\ 0 & 0 & I & A_1 & \ldots & A_{n-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & I \end{pmatrix}$$

*where*

$$A_j = \sum_{p=1}^{j} 1/p! \sum_{\substack{m_1+\ldots+m_p=j \\ 1\leqq m_i\leqq j}} m_1!^{-1} X_{m_1} m_2!^{-1} X_{m_2} \ldots m_p!^{-1} X_{m_p}.$$

*Proof*. It is left to the reader.

*Remark 2.4*. Let $H$ be the group generated (algebraically) by all matrices of the form $\exp A(X_1, X_2, \ldots, X_n)$, $X_i \in \mathscr{G}$, $i=1,2\ldots n$. Its Lie algebra is isomorphic with $\mathscr{h}^{(n)}$. Let $G$ be the simply connected group for which the Lie algebra is $\mathscr{G}$. Then for any $X_0 \in \mathscr{G}$, the element $\exp X_0$ of $G$ acts as an automorphism of $H$ as follows:

$$\begin{aligned}&\exp X_0 : \exp A(X_1, X_2, \ldots, X_n)\\ &\to \exp A(e^{\operatorname{ad} X_0}(X_1), e^{\operatorname{ad} X_0}(X_2), \ldots, e^{\operatorname{ad} X_0}(X_n)).\end{aligned}$$

Hence we can form the semi-direct product $G \odot H$ of the two groups $G$ and $H$. $G \odot H$ consists of all pairs $(g,h)$, $g \in G$, $h \in H$. The multiplication operation is defined by

$$(g,h)\cdot(g',h') = (gg', h\cdot g(h')),$$

where $h' \to g(h')$ is the automorphism of $H$ induced by $g$. The Lie algebra of the group $G \odot H$ is then isomorphic to the Lie algebra $\mathscr{G}_n$. In particular $\mathscr{G}_1$ is the Lie algebra of the semidirect product of $G$ and the additive group $\mathscr{G}$, where $G$ acts as the adjoint representation in $\mathscr{G}$.

**Lemma 2.4.** *For any* $X = (X_0, X_1, \ldots, X_n) \in \mathscr{G}_n$, *the exponential map from* $\mathscr{G}_n$ *into* $G \odot H$ *is defined as follows: let*

$$A_j(t) = \sum_{p=1}^{j} \sum_{\substack{m_1+\ldots+m_p=j \\ 1\leqq m_i\leqq j}} \int_{0<t_1<t_2<\ldots<t_p<t} \left( \prod_{k=1}^{p} e^{t_k \operatorname{ad} X_0}(m_k!^{-1} X_{m_k}) \right) dt_1 \ldots dt_p \quad (2.2)$$

*for* $j = 1, 2 \ldots n$. *Let*

$$A(t) = \begin{pmatrix} I & A_1(t) & A_2(t) & \ldots A_n(t) \\ 0 & I & A_1(t) & \ldots A_{n-1}(t) \\ 0 & 0 & I & \ldots A_{n-2}(t) \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\ 0 & 0 & \ldots\ldots\ 0 & I \end{pmatrix}.$$

*Then*

$$\exp tX = (\exp tX_0, A(t)) \quad \textit{for all} \quad t \in R.$$

*Proof*. Indeed, differentiating (2.2) at $t=0$, we get

$$dA_j/dt|_{t=0} = j!^{-1} X_j.$$

Thus

$$dA(t)/dt|_{t=0} = \begin{pmatrix} 0 & X_1/1! & \dots & X_n/n! & \\ 0 & 0 & X_1/1! & \dots & X_{n-1}/(n-1)! \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Further

$$(\exp tX_0, A(t))\cdot(\exp sX_0, A(s))$$
$$=(\exp(t+s)X_0, A(t)\cdot\exp tX_0(A(s))\,,$$

where

$$\exp tX_0(A(s)) = \begin{pmatrix} I & B_1 & B_2 & \dots & B_n \\ 0 & I & B_1 & \dots & B_{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I \end{pmatrix}$$

and

$$B_j = B_j(t,s) = e^{tX_0}A_j(s)e^{-tX_0}$$
$$= \sum_{p=1}^{j} \sum_{\substack{m_1+\dots+m_p=j \\ m_i \geqq 1 \text{ for all } i}} \int_{0<t_1<t_2<\dots<t_p<s} \prod_{k=1}^{p} e^{(t_k+t)\operatorname{ad}X_0}(k!^{-1}X_{m_k})dt_1\dots dt_p$$
$$= \sum_{p=1}^{j} \sum_{\substack{m_1+\dots+m_p=j \\ m_i \geqq 1 \text{ for all } i}} \int_{0<t_1<t_2<\dots<t_p<t+s} \prod_{k=1}^{p} e^{t_k\operatorname{ad}X_0}(k!^{-1}X_{m_k})dt_1\dots dt_p\,. \quad (2.3)$$

A straightforward matrix multiplication shows that

$$A(t)\cdot\exp tX_0(A(s)) = \begin{pmatrix} I & C_1 & C_2 & \dots & C_n \\ 0 & I & C_1 & \dots & C_{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I \end{pmatrix},$$

where

$$C_j = \sum_{r=0}^{j} A_r(t)B_{j-r}(t,s)\,,$$

$$A_0(t) = B_0(t,s) = I\,,$$

and where $A_r$ and $B_r$ is defined by (2.2) and (2.3) respectively. Now an easy computation gives $C_j = A_j(t+s)$. This shows that $(\exp tX_0, A(t))$ is a one parameter group with the generator $(X_0, X_1, X_2, \dots, X_n)$. The proof is complete.

**Corollary 2.5.** *When $n=1$ and $G \odot H$ is identified with the semidirect product of $G$ and the additive group $\mathscr{G}$, where $G$ acts as adjoint representation in $\mathscr{G}$, we have*

$$\exp t(X_0, X_1) = \left(\exp tX_0, \frac{e^{t\,\mathrm{ad}\,X_0 - 1}}{t\,\mathrm{ad}\,X_0}(X_1)\right)$$

*for all $t \in R$.*

*Proof.* This follows from the preceding lemma by noting that

$$\int_0^t e^{t_1\,\mathrm{ad}\,X_0}(X_1)dt_1 = \frac{e^{t\,\mathrm{ad}\,X_0} - 1}{t\,\mathrm{ad}\,X_0}(X_1).$$

## § 3. Representation of Current Algebras and Current Groups

In the preceding section we gave a complete description of the group associated with the $n$-th Leibnitz extension $\mathscr{G}_n$ of a Lie algebra $\mathscr{G}$. The following lemma yields the required embedding of $C_0^\infty(R, \mathscr{G})$ into $C_0^\infty(R, \mathscr{G}_n)$ for writing down our representations.

**Lemma 3.1.** *Let $\Pi_n$ be the map from $C_0^\infty(R, \mathscr{G})$ into $C_0^\infty(R, \mathscr{G}_n)$ defined by*

$$\Pi_n(f)(x) = (f(x), f'(x), f''(x), \ldots, f^{(n)}(x))$$

*for all $x \in R$, $f \in C_0^\infty(R, \mathscr{G})$.*

*Then $\Pi_n$ is a Lie algebra isomorphism of $C_0^\infty(R, \mathscr{G})$ into $C_0^\infty(R, \mathscr{G}_n)$.*

*Proof.* This follows immediately from the fact that

$$d^j[f, g]/dx^j = \sum_{r=0}^{j} \binom{j}{r} [f^{(r)}(x), g^{(j-r)}(x)]$$

and the commutation rules in $\mathscr{G}_n$ are defined by (2.1).

As mentioned in § 1, we define for any $f \in C_0^\infty(R, \mathscr{G})$, $\operatorname{Exp} f$ as the element in $C_e^\infty(R, G)$ with the property

$$(\operatorname{Exp} f)(x) = \exp f(x), \; x \in R.$$

Consider the group $G \odot H$ described in Remark 2.4. We shall call it the $n$-th Leibnitz extension of the group $G$. For any $f \in C_0^\infty(R, \mathscr{G})$, we define $\operatorname{Exp}_n f$ as the element in $C_e^\infty(R, G \odot H)$ with the property

$$(\operatorname{Exp}_n f)(x) = (\exp f(x), A^f(x)),$$

where

$$A^f(x) = \begin{pmatrix} I & A_1^f(x) & A_2^f(x) & \ldots & A_n^f(x) \\ 0 & I & A_1^f(x) & \ldots & A_{n-1}^f(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & I \end{pmatrix}$$

$$A_j^f(x) = \sum_{p=1}^{j} \sum_{\substack{m_1+\ldots+m_p=j \\ m_i \geqq 1 \text{ for all } i}} \int_{0<t_1<t_2<\ldots<t_p<1} \prod_{k=1}^{p} e^{t_k \operatorname{ad} f(x)}$$

$$\cdot m_k!^{-1} f^{(m_k)}(x)\, dt_1\, dt_2 \ldots dt_p\,, \tag{3.1}$$

for $j=1,2\ldots n$. With this notation we have the following corollary to Lemma 3.1.

**Theorem 3.2.** *Let $G$ be a connected and simply connected Lie group whose $n$-th Leibnitz extension is $G_n$. Suppose $\varphi \to U_\varphi$ is a factorisable representation of the current group $C_e^\infty(R, G_n)$. Then the map*

$$U^{(n)}: \operatorname{Exp} f \to U_{\operatorname{Exp}_n f},\ f \in C_0^\infty(R, \mathscr{G})$$

*determines a factorisable representation of the current group $C_e^\infty(R, G)$. In particular this determines a factorisable representation of the current algebra $C_0^\infty(R, \mathscr{G})$.*

*Remark 3.3.* To construct a factorisable representation $U$ of the current group $C_e^\infty(R, G_n)$ one may start with a projection valued measure on the Borel subsets of $R$, a unitary represention $V$ of the group $G_n$ commuting with the projection valued measure and a first order cocycle for the representation $V$, and adopt the procedure outlined in [4]. Since $G$ is a subgroup of $G_n$ it follows that $C_e^\infty(R, G)$ is a subgroup of $C_e^\infty(R, G_n)$. Hence the restriction of $U$ to $C_e^\infty(R, G)$ yields a representation $U^{(0)}$ of $C_e^\infty(R, G)$. The representation $U^{(n)}$ of Theorem 3.1 obtained from $U$ may be considered as the $n$-th derivative of the representation $U^{(0)}$.

*Example 3.4.* We shall now illustrate the procedure outlined in the preceding remark in a special case. Let $G$ be a compact, connected, simply connected and semi-simple Lie group with Lie algebra $\mathscr{G}$ and Cartan Killing form $B(X, Y)$, $X, Y \in \mathscr{G}$. Let $g \to \operatorname{Ad} g$ be the adjoint representation of $G$ acting in $\mathscr{G}$. Let $G_1$ denote the first Leibnitz extension of $G$. Then $G_1$ is the semi direct product of $G$ and the additive group $\mathscr{G}$ in which $G$ acts as a group of automorphisms through the adjoint representation. Any element of $G_1$ can be expressed as a pair $(g, X)$, $g \in G$, $X \in \mathscr{G}$. Then $(g, X) \to \operatorname{Ad} g$ is an irreducible unitary representation $U$ of $G_1$ acting in the vector space $\mathscr{G}$ with the positive definite inner product $-B$. Define the map $\delta: G_1 \to \mathscr{G}$ by

$$\delta(g, X) = X\,.$$

Then $\delta$ is a first order cocycle for the representation $U$. Hence the function

$$\Phi(g, X) = \exp \tfrac{1}{2} B(X, X)$$

is an infinitely divisible positive definite function on the group $G_1$.

Let now $\varphi: R \to \mathscr{G}$ be a $C_0^\infty$ map from $R$ into $\mathscr{G}$. Then the map $t \to (\varphi(t), \varphi'(t))$ is a $C_0^\infty$ map from $R$ into $\mathscr{G}_1$ the Lie algebra of $G_1$. Let

$$\psi(t) = \frac{e^{\operatorname{ad} \varphi(t)} - 1}{\operatorname{ad} \varphi(t)} (\varphi'(t))\,,$$

and let

$$K(\operatorname{Exp} \varphi) = \exp \tfrac{1}{2} \int B(\psi(t), \psi(t))\, dt\,. \tag{3.2}$$

Then $K$ is an infinitely divisible positive definite functional on $C_e^\infty(R, G)$ which extends to $C_e^1(R, G)$, the group of all $C^1$ maps from $R$ into $G$ with compact support. This positive definite functional defines a factorisable representation of $C_e^1(R, G)$ which cannot be extended to all bounded borel maps from $R$ into $G$ with compact support.

Since the factorisable representation corresponding to (3.2) is in a sense a continuous tensor product of copies of the irreducible adjoint representation of $G$ one is tempted to conjecture that (3.2) yields an irreducible factorisable representation of $C_e^1(R, G)$.

*Remark 3.5.* The theory outlined above extends in a natural manner when $R$ is replaced by $R^m$ and one considers current groups $C_e^\infty(R^m, G)$. To describe this extension we adopt the following conventions. Let, for any positive integer $N$, $F_N$ denote the set of all ordered $m$-tuples $\underline{j}=(j_1, j_2, \ldots, j_m)$ of non-negative integers such that $j_1+j_2+\ldots+j_m<N$. For any $\underline{j}\in F_N$, let $\underline{j}!=j_1!j_2!\ldots j_m!$, where $0!=1$. A general point of $R^m$ will be denoted by $x=(x_1, x_2, \ldots, x_m)$. Let $|\underline{j}|=j_1+j_2+\ldots+j_m$. For any $C^\infty$ map $f$ from $R^m$ into the Lie algebra $\mathcal{G}$, let

$$f^{(\underline{j})}=\partial^{|\underline{j}|}f/\partial x_1^{j_1}\partial x_2^{j_2}\ldots\partial x_m^{j_m}.$$

We now define the $N$-th Leibnitz extension $\mathcal{G}_N$ of $\mathcal{G}$ as the set of all maps $\boldsymbol{X}$ from $F_N$ into $\mathcal{G}$ with Lie bracket $[\boldsymbol{X}, \boldsymbol{Y}]$ defined by

$$[\boldsymbol{X}, \boldsymbol{Y}](\underline{j})=\sum(\underline{j}!/\underline{r}!(\underline{j}-\underline{r})!\,[\boldsymbol{X}(\underline{r}), \boldsymbol{Y}(\underline{j}-\underline{r})]$$

where the summation is over all $\underline{0}\leqq\underline{r}\leqq\underline{j}$. Here $\underline{r}\leqq\underline{j}$ means that $r_i\leqq j_i$ for all $i=1, 2, \ldots, m$. Then $\mathcal{G}_N$ is a Lie algebra. As before $\mathcal{G}$ may be embedded in $\mathcal{G}_N$ by mapping any $X\in\mathcal{G}$ to the element $\boldsymbol{X}$ with $\boldsymbol{X}(\underline{0})=X$, $\boldsymbol{X}(\underline{i})=0$ for $\underline{i}\neq\underline{0}$. Let us say that $\underline{i}<\underline{j}$ if $\underline{i}\neq\underline{j}$ but $\underline{i}\leqq\underline{j}$. As before all elements $\boldsymbol{X}$ such that $\boldsymbol{X}(\underline{0})=0$ constitute a nilpotent Lie subalgebra $\mathscr{h}^{(N)}$ of $\mathcal{G}_N$. $\mathcal{G}_N$ is a semidirect sum of $\mathcal{G}$ and $\mathscr{h}^{(N)}$. For $\boldsymbol{X}\in\mathscr{h}^{(N)}$, we define the matrix $A(\boldsymbol{X})$ whose $(\underline{i},\underline{j})^{\text{th}}$ element is $\boldsymbol{X}(\underline{i}+\underline{j})$ if $\underline{j}>\underline{i}$ and 0 otherwise. The order of the matrix is $ck\times ck$ where $c$ is the cardinality of $F_N$ and $k$ is the order of the matrices which constitute the Lie algebra $\mathcal{G}$. Lemma 2.3 now holds with the convention

$$A_{\underline{j}}=\sum_{p=1}^{|\underline{j}|}p!^{-1}\sum_{\underline{m}_1+\ldots+\underline{m}_p=\underline{j}}\underline{m}_1!^{-1}\boldsymbol{X}(\underline{m}_1)\ldots\underline{m}_p!^{-1}\boldsymbol{X}(\underline{m}_p).$$

Lemma 2.4 holds with the condition

$$A_{\underline{j}}(t)=\sum_{p=1}^{|\underline{j}|}\sum_{\underline{m}_1+\ldots+\underline{m}_p=\underline{j}}\int_{0<t_1<t_2\ldots<t_p<t}\prod_{i=1}^{p}e^{t_i\,\mathrm{ad}\,\boldsymbol{X}(0)}\cdot(\underline{m}_i!^{-1}\boldsymbol{X}(\underline{m}_i))\,dt_1\ldots dt_p.$$

Then Theorem 3.2 holds with the condition that in defining the map $f \to \mathrm{Exp}_n f$ we change (3.1) to

$$A_{\underline{j}}^f = \sum_{p=1}^{|\underline{j}|} \sum_{\underline{m}_1+\ldots+\underline{m}_p=\underline{j}} \int_{0<t_1<t_2\ldots<t_p<1} \prod_{i=1}^{p} e^{t_i \,\mathrm{ad}\, f(x)} (\underline{m}_i!^{-1} f^{(\underline{m}_i)}(x))\, dt_1 \ldots dt_p \,.$$

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