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A Possible Constructive Approach to ϕ_4^4

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Abstract. We propose a constructive approach to ϕ_4^4. It is based on formulating the ϕ_4^4 theory as an implicit function problem using multiplicative renormalization. For the corresponding lattice formulation we reduce the problem to verifying three conjectures. One conjecture is a regularity condition. The remaining two concern properties of the classical Ising ferromagnet, one of which we discuss in the frame work of critical point analysis.

I. The Approach (Formal Considerations)

In recent years constructive field theory has made tremendous progress by using euclidean methods (see e.g. [12, 13, 22], and the literature quoted there). However, so far only superrenormalizeable theories have been successfully treated, since the techniques involved mostly rely on additive renormalization. In this article we propose the use of multiplicative renormalization. We have the philosophy respectively the rigourous result in mind that in perturbation theory additive renormalization, multiplicative renormalization and the BPHZ formulation are equivalent (see e.g. [14, 23]). Now in the ϕ_4^4 theory there are three renormalization constants entering the multiplicative renormalization procedure

(i) the mass counterterm δm^2;

(ii) the amplitude renormalization constant $Z_3 \geqq 0$;

(iii) the vertex function renormalization constant $Z_4 \geqq 0$.

On the other hand, there are three normalization conditions for the theory. Two involve the point function and one the four point function. Our central idea is simply to try to determine the renormalization constants for given normalization constants. Now usually the relativistic two point function is normalized by requiring a pole with residue 1 at (relativistic) $p^2 = m^2 > 0$. Since we are interested in formulating and solving the theory in the euclidean framework, we will instead work with the intermediate renormalization [2] or more precisely a generalization of it. There the two point function is normalized at $p^2 = 0$. We note that in perturbation theory, it is irrelevant, where the normalization is done (see e.g. [14]).

Formally our theory will thus be given as follows. Let ϕ be the euclidean field, i.e. for each $f \in \mathscr{S}'(\mathbb{R}^4)$, $\phi(f)$ is the linear function

$$\phi(f): f' \mapsto \langle f', f \rangle$$

on $\mathscr{S}'(\mathbb{R}^4)$. Here $\langle\,,\rangle$ denotes the canonical pairing on $\mathscr{S}'(\mathbb{R}^4) \times \mathscr{S}(\mathbb{R}^4)$. We write $\phi(f) = \int \phi(x) f(x) dx$. The ϕ_4^4 theory is then given in terms of its euclidean Green's functions which are the moments of a euclidean invariant measure μ on $\mathscr{S}'(\mathbb{R}^4)$. $d\mu$ is given by normalizing

$$d\mu' = e^{-\lambda Z_4 \int \vdots \phi^4 \vdots (x) dx + \frac{\delta m^2}{2} \int \vdots \phi^2 \vdots (x) dx} d\mu^0_{(Z_3, m)} \,. \tag{1.1}$$

Here $d\mu^0_{(Z_3,m)}$ is the Gaussian measure on $\mathscr{S}'(\mathbb{R}^4)$ with covariance $(Z_3(-\Delta + m^2))^{-1}$ where Δ is the Laplacian on $\mathbb{R}^4$ and $m^2 > 0$. $\vdots \; \vdots$ denotes normal ordering w.r.t. μ, thus

$$\vdots \phi^2 \vdots (x) = \lim_{x_j \to x} \phi(x_1)\phi(x_2) - \langle \phi(x_1)\phi(x_2) \rangle$$

and

$$\begin{aligned} \vdots \phi^4(x) \vdots = \lim_{x_j \to x} \{ &\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \\ &- \sum_{\substack{i<j,\,k<l \\ (i,j) \neq (k,l)}} \langle \phi(x_i)\phi(x_j) \rangle \vdots \phi(x_k)\phi(x_l) \vdots \\ &- \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle \} \end{aligned}$$

and we have written

$$\langle \cdot \rangle = \int \cdot \, d\mu \,.$$

If we write

$$\tilde{\phi}(p) = (2\pi)^{-2} \int \phi(x) e^{ipx} dx \qquad (p \in \mathbb{R}^4;\ p \cdot x \text{ euclidean scalar product})$$

the normalization conditions are

$$\tilde{\Delta}(p)^{-1} = (2\pi)^{-2} \langle \phi(0) \tilde{\phi}(p) \rangle^{-1} \approx p^2 + m^2 \quad \text{(Intermediate renormalization)} \tag{1.2}$$

$(p^2$ small)

and

$$-\frac{(2\pi)^6}{4!} \langle \phi(0); \tilde{\phi}(0); \tilde{\phi}(0); \tilde{\phi}(0) \rangle \tilde{\Delta}(0)^{-4} = \lambda \tag{1.3}$$

$(\lambda =$ renormalized coupling constant$)$.

Here $\langle A_1; A_2; \ldots; A_n \rangle$ denotes the n-fold truncated expectations (w.r.t. μ) of the random variables $A_1, \ldots, A_n$. We note that m^2 in (1.1) is then not necessarily the physical mass. Also Z_3 is then not necessarily smaller than 1. Now define

$$\begin{aligned} y_1 &= \tilde{\Delta}(0) = \int \langle \phi(0)\phi(x) \rangle dx \\ y_2 &= -\left(\frac{\partial}{\partial p^2} \tilde{\Delta} \right)(0) = \int x^2 \langle \phi(0)\phi(x) \rangle dx \\ y_3 &= -(2\pi)^6 \langle \phi(0); \tilde{\phi}(0); \tilde{\phi}(0); \tilde{\phi}(0) \rangle \\ &= -\langle \phi(0); \int \phi(x) dx; \int \phi(x) dx; \int \phi(x) dx \rangle \,. \end{aligned} \tag{1.4}$$

Our aim is to construct a theory for certain prescribed $y=(y_1, y_2, y_3)$ ($y_i>0$) in particular for $y_3=4!\lambda\tilde{\Delta}(0)^4$. To understand the ansatz of the next chapter, the following remarks are useful.

1) When y varies, $(Z_3, Z_4, \delta m^2)$ also vary. Hence we may equivalently consider the measures μ parametrized by $(\lambda_0, Z_3, \varepsilon)$ $(\lambda_0, Z_3>0, \varepsilon$ real) with

$$d\mu=(\text{Normalization})^{-1}\, e^{-\lambda_0\int\phi(x)^4dx+\varepsilon\int\phi(x)^2dx}\, d\mu^0_{(Z_3,m)} \tag{1.5}$$

i.e. we may drop the normal ordering. Thus μ runs through a set which looks like $(\mathbb{R}^+)^2\times\mathbb{R}$.

2) y_1, y_2, and y_3 have the dimensions $(\mathrm{cm})^2$, $(\mathrm{cm})^4$, and $(\mathrm{cm})^{4+d}$ respectively m and λ have the dimensions $(\mathrm{cm})^{-1}$ and $(\mathrm{cm})^{d-4}$ respectively. Here d denotes the euclidean space dimensions (in our case $d=4$).

3) $y_3>0$ guarantees that μ is not Gaussian, i.e. the theory is non-trivial (for a lattice proof see [17]).

4) The above set of measures satisfies the Griffiths and Lebowitz inequalities (and many more) (see e.g. [1, 22]).

In particular

$$\langle\phi(x)\phi(x')\rangle\geqq 0 \quad \text{for all} \quad x, x'$$

and thus due to translation invariance

$$\begin{aligned}|\langle\phi(f)\phi(g)\rangle| &= \left|\int\langle\phi(x)\phi(x')\rangle f(x)g(x')dxdx'\right| \leqq y_1 \sup_x|f(x)|\int|g(x')|dx' \\ &\leqq y_1\|f\|\,\|g\|\end{aligned} \tag{1.6}$$

with

$$\|f\|=\max\left(\sup_x|f(x)|, \int|f(x)|dx\right)$$

being an $\mathscr{S}(\mathbb{R}^4)$-norm. Hence the two-point function is a tempered distribution and by an extension of the Lebowitz inequalities due to Glimm and Jaffe [8], the higher moments of μ satisfy axiom (E0′) of [18].

5) $(Z_3, Z_4, \delta m^2)$ may be expressed in terms of appropriate moments of μ (see e.g. [20], and the authors contribution in [12]).

Now these considerations are highly formal, so what we intend to do is to look at the corresponding formulation in a lattice theory. We will take a lattice on a torus in d dimensions. This guarantees translation invariance. Thus for fixed, y in a certain set $\mathscr{P}$, the problem will be to solve the relations corresponding to (1.4) on the lattice for all sufficiently small lattice spacings a and all sufficiently large tori.

Due to estimate (1.6) and the remark following it, this will for each n give a uniformly bounded family of euclidean Green's functions of order n in $\mathscr{S}'(\mathbb{R}^{dn})$. Considering a convergent subsequence we obtain (as in [8]) a limiting family of distributions satisfying (E0′) [18] and which by Minlos' theorem (see e.g. [11]) are the moments of a unique measure μ on $\mathscr{S}'(\mathbb{R}^d)$. It remains to verify the other euclidean axioms and to prove the nontriviality of the theory thus obtained. The last property would follow, if relations (1.4) on the tori would in the limit lead to relations (1.4) for the limiting theory. This has been shown in a second paper [21].

We have at the moment no idea how to prove the uniqueness of the solutic⁻ of (1.4) for the lattice case. Such a result would be very welcome, since it wou, show that the normalization uniquely fixes the theory and hence coincides wit the theory discussed in perturbation theory (as e.g. in [24]). Finally we note tha we expect only two physically relevent parameters, the mass and the couplin constant. This, however, is consistent with the above picture: We say that two measures are physically equivalent, if there is a constant $\varrho > 0$ such that the corresponding euclidean Green's functions of order n differ by the factor ϱ^n. This is a renormalization group relation of the simplest form.

A slightly different approach was suggested by the author at the 1975 Marseille conference on Mathematical Methods in Quantum Field Theory [12]. We suggested to solve ϕ_4^4 by a combination of an implicit function theorem and a fixed point problem. However, we consider the present approach more amenable.

Implicit function arguments have also been emplyoed by Baker in the similar context of determining δm^2 for given physical mass ([1], see also [19]).

We note that our present approach is suited for the single phase region. However, this discussion may also be extended to cover the (expected) two-phase region. Without going into details, we outline the idea:

Add a term $h \int \phi(x) dx$ to the exponent entering μ and let y_4 be given as the magnetization $\langle \phi(x) \rangle$. Also y_1 and y_2 are now defined using the truncated two-point function. Then the problem is to construct a theory for given y_i $(i=1 \ldots 4)$ $(y_i > 0,\ i=1,2;\ y_3 \geqq 0)$. Spontaneous magnetization and hence the existence of (at least) two phases would manifest itself in the fact that for certain given y_i $(i=1 \ldots 3)$, y_4 cannot take values in an interval symmetric around $y_4=0$, except for $y_4=0$.

II. The Lattice Theory

In this chapter we go the first steps in solving ϕ_4^4 on the lattice. We assume the reader to be familiar with the euclidean formulation of ϕ_4^4 on the lattice (see e.g. [13, 22, 23]).

Let $\mathscr{T}$ be a unit lattice on a torus in d dimensions, i.e.

$$\mathscr{T} = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_d}$$

where $\mathbb{Z}_n$ denotes the set of integers modulo n. (For technical reasons we will assume all n_e to be odd.) $|\mathscr{T}| = \prod_{e=1}^{d} n_e$ is the number of points on $\mathscr{T}$. These points we denote by $i, j \ldots$ and we also will call them modes adhering to the physical picture.

For $i=(i_1 \ldots i_d)$; $j=(j_1 \ldots j_d) \in \mathscr{T}$ we let $(i-j)^2 = \sum_{e=1}^{d} (i_e - j_e)^2$ be the translation invariant distance square on $\mathscr{T}$, with $(i_e - j_e)^2 = \mathrm{Min}(|i_e - j_e|^2, (|i_e - j_e| - n_e)^2)$ $(0 \leqq i_e, j_e \leqq n_e - 1)$. Two points i and j on $\mathscr{T}$ are called nearest neighbors (N.N.), if $(i-j)^2 = 1$. In d dimensions, obviously each point has $2d$ next neighbors whenever $n_e > 2$ for all e.

Now for each $\mathcal{T}$ and each $\alpha=(\alpha_1,\alpha_2,\alpha_3)\in \mathbb{R}^+\times\overline{\mathbb{R}^+}\times\mathbb{R}$ we define a probability measure μ on $\mathbb{R}^{|\mathcal{T}|}$

$$d\mu(\{x_j\}_{j\in\mathcal{T}})=\frac{1}{N}\prod_{i,j\,\mathrm{N.N.}} e^{\alpha_2 x_i x_j}\prod_{i\in\mathcal{T}} e^{-\alpha_1 x_i^4+\alpha_3 x_i^2}dx_i\,. \tag{2.1}$$

Here and in what follows N will always denote a normalizing factor which makes the measure in question a probability measure. The measure (2.1) is a discrete version of [1.5] (see e.g. [13, 22]).

Again $\langle\ \rangle$ will denote expectations w.r.t. μ. Now to each $(\mathcal{T},a)$ ($a>0$, the lattice spacing) we define a C^∞-map $T=T(\mathcal{T},a)$ from $\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}$ into $(\mathbb{R}^+)^3$ by $\alpha\to y=T(\alpha)$

$$\begin{aligned}
y_1&=y_1(\alpha)=a^2\frac{1}{|\mathcal{T}|}\Big\langle\sum_{i,j\in\mathcal{T}} x_i x_j\Big\rangle\\
y_2&=y_2(\alpha)=a^4\frac{1}{|\mathcal{T}|}\Big\langle\sum_{i,j\in\mathcal{T}} (i-j)^2 x_i x_j\Big\rangle\\
y_3&=y_3(\alpha)=-a^{4+d}\frac{1}{|\mathcal{T}|}\Big\langle\sum_{i\in\mathcal{T}} x_i;\sum_{j\in\mathcal{T}} x_j;\sum_{k\in\mathcal{T}} x_k;\sum_{l\in\mathcal{T}} x_l\Big\rangle
\end{aligned} \tag{2.2}$$

The relations $y_1\geqq 0$; $y_2\geqq 0$ are a consequence of Griffiths first inequality; $y_3\geqq 0$ follows from the Lebowitz inequality (see e.g. [22]).

The following quantities will play a rôle in our discussion

$$\begin{aligned}
D(\mathcal{T},a)&=\frac{a^2}{|\mathcal{T}|^2}\sum_{i,j\in\mathcal{T}}(i-j)^2\\
V(\mathcal{T},a)&=a^d|\mathcal{T}|\,.
\end{aligned} \tag{2.3}$$

$V(\mathcal{T},a)$ is the volume and $D(\mathcal{T},a)$ the mean square distance. If $\mathcal{T}$ goes to infinity in a regular sense (say $n_e=n\to\infty$) then with

$$\frac{V(\mathcal{T},a)}{a_0^d}\to\infty \quad \text{also} \quad \frac{D(\mathcal{T},a)}{a_0^2}\to\infty$$

and (2.4)

$$\frac{V(\mathcal{T},a)}{a_0^{d-2}D(\mathcal{T},a)}\to\infty\ (d\geqq 3)\ \text{uniformly}$$

for all $0<a\leqq a_0$. Here a_0 is arbitrary but fixed and plays the rôle of a unit length.

Our main result is the

Main Theorem. *Suppose the* 3 *conjectures listed below are true.*

Then there is an open manifold $\mathcal{P}$ *in* $(\mathbb{R}^+)^3$ *with the following two properties*

(i) *The set* $\{y|y_1\geqq 0, y_2\geqq 0, y_3=0\}$ *is contained in the boundary* $\partial\mathcal{P}$ *of* $\mathcal{P}$.

(ii) *Any* $y^0\in\mathcal{P}$ *is in the image of* $T(\mathcal{T},a)$ *for all large* $V(\mathcal{T},a)$.

To start the proof, we analyze the image of the boundary of $\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}$ under $T=T(\mathcal{T},a)$ as well as the image at infinity. We set

$$\mathcal{M}_1=\{y=T(\alpha)|\alpha_2=0;\alpha_1>0,\alpha_3\ \text{real}\}\,.$$

This situation corresponds to a theory of uncoupled modes.

$$\mathcal{M}_2 = \{y = T(\alpha) | \alpha_1 = 0;\ -\alpha_3 > d\alpha_2 \geqq 0\}.$$

This situation corresponds to the so called Gaussian measures of ferromagnetic type [13]. That $\mathcal{M}_2$ is well defined will follow from the discussion below.

Proposition 2. *$\mathcal{M}_1$ is the set of all $y \in \mathbb{R}^3$ with*

$$y_1 > 0;\ y_2 = 0;\ 0 \leqq y_3 < 2a^d (y_1)^2 \tag{2.5}$$

and T defines a diffeomorphism of

$$\{\alpha | \alpha_2 = 0;\ \alpha_1 > 0, \alpha_3 \text{ real}\}$$

onto $\mathcal{M}_1$.

Proposition 3. *$\mathcal{M}_2$ is the set of all y with*

$$y_1 > 0;\ 0 \leqq y_2 < D(\mathcal{T}, a) y_1;\ y_3 = 0 \tag{2.6}$$

and T defines a diffeomorphism of

$$\{\alpha | \alpha_1 = 0,\ -\alpha_3 > d\alpha_2 \geqq 0\} \text{ onto } \mathcal{M}_2\,.$$

We prove Proposition 2 first: For uncoupled modes we have

$$\begin{aligned} y_1 &= y_1(\alpha_1, 0, \alpha_3) = a^2 \langle x^2 \rangle' \\ y_2 &= y_2(\alpha_1, 0, \alpha_3) = 0 \\ y_3 &= y_3(\alpha_1, 0, \alpha_3) = -a^{4+d} \langle x; x; x; x \rangle' \\ &= -a^{4+d}(\langle x^4 \rangle' - 3(\langle x^2 \rangle')^2) \end{aligned} \tag{2.7}$$

where $\langle\ \rangle'$ denotes the expectation w.r.t. the probability measure on $\mathbb{R}$ given by

$$d\varrho(x) = N^{-1} \exp(-\alpha_1 x^4 + \alpha_3 x^2) dx\,.$$

We claim that the map $\mathbb{R}^+ \times \mathbb{R} \to (\overline{\mathbb{R}^+})^2$ given by $(\alpha_1, \alpha_3) \mapsto (y_1, y_3)$ in (2.7) is a diffeomorphism of $\mathbb{R}^+ \times \mathbb{R}$ onto the image of this map. In fact, let

$$\begin{aligned} w_1(\alpha_1, \alpha_3) &= y_1(\alpha_1, 0, \alpha_3) \\ w_3(\alpha_1, \alpha_3) &= a^{4+d} \langle x^4 \rangle' \end{aligned}$$

Then $\frac{\partial w_i}{\partial \alpha_1} \leqq 0$; $\frac{\partial w_i}{\partial \alpha_3} \geqq 0$ by Griffiths second inequality and

$$\begin{aligned} \frac{\partial(w_1, w_3)}{\partial(\alpha_1, \alpha_3)} &= -a^{6+d}(\langle x^2; x^2 \rangle' \langle x^4; x^4 \rangle' - (\langle x^2; x^4 \rangle')^2) \\ &\leqq 0 \end{aligned}$$

by Schwarz inequality. The inequality is even strict, since equality would imply $x^4 + \tau x^2 = \text{const}$ a.e. for some τ which is impossible. Thus the map $(\alpha_1, \alpha_3) \mapsto (w_1, w_3)$ is a diffeomorphism of $\mathbb{R}^+ \times \mathbb{R}$ onto the image due to the following lemma, a proof of which we present in the appendix.

Lemma 4. *Let S be a C^∞-map from $\mathbb{R}^+ \times \mathbb{R}$ into $\mathbb{R}^2 : x=(x_1, x_2) \mapsto y=(y_1, y_2)$ such that*

$$\frac{\partial y_i}{\partial x_j} \geqq 0 \quad \text{for all} \quad i, j.$$

Then S is injective and hence a diffeomorphism of $\mathbb{R}^+ \times \mathbb{R}$ onto the image if and only if S is everywhere locally injective, i.e. of maximal rank 2 *everywhere.*

Next the map $w=(w_1, w_3) \mapsto y=(y_1, y_3)$ defined by

$$y_1 = w_1$$
$$y_3 = -w_3 + 3a^d w_1^2$$

is a diffeomorphism of $\mathbb{R}^2$ onto $\mathbb{R}^2$. This proves that the map $(\alpha_1, \alpha_3) \mapsto (y_1, y_3)$ is a diffeomorphism of $\mathbb{R}^+ \times \mathbb{R}$ onto the image. To determine the image, we note that by Schwarz inequality and $y_3 \geqq 0$ we have

$$a^d(w_1)^2 \leqq w_3 \leqq 3a^d(w_1)^2 \tag{2.8}$$

and Proposition 2 will be proved, if we can show that w_3 runs through the whole range given by (2.8) w_1 stays fixed. Now we have

$$w_1(\alpha_1, \alpha_3) = \alpha_1' \cdot w_1(1, \alpha_3')$$
$$w_3(\alpha_1, \alpha_3) = (\alpha_1')^2 w_3(1, \alpha_3')$$
$$\alpha_1' = \alpha_1^{-\frac{1}{2}}; \; \alpha_3' = \alpha_3 \alpha_1^{-\frac{1}{2}}$$

and thus

$$w_3(\alpha_1, \alpha_2) = w_1^2(\alpha_1, \alpha_3) \frac{w_3(1, \alpha_3')}{(w_1(1, \alpha_3'))^2}. \tag{2.9}$$

By Schwarz inequality

$$\frac{\partial}{\partial \alpha_3'} w_1(1, \alpha_3') > 0$$

and

$$\lim_{\alpha_3' \to -\infty} w_1(1, \alpha_3') = 0; \; \lim_{\alpha_3' \to \infty} w_1(1, \alpha_3') = \infty.$$

Hence $\alpha_3' \mapsto \tilde{w}_1(\alpha_3') = w_1(1, \alpha_3')$ is a diffeomorphism of $\mathbb{R}$ onto $\mathbb{R}^+$ and hence we may take α_1' and $\tilde{w}_1$ as new variables. But then we may also take $w_1 = \alpha_1' \tilde{w}_1$ and $\tilde{w}_1$ as new variables. We write $w_3(1, \alpha_3') = \tilde{w}_3(\tilde{w}_1)$. Thus by combining (2.8) and (2.9) we have to show that $\tilde{w}_3(\tilde{w}_1)\tilde{w}_1^{-2}$ runs through the interval $(a^d, 3a^d)$. By going back to (2.9) again it is sufficient to show that $w_3(\alpha_1, \alpha_3) w_1(\alpha_1, \alpha_3)^{-2}$ runs through the interval $(a^d, 3a^d)$ when α_1 and α_3 vary. Now the upper limit is attained when $\alpha_1 = 0$ (since then $y_3 = 0$). Also the lower limit is attained when $t \to +\infty$ with $\alpha_1 = t$, $\alpha_3 = 2t\tau$ ($\tau > 0$ arbitrary). This concludes the proof of Proposition 2.

We turn to a proof of Proposition 3. Let $\mathcal{T}'$ be the lattice dual to $\mathcal{T}$, i.e.

$$\mathcal{T}' = \left\{ q \in \mathbb{R}^d \,\middle|\, q_e = \frac{2\pi}{n_e} p_e; 0 \leqq p_e \leqq n_e - 1 \right\}.$$

We define

$$F(q)=\frac{1}{|\mathcal{T}|^2}\sum_{i,j\in\mathcal{T}}\langle x_i x_j\rangle e^{iq(i-j)}; \qquad q\in\mathcal{T}'$$

such that

$$\langle x_i x_j\rangle=\sum_{q\in\mathcal{T}'}F(q)e^{-iq(i-j)}.$$

Now

$$F(q)=\frac{1}{2|\mathcal{T}|}\left(-\alpha_3-\alpha_2\sum_{e=1}^{d}\cos q_e\right)^{-1}. \tag{2.10}$$

This follows from the fact that

$$-\alpha_3-\alpha_2\sum_{e=1}^{d}\cos q_e; \qquad q\in\mathcal{T}'$$

are the eigenvalues of the matrix

$$A_{i,j}=-\alpha_3\delta_{i,j}-\alpha_2 2^{-1}\delta_{i,j}^{\text{NN.}}; \qquad i,j\in\mathcal{T}$$

with

$$\delta_{i,j}^{\text{N.N.}}=\begin{cases}1 & i \text{ and } j \text{ next neighbours}\\ 0 & \text{otherwise}\end{cases}$$

and standard calculations on Gaussian measures. Inserting (2.10) gives

$$\begin{aligned} y_1&=\frac{a^2}{2(-\alpha_3-d\alpha_2)}=\frac{a^2}{2\alpha'}; \; \alpha'=-\alpha_3-d\alpha_2\\ y_2&=\frac{a^4}{2|\mathcal{T}|^2}\sum_{\substack{i,j\in\mathcal{T}\\ q\in\mathcal{T}'}}\frac{(i-j)^2\cos q(i-j)}{\left(\alpha'+\alpha_2\sum_{e=1}^{d}(1-\cos q_e)\right)}.\end{aligned} \tag{2.11}$$

Now for given y_1 and hence α' we have $y_2(\alpha',\alpha_2=0)=0$ and

$$y_2(\alpha',\alpha_2=\infty)=D(\mathcal{T},a)y_1.$$

Hence for fixed y_1 we see from (2.11) that y_2 covers at least the interval

$$0\leqq y_2<D(\mathcal{T},a)y_1.$$

To see that exactly these values are taken we show that y_2 is monotone in α_2 for fixed α'. Indeed, after some elementary computation using the definition of $(i-j)^2$ we obtain

$$\left.\frac{\partial y_2}{\partial\alpha_2}\right|_{\alpha'\text{ fixed}}=a^4\sum_{e=1}^{d}\sum_{p=1}^{n_e-1}\frac{(-1)^{p+1}\cos\frac{p\pi}{n_e}}{\left[\alpha'+\alpha_2\left(1-\cos\frac{2p\pi}{n_e}\right)\right]^2}. \tag{2.12}$$

(Note that the n_e are assumed to be odd.)

This expression is positive since

$$\frac{\cos\frac{p\pi}{n_e}}{\left[\alpha'+\alpha_2\left(1-\cos\frac{[2p\pi]}{n_e}\right)\right]^2}$$

is monotone decreasing in p in the relevant range. This concludes the proof of Proposition 3, since by the last arguments we have also shown that

$$\left.\frac{\partial(y_1, y_2)}{\partial(\alpha_2, \alpha_3)}\right|_{\alpha_1=0} \neq 0 \text{ everywhere}.$$

We now turn to an analysis of the image at infinity.

For this purpose we introduce two new sets $\mathscr{M}_3(t)$ and $\mathscr{M}_4(t)$ depending on the parameter $t>0$:

$$\mathscr{M}_3(t)=\{y=T(\alpha)|\alpha_2=t\}$$
$$\mathscr{M}_4(t)=\{y=T(\alpha)|\alpha_1=t\}$$

and we are interested in the limiting sets when $t\to+\infty$. To determine these sets we introduce new variables

$$\alpha_1=\frac{\beta_1}{|\mathscr{T}|}$$
$$\alpha_2=t \qquad (\beta_1>0;\ \beta_3 \text{ real}) \tag{2.13}$$
$$\alpha_3=\frac{\beta_3}{|\mathscr{T}|}-dt$$

and

$$\alpha_1=t$$
$$\alpha_2=\gamma_2 \qquad (\gamma_2>0;\ \gamma_3 \text{ real}) \tag{2.14}$$
$$\alpha_3=2t\gamma_3$$

and by abuse of notation we rewrite the measure $\mu=\mu(\alpha)$ in (2.1) as $\mu(t, \beta_1, \beta_3)$ and $\mu(t, \gamma_2, \gamma_3)$ respectively. It is easily seen that for $t\to+\infty$ we obtain limiting measures $\mu_3(\beta_1, \beta_3)$ and $\mu_4(\gamma_2, \gamma_3)(\gamma_3>0)$ respectively such that

$$d\mu_3(\beta_1, \beta_3)(\{x_j\}_{j\in\mathscr{T}})=N^{-1}\prod_{j\neq 0}\delta(x_j-x_0)dx_j e^{-\beta_1 x_0^4+\beta_3 x_0^2}dx_0\,. \tag{2.15}$$

This corresponds to the situation where all modes are coupled infinitely strongly to each other: The system has essentially only one mode of freedom.

Also

$$d\mu_4(\gamma_2, \gamma_3)=N^{-1}\prod_{i,j\text{N N.}} e^{\gamma_2 x_i x_j}\prod_{i\in\mathscr{T}}(\tfrac{1}{2}\delta(x_i-\sqrt{\gamma_3})+\tfrac{1}{2}\delta(x_i+\sqrt{\gamma_3}))dx_i\,. \tag{2.16}$$

This is nothing but an Ising ferromagnet on $\mathscr{T}$ with $\sigma_i=\frac{x_i}{\sqrt{\gamma^3}}=\pm 1$ as spin variables and $J=\gamma_2\gamma_3>0$ as interaction strength. Thus we have the following two lemmas.

Lemma 5. *$\mathcal{M}_3 = \lim_{t\to\infty} \mathcal{M}_3(t)$ consists of the points $y \in \mathbb{R}^3$ of the form*

$$\begin{aligned} y_1 &= \hat{y}_1(\beta_1, \beta_3) = a^2|\mathcal{T}|\langle x^2\rangle'' \\ y_2 &= \hat{y}_2(\beta_1, \beta_3) = a^2|\mathcal{T}|D(\mathcal{T}, a)\langle x^2\rangle'' \\ y_3 &= \hat{y}_3(\beta_1, \beta_3) = -a^4|\mathcal{T}|^2 V(\mathcal{T}, a)\langle x; x; x; x\rangle'' \end{aligned} \tag{2.17}$$

where $\langle\ \rangle''$ denotes the expectation w.r.t. the measure $\kappa = \kappa(\beta_1, \beta_3)$ on $\mathbb{R}$ given by

$$d\kappa(x) = N^{-1}\exp(-\beta_1 x^4 + \beta_3 x^2)dx\,;\ \beta_1 > 0,\ \beta_3 \text{ real}. \tag{2.18}$$

Note that $\frac{\partial \hat{y}_i}{\partial \beta_1} \leqq 0$ and $\frac{\partial \hat{y}_i}{\partial \beta_2} \geqq 0$ $(i=1,2)$ by Griffiths second inequality.

Lemma 6. *The set $\mathcal{M}_4 = \lim_{t\to\infty} \mathcal{M}_4(t)$ consists of all $y \in \mathbb{R}^3$ of the form*

$$\begin{aligned} y_1 &= \tilde{y}_1(\gamma_3, J) = a^2 \frac{\gamma_3}{|\mathcal{T}|}\left\langle \sum_{i\in\mathcal{T}} \sigma_i \sum_{j\in\mathcal{T}} \sigma_j \right\rangle_J \\ y_2 &= \tilde{y}_2(\gamma_3, J) = a^4 \frac{\gamma_3}{|\mathcal{T}|}\left\langle \sum_{i,j\in\mathcal{T}} (i-j)^2 \sigma_i\sigma_j \right\rangle_J \\ y_3 &= \tilde{y}_3(\gamma_3, J) = -a^{4+d} \frac{\gamma_3^2}{|\mathcal{T}|}\left\langle \sum_{i\in\mathcal{T}} \sigma_i; \sum_{j\in\mathcal{T}} \sigma_j; \sum_{k\in\mathcal{T}} \sigma_k; \sum_{l\in\mathcal{T}} \sigma_l \right\rangle_J \end{aligned} \tag{2.19}$$

where $\langle\ \rangle_J$ denotes the expectation w.r.t. the Ising model on $\mathcal{T}$ with coupling strength J.

From the proof of Proposition 3 we have the

Proposition 7. *$\mathcal{M}_3$ consists of all points $y \in \mathbb{R}^3$ of the form*

$$\begin{aligned} &y_1 \geqq 0 \\ &y_2 = D(\mathcal{T}, a)y_1 \\ &0 \leqq y_3 < 2V(\mathcal{T}, a)(y_1)^2\,. \end{aligned} \tag{2.20}$$

We turn to a discussion of $\mathcal{M}_4$.

First we note that $\frac{\partial \tilde{y}_i}{\partial a} > 0$ $(i=1,2,3)$ due to Griffiths first inequality and the Lebowitz inequality. Also $\frac{\partial \tilde{y}_i}{\partial J} \geqq 0$ $(i=1,2)$ due to Griffiths second inequality. Setting $J=0$ in (2.19), we have

$$\begin{aligned} &\tilde{y}_1(\gamma_3, J=0) = a^2\gamma_3 \\ &\tilde{y}_2(\gamma_3, J=0) = 0 \\ &\tilde{y}_3(\gamma_3, J=0) = 2\gamma_3^2 a^{4+d} \end{aligned} \tag{2.21}$$

which are points in $\mathcal{M}_4$. For $J \to \infty$ we obtain the following points in $\bar{\mathcal{M}}_4$:

$$\begin{aligned} &\tilde{y}_1(\gamma_3, J=\infty) = a^2\gamma_3|\mathcal{T}| \\ &\tilde{y}_2(\gamma_3, J=\infty) = a^2\gamma_3|\mathcal{T}|D(\mathcal{T}, a) \\ &\tilde{y}_3(\gamma_3, J=\infty) = 2(a^2\gamma_3|\mathcal{T}|)^2 V(\mathcal{T}, a)\,. \end{aligned} \tag{2.22}$$

Collecting these informations on $\mathscr{M}_i$ $(i=1,2,3,4)$ we obtain

$$\mathscr{M}_1 \cap \mathscr{M}_2 = \{y | y_1 > 0, y_2 = y_3 = 0\} \tag{2.23a}$$

$$\bar{\mathscr{M}}_2 \cap \mathscr{M}_3 = \{y | y_1 \geqq 0; y_2 = D(\mathscr{T}, a) y_1; y_3 = 0\} \tag{2.23b}$$

$$\bar{\mathscr{M}}_3 \cap \bar{\mathscr{M}}_4 \supsetneqq \{y | y_1 \geqq 0; y_2 = D(\mathscr{T}, a) y_1; y_3 = 2V(\mathscr{T}, a) y_1^2\} \tag{2.23c}$$

$$\bar{\mathscr{M}}_1 \cap \mathscr{M}_4 = \{y | y_1 \geqq 0; y_2 = 0; y_3 = 2a^d y_1^2\}. \tag{2.23d}$$

The last relation follows from the fact that $y_1(\gamma_3, J) > 0$ and $y_2(\gamma_3, J) > 0$ for $\gamma_3 > 0$ and $J > 0$ due to Griffiths inequalities and the analyticity in J. We would like $\mathscr{M}_4$, which is connected, to have properties similar to those of $\mathscr{M}_i$ $(i=1,2,3)$. More precisely let $\pi: \mathbb{R}^3 \to \mathbb{R}^3$ be given by $\pi(y_1, y_2, y_3) = (y_1, y_2, 0)$.

Conjecture 1. *π restricted to $\mathscr{M}_4$ defines a diffeomorphism of $\mathscr{M}_4$ onto $\mathscr{M}_2$. Thus $\mathscr{M}_4$ is a manifold (with boundary) of dimension two and relation* (2.23c) *is an equality.*

Conjecture 1 says in particular that the tangent vectors $\frac{\partial}{\partial y_3} \tilde{y}$ and $\frac{\partial}{\partial J} \tilde{y}$ to $\mathscr{M}_4$ are everywhere linearly independent and the C^∞-map defined by (2.19) is a diffeomorphism. With this conjecture the set $\bar{\mathscr{M}}_1 \cup \bar{\mathscr{M}}_2 \cup \bar{\mathscr{M}}_3 \cup \bar{\mathscr{M}}_4$ forms the boundary ∂K of an open (nonempty) set $K = K(\mathscr{T}, a) \subset (\mathbb{R}^+)^3$. Due to the construction of ∂K we expect that K is in the image of T. Or speaking in geometrical terms we expect $\bar{\mathscr{M}}_3 \cup \bar{\mathscr{M}}_4$ to be the complete image of infinity. Indeed, we may prove this under the additional conjecture:

Conjecture 2. *For all $(\mathscr{T}, a)$ the map $T = T(\mathscr{T}, a)$ is everywhere locally injective, i.e. has maximal* rank 3 *everywhere.*

Theorem 8. *If conjectures* 1 *and* 2 *hold, then $K = K(\mathscr{T}, a)$ is in the image of $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$ under $T = T(\mathscr{T}, a)$ for every $T = T(\mathscr{T}, a)$.*

The proof will be given in the next chapter. Actually we expect K to be the entire image of T, but for our purpose the statement of Theorem 8 is sufficient.

Now if $\mathscr{M}_4(\mathscr{T}, a)$ does not move too much towards $\mathscr{M}_2(\mathscr{T}, a)$ when $V(\mathscr{T}, a) \to \infty$, $K(\mathscr{T}, a)$ will stay sufficiently large and we may prove the main theorem. More precisely we make the

Conjecture 3. *There is a two-dimensional C^∞-manifold $\mathscr{M}_5$ (independent of $(\mathscr{T}, a)$) with a boundary consisting of $\{y | y_1 \geqq 0; y_2 = y_3 = 0\}$ and a smooth curve on $\{y | y_1 = 0, y_2 \geqq 0, y_3 \geqq 0\}$, having the following properties*

(i) *$y_i \geqq 0$ $(i=1,2,3)$ for all $y \in \mathscr{M}_5$ and $y_2 > 0$ implies $y_3 > 0$. π defines a diffeomorphism of $\mathscr{M}_5$ onto $\{y | y_1 \geqq 0, y_2 \geqq 0; y_3 = 0\}$.*

(ii) *The relations $y = (y_1, y_2, y_3) \in \mathscr{M}_5$ and $y' = (y_1, y_2, y_3') \in \mathscr{M}_4(\mathscr{T}, a)$ imply $y_3 < y_3'$ for all large $V(\mathscr{T}, a)$.*

Note that Conjecture 3 is a statement about the Ising model. Geometrically speaking it says that

$$\mathscr{M}_5(\mathscr{T}, a) \underset{\text{def}}{=} \mathscr{M}_5 \cap K(\mathscr{T}, a)$$

lies between $\mathscr{M}_2(\mathscr{T}, a)$ and $\mathscr{M}_4(\mathscr{T}, a)$ for all large $V(\mathscr{T}, a)$.

We note that Conjecture 3 is the physically most interesting conjecture. It says that in the thermodynamic limit the Ising model does not fall into the Gaussian theory. Thus it relates ϕ_4^4 and the corresponding Ising model directly. To check Conjecture 3 it would of course be sufficient to know the form of

$$\mathcal{M}_4^\infty = \lim_{V(\mathcal{T},a)\to\infty} \mathcal{M}_4(\mathcal{T},a)$$

(if it exists). Now in the thermodynamic limit, the behaviour of the Ising model near its critical point should determine the shape of $\mathcal{M}_4^\infty$. In the Appendix B we present such an analysis of $\mathcal{M}_4^\infty$ based on standard assumptions for the correlation functions near the critical point (as given e.g. in [7]). This discussion relates our approach to and supports recent efforts by Glimm and Jaffe on ϕ^4 [10], see also the discussion in [15].

In particular we obtain the result that for Conjecture 3 to be valid, the Buckingham-Gunton inequality ([4, 6]) has actually to be an equality.

This again would be a consequence of the scaling hypothesis (see e.g. [5] for a discussion of this point in the Ising model).

With our result and conjectures at hand, we may now prove the main theorem. We define $\mathcal{P}$ to be the open set in $(\mathbb{R}^+)^3$ having $\mathcal{M}_5$, $\{y|y_1 \geqq 0, y_2 \geqq 0, y_3 = 0\}$ and the appropriate part of $\partial(\mathbb{R}^+)^3$ as boundary. Let also $K'(\mathcal{T},a)$ be the set having $\mathcal{M}_2(\mathcal{T},a) \cup \mathcal{M}_5(\mathcal{T},a)$ and the appropriate part of $\mathcal{M}_3(\mathcal{T},a)$ as boundary. By Conjecture 3

$$K'(\mathcal{T},a) \subset K(\mathcal{T},a).$$

Now any $y^0 \in \mathcal{P}$ is in $K'(\mathcal{T},a)$ for all sufficiently large $V(\mathcal{T},a)$. This follows from Conjectures 2 and 3 and the established properties of $\mathcal{M}_2(\mathcal{T},a)$. But then y^0 is also in $K(\mathcal{T},a)$ so the main theorem is now an immediate consequence of Theorem 8.

We conclude this section with a remark. We expect the map $T = T(\mathcal{T},a)$ always to be one-to-one. For this to be true, it is of course necessary that Conjecture 2 be valid. Conversely, it would have been convenient to have the analogue of Lemma 4 for higher dimensions. This would have enabled us to deduce injectivity from local injectivity, which is Conjecture 2 (see the proof of Proposition 2). Unfortunately, however, such an analogue is not valid in higher dimensions [3].

III. Proof of Theorem 8

For given $y^0 \in K$ we will construct a 2-dimensional C^0-manifold $S \subset \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$ without boundary such that $T(S)$ "encloses" y^0. y^0 will then be in the image of T using arguments from singular homology theory, in particular a three dimensional version of the winding number. (The author would like to thank R. Bott for a discussion on this point.) The strategy for constructing S will be as follows: Let

$$\gamma' = \operatorname{dist}(y^0, \partial K) > 0 \qquad \gamma = \tfrac{1}{4}\min(\gamma', y_1^0). \tag{3.1}$$

S will then be the union of C^∞-manifolds (with boundaries) $\mathcal{N}_i (i=1 \dots 6)$ such that

$$(1)\quad T(\mathcal{N}_i) \subset \mathcal{M}_i \ (i=1,2) \tag{3.2}$$

$$(2)\quad \operatorname{dist}(T(\mathcal{N}_i), \mathcal{M}_i) \leqq \gamma \ (i=3,4) \tag{3.3}$$

$$(3)\quad y_1 < y_1^0 \quad \text{for all} \quad y \in T(\mathcal{N}_5) \tag{3.4}$$

$$y_1^0 < y_1 \quad \text{for all } y \in T(\mathcal{N}_6)\,. \tag{3.5}$$

$\mathcal{N}_4$ and $T(\mathcal{N}_4)$ will have "corners" $\alpha^{(k)}$ and $y^{(k)} = T(\alpha^{(k)})$ respectively $(k=1 \dots 4)$. $\mathcal{N}_2$ and $T(\mathcal{N}_2)$ will have "corners" $\alpha^{(k)}$ and $y^{(k)} = T(\alpha^{(k)})$ respectively $(k=5 \dots 8)$. Also

$$y^{(2i-1)} \in T(\mathcal{N}_5);\ y^{(2i)} \in T(\mathcal{N}_6) \qquad (i=1 \dots 4)\,.$$

Furthermore for the curves $I_{(j,k)} = \mathcal{N}_j \cap \mathcal{N}_k$ we will have:

The boundary $\partial \mathcal{N}_k$ of $\mathcal{N}_k$ is $\bigcup_j I_{(j,k)}$ (Hence S will have no boundary). With the exception of $I_{(1,6)}$, $I_{(2,6)}$, and $I_{(3,6)}$, each $I_{(j,k)}$ is either empty or a closed interval of a straight line in $\mathbb{R}^3$.

From these properties of course the desired structures of S and $T(S)$ follows. We note (without mentioning it further) that Conjecture 2 will be used to ensure that the $\mathcal{N}_i$ are twodimensional manifolds.

We start by constructing $\mathcal{N}_4$, $I_{(4,l)}$ $(l=1, 3, 5, 6)$ and $y^{(k)}$ $(k=1 \dots 4)$. For this we need some preparation.

Let first

$$\begin{aligned} &\alpha(s,t,u) = (\alpha_1 = t, \alpha_2 = s, \alpha_3 = 2ta(u) - ds) \\ &\qquad a(u) = (1-u)a_1 + ua_2 \\ &y(s,t,u) = T(\alpha(s,t,u)) \\ &(0 \leqq t < \infty; 0 \leqq s < \infty; 0 \leqq u \leqq 1) \end{aligned} \tag{3.6}$$

where $a_1 > 0$ and $a_2 > 0$ will be fixed in a moment. Rewriting the probability measure (2.1) in terms of (s,t,u) we have

$$d\mu(\{x_j\}_{j\in\mathcal{T}}) = N^{-1} e^{-\frac{s}{2} \sum\limits_{i,j\text{N N.}} (x_i - x_j)^2 - t \sum\limits_{i\in\mathcal{T}} (x_i^2 - a(u))^2} dx_i\,. \tag{3.7}$$

Next write

$$\hat{y}'(t,u) = \hat{y}(\beta_1 = t|\mathcal{T}|, \beta_2 = 2t|\mathcal{T}|a(u)) \tag{3.8}$$

[see (2.17)]. Then $\hat{y}$ is obtained from the measure [see (2.17) and 2.18)]

$$\begin{aligned} d\kappa(x) &= N^{-1} \exp(-t|\mathcal{T}|x^4 + 2|\mathcal{T}|ta(u)x^2)dx \\ &= N^{-1} \exp(-t|\mathcal{T}|(x^2 - a(u))^2)dx\,. \end{aligned} \tag{3.9}$$

Also let [see (2.19)]

$$\tilde{y}'(s,u) = \tilde{y}(\gamma_3 = a(u), J = a(u)s)\,. \tag{3.10}$$

Comparing (3.7)–(3.10) the following lemma follows easily

Lemma 9. *With the notation as above and fixed* $a_1, a_2 > 0$

$$\lim_{t\to\infty} y(s,t,u) = \tilde{y}'(s,u) \tag{3.11}$$

$$\lim_{s\to\infty} y(s,t,u) = \hat{y}'(t,u) \tag{3.12}$$

uniformly in $0 \leqq s < \infty$, $0 \leqq u \leqq 1$, *and* $0 \leqq t < \infty$, $0 \leqq u \leqq 1$ *respectively.*

Now we fix a_1 and a_2 by [see (3.1)]

$$a_1 = \frac{\gamma}{a^2 |\mathscr{T}|}; \qquad a_2 = \frac{2y_1^0}{a^2}. \tag{3.13}$$

Then we have

$$\begin{aligned} &\tilde{y}_1'(s,0) \leqq \gamma \leqq \frac{y_1^0}{4} \\ &\tilde{y}_1'(s,1) \geqq 2y_1^0 \end{aligned} \tag{3.14}$$

for all $0 \leqq s < \infty$ by Griffiths' second inequality [see (2.21) and (2.22)]. Let now s_0, t_0 be so large that

$$|y(s,t,u) - \tilde{y}'(s,u)| < \frac{\gamma}{2}; \; t \geqq t_0; \; 0 \leqq s < \infty$$
$$0 \leqq u \leqq 1 \tag{3.15a}$$

$$|y(s,t,u) - \hat{y}'(t,u)| < \frac{\gamma}{2}; \; s \geqq s_0; \; 0 \leqq t < \infty$$
$$0 \leqq u \leqq 1. \tag{3.15b}$$

Now we define

$$\begin{aligned} \mathscr{N}_4 &= \{\alpha(s,t_0,u) | 0 \leqq s \leqq s_0; 0 \leqq u \leqq 1\} \\ T(\mathscr{N}_4) &= \{y(s,t_0,u) | 0 \leqq s \leqq s_0; 0 \leqq u \leqq 1\} \\ \alpha^{(1)} &= \alpha(s_0,t_0,0),\; y^{(1)} = y(s_0,t_0,0) \\ \alpha^{(2)} &= \alpha(s_0,t_0,1);\; y^{(2)} = y(s_0,t_0,1) \\ \alpha^{(3)} &= \alpha(0,t_0,0);\; y^{(3)} = y(0,t_0,0) \\ \alpha^{(4)} &= \alpha(0,t_0,1);\; y^{(4)} = y(0,t_0,1) \\ I_{(1,4)} &= \{\alpha(0,t_0,u) | 0 \leqq u \leqq 1\} \\ I_{(3,4)} &= \{\alpha(s_0,t_0,u) | 0 \leqq u \leqq 1\} \\ I_{(4,5)} &= \{\alpha(s,t_0,0) | 0 \leqq s \leqq s_0\} \\ I_{(4,6)} &= \{\alpha(s,t_0,1) | 0 \leqq s \leqq s_0\}. \end{aligned} \tag{3.16}$$

Due to (3.15a), relation (3.3) is valid for $i = 4$, and from (3.14) we have

$$y_1 < \tfrac{3}{8} y_1^0 (< y_1^0); \; y \in I_{(4,5)} \tag{3.17}$$

$$y_1 > \tfrac{15}{8} y_1^0 (> y_1^0); \; y \in I_{(4,6)}. \tag{3.18}$$

To construct $\mathcal{N}_3$, consider the points $\hat{y}'(t_0, 0)$ and $\hat{y}'(t_0, 1)$ in $\mathcal{M}_3$. By (3.15b) we have

$$|\hat{y}'(t_0, 0) - y(s_0, t_0, 0)| < \frac{\gamma}{2}$$

$$|\hat{y}'(t_0, 1) - y(s_0, t_0, 1)| < \frac{\gamma}{2}.$$

Combining this with (3.17) and (3.18) we have

$$\begin{aligned} &\hat{y}_1(t_0|\mathcal{T}|, 2t_0|\mathcal{T}|a_1) < \tfrac{1}{2} y_1^0 \\ &\hat{y}_1(t_0|\mathcal{T}|, 2t_0|\mathcal{T}|a_2) > \tfrac{7}{4} y_1^0 . \end{aligned} \tag{3.19}$$

Thus, by continuity, there is $t_1 < t_0$ such that

$$\hat{y}_1(t|\mathcal{T}|, 2t_0|\mathcal{T}|a_1) < \tfrac{1}{2} y_1^0 \tag{3.20}$$

$$\hat{y}_1(t|\mathcal{T}|, 2t_0|\mathcal{T}|a_2) > \tfrac{7}{4} y_1^0 \tag{3.21}$$

for all $t_1 \leqq t \leqq t_0$.

Now choose $a_3 < 0$ such that

$$\hat{y}_1(t|\mathcal{T}|, 2t_0|\mathcal{T}|a_1 + a_3) \leqq \tfrac{1}{2} y_1^0 \tag{3.22}$$

for $0 \leqq t \leqq t_1$ and set $a_4 = (t_0 - t_1)^{-1} a_3$. Then by Griffiths second inequality the combination of (3.20) and (3.22) gives

$$\hat{y}_1(t|\mathcal{T}|, 2t_0|\mathcal{T}|a_1 + (t_0 - t)a_4) \leqq \tfrac{1}{2} y_1^0 \quad \text{for all} \quad 0 \leqq t \leqq t_0 . \tag{3.23}$$

Next with the help of the proof of Proposition 2 it is easy to construct a C^∞-curve $t \mapsto \beta_3(t)$ $(0 \leqq t \leqq t_0)$ with the properties

$$\beta_3(t_0) = 2t_0|\mathcal{T}|a_2$$

$$\hat{y}_1(\beta_1 = t|\mathcal{T}|, \beta_3 = \beta_3(t)) > \tfrac{3}{2} y_1^0$$

$$(0 \leqq t \leqq t_0) .$$

Now set

$$\begin{aligned} \beta_3(t, u) = (1 - u)&[2t_0|\mathcal{T}|a_1 + (t_0 - t)a_4] \\ &+ u\beta_3(t) \end{aligned} \tag{3.25}$$

and let

$$\hat{y}''(t, u) = \hat{y}(\beta_1 = t|\mathcal{T}|, \beta_3 = \beta_3(t, u)) .$$

Also define

$$\begin{aligned} &\alpha''(s, t, u) = \left(\alpha_1 = t, \alpha_2 = s, \alpha_3 = \frac{\beta_3(t, u)}{|\mathcal{T}|} - ds\right) \\ &y''(s, t, u) = T(\alpha''(s, t, u)) \end{aligned} \tag{3.26}$$

and let s_1 be so large that

$$|y''(s,t,u)-\hat{y}''(t,u)|<\frac{\gamma}{2} \quad \text{for all} \quad s\geqq s_1,\ 0\leqq t\leqq t_0,\ 0\leqq u\leqq 1\,. \tag{3.27}$$

Now let $s_2=\operatorname{Max}(s_0,s_1)$. Due to (3.15b) we have

$$|y(s_0,t_0,u)-\hat{y}'(t_0,u)|\leqq\frac{\gamma}{2} \tag{3.28}$$

for all $s\geqq s_0$, $0\leqq u\leqq 1$. By construction

$$y(s_0,t_0,u)=y''(s_0,t_0,u)\,.$$

Hence by continuity and compactness arguments there is $t_2<t_0$ such that

$$|y''(s,t,u)-\hat{y}'(t,u)|\leqq\tfrac{3}{4}\gamma \tag{3.29}$$

for all $s_0\leqq s\leqq s_2$, all $t_2\leqq t\leqq t_0$ and all $0\leqq u\leqq 1$.

Now define

$$\alpha''(t,u)=\alpha''\left(s=s_0+\frac{t_0-t}{t_0-t_2}(s_2-s_0),t,u\right)$$
$$y''(t,u)=T(\alpha''(t,u))\,.$$

We set

$$\begin{aligned}\mathcal{N}_3&=\{\alpha''(t,u)|0\leqq t\leqq t_0;0\leqq u\leqq 1\}\\ T(\mathcal{N}_3)&=\{y''(t,u)|0\leqq t\leqq t_0;0\leqq u\leqq 1\}\\ \alpha^{(5)}&=\alpha''(0,0);\,y^{(5)}=y''(0,0)\\ \alpha^{(6)}&=\alpha''(0,1);\,y^{(6)}=y''(0,1)\\ I_{(2,3)}&=\{\alpha=\alpha''(0,u)|0\leqq u\leqq 1\}\\ I_{(3,5)}&=\{\alpha=\alpha''(t,0)|0\leqq t\leqq t_0\}\\ I_{(3,6)}&=\{\alpha=\alpha''(t,1)|0\leqq t\leqq t_0\}\,.\end{aligned} \tag{3.30}$$

Also $I_{(3,4)}$ may be rewritten as $I_{(3,4)}=\{\alpha=\alpha''(t_0,u)|0\leqq u\leqq 1\}$. By (3.27–(3.29) we have dist $(T(\mathcal{N}_3),\mathcal{M}_3)\leqq\frac{3}{4}\gamma$ proving (3.3) for $i=3$. Combining (3.23), (3.24) with (3.28) and (3.29) gives

$$\begin{aligned}&y_1<\tfrac{3}{4}y_1^0(<y_1^0),\ y\in I_{(3,5)}\\ &y_1>\tfrac{5}{4}y_1^0(>y_1^0),\ y\in I_{(3,6)}\,.\end{aligned} \tag{3.31}$$

We turn to a construction of $\mathcal{N}_5$, $\alpha^{(7)}$, $y^{(7)}$, $I_{(1,5)}$ and $I_{(2,5)}$. First we define $I_{(1,5)}$ by

$$I_{(1,5)}=\left\{\alpha|\alpha=\alpha'''(t)=\frac{t_0-t}{t_0}(0,0,a_5)+\frac{t}{t_0}\alpha^{(3)},0\leqq t\leqq t_0\right\}$$

where $a_5<0$ will be fixed in a moment. Let $\mathcal{N}_5$ be the manifold connecting $I_{(1,5)}$ and $I_{(3,5)}$, i.e. define

$$\mathcal{N}_5=\{\alpha|\alpha=\check{\alpha}(t,v)=(1-v)\alpha'''(t)+v\alpha''(t,0)\}$$
$$0\leqq t\leqq t_0,\ 0\leqq v\leqq 1\,.$$

Setting $\check{y}(t,v)=T(\check{\alpha}(t,v))$, we have $T(\mathcal{N}_5)=\{\check{y}(t,v)|0\leqq t\leqq t_0;0\leqq v\leqq 1\}$ and we let

$$I_{(2,5)}=\{\alpha=\check{\alpha}(0,v)\,0\leqq v\leqq 1\}$$
$$\alpha^{(7)}=\check{\alpha}(0,0);\,y^{(7)}=T(\alpha^{(7)})=\check{y}(0,0)$$
$$=(0,0,a_5)$$

and we may rewrite $I_{(1,5)}$, $I_{(3,5)}$, and $I_{(4,5)}$ as

$$I_{(1,5)}=\{\alpha=\check{\alpha}(t,0)|0\leqq t\leqq t_0\}$$
$$I_{(3,5)}=\{\alpha=\check{\alpha}(t,1)|0\leqq t\leqq t_0\}$$
$$I_{(4,5)}=\{\alpha=\check{\alpha}(t_0,v)|0\leqq v\leqq 1\}\,.$$

By Griffiths second inequality and by analyticity, $\check{y}_1(t,v)$ is strictly increasing function of a_5 for any fixed $t<t_0$ and $v<1$. Also by (3.17) and (3.31) there is $t_3<t_0$ and $v_1<1$ such that

$$\check{y}_1(t,v)<\tfrac{4}{5}y_1^0 \quad \text{for } t_3\leqq t\leqq t_0 \text{ and all } 0\leqq v\leqq 1$$
$$\check{y}_1(t,v)<\tfrac{4}{5}y_1^0 \quad \text{for } v_1\leqq v\leqq 1 \text{ and all } 0\leqq t\leqq t_0$$

if we choose $a_5=-1$ say. Hence by standard compactness arguments, there is $a_5\leqq -1$ sufficiently negative such that

$$\check{y}_1(t,v)<\tfrac{4}{5}y_1^0(<y_1^0) \quad \text{for all} \quad 0\leqq t\leqq 1, \quad 0\leqq v\leqq 1\,. \tag{3.34}$$

With this choice of a_5 we have therefore proved (3.4).

Next let $t\mapsto\bar{\alpha}_3(t)$ $(0\leqq t\leqq t_0)$ be a C^∞-curve such that
(i) $\bar{\alpha}_3(0)<0$; $\bar{\alpha}_3(t_0)=2t_0a_2$,
(ii) $\bar{y}(t)=T(\bar{\alpha}(t))$,
where $\bar{\alpha}(t)=(t,0,\bar{\alpha}_3(t)$, satisfies

$$\bar{y}_1(t)>\tfrac{15}{8}y_1^0 \quad \text{for all} \quad 0\leqq t\leqq t_0 \tag{3.35}$$

[compare (3.18)] and let $s\mapsto\bar{\bar{\alpha}}_3(s)$ $(0\leqq s\leqq\alpha_2^{(6)})$ be a C^∞-curve such that
(i) $\bar{\bar{\alpha}}_3(0)=\bar{\alpha}_3(0)$, $\bar{\bar{\alpha}}_3(\alpha_2^{(6)})=\bar{\alpha}_3^{(6)}$,
$-\bar{\bar{\alpha}}_3(s)>ds$ $(0\leqq s\leqq\alpha_2^{(6)})$,
(ii) $\bar{\bar{y}}(s)=T(\bar{\bar{\alpha}}(s))$,
with $\bar{\bar{\alpha}}(s)=(0,s,\bar{\bar{\alpha}}_3(s))$, satisfies

$$\bar{\bar{y}}_1(s)>\tfrac{5}{4}y_1^0 \quad \text{for all} \quad 0\leqq s\leqq\alpha_2^{(6)} \tag{3.36}$$

[compare (3.30)–(3.31)].

Using the proofs of Proposition 2 and 3 it is easily checked that such $\bar{\alpha}_3(\cdot)$ and $\bar{\bar{\alpha}}_3(\cdot)$ may indeed be found.

We set

$$\alpha^{(8)}=\bar{\bar{\alpha}}(0);\,y^{(8)}=\bar{\bar{y}}(0)$$
$$I_{(1,2)}=(\alpha=(0,0,\alpha_3)|a_5\leqq\alpha_3\leqq\bar{\alpha}_3(0)\}\,.$$

Note that $a_5 < \bar{\alpha}_3(0)$ due to (3.34), (3.35) and Griffiths inequality.

$$I_{(1,6)} = \{\alpha = \bar{\alpha}(t) | 0 \leqq t \leqq t_0\}$$

$$I_{(2,6)} = \{\alpha = \bar{\bar{\alpha}}(s) | 0 \leqq s \leqq \alpha_2^{(6)}\}.$$

We define $\mathcal{N}_1$ to be the area contained in the hyperplane $\alpha_2 = 0$ enclosed by $I_{(1,2)}$, $I_{(1,5)}$, $I_{(1,4)}$, and $I_{(1,6)}$ such that $T(\mathcal{N}_1) \subset \mathcal{M}_1$. Let $\mathcal{N}_2$ be the area enclosed by $I_{(1,2)}$, $I_{(2,5)}$, $I_{(2,3)}$, and $I_{(2,6)}$ contained in the hyperplane $\alpha_1 = 0$ such that $T(\mathcal{N}_2) \subset \mathcal{M}_2$.

Thus we are left with a construction of $\mathcal{N}_6$ satisfying (3.5) and having prescribed boundary $\Gamma = I_{(1,6)} \cup I_{(4,6)} \cup I_{(3,6)} \cup I_{(2,6)}$. The construction will be geometrical. The delicate part comes from $I_{(2,6)}$ since $T(\alpha)$ for $\alpha_1 = 0$ is defined only when $-\alpha_3 > d\alpha_2$. However, by continuity there is a "tubular" neighborhood U_0 of $I_{(2,6)}$ in $\bar{\mathbb{R}}^+ \times \bar{\mathbb{R}}^+ \times \mathbb{R}$ such that $y_1 > \frac{5}{4} y_1^0$ for $y \in T(U_0)$. Looking at the geometry of Γ we see there is a smooth curve Γ' in U_0 with endpoints $\alpha^{(9)}$ and $\alpha^{(10)}$ on $I_{(3,6)}$ and $I_{(1,6)}$ respectively having the following properties

(i) $\inf_{\alpha \in \Gamma'} \alpha_1 > 0$;

(ii) $\Gamma \cap \Gamma' = \{\alpha^{(9)}, \alpha^{(10)}\}$;

(iii) If $I'_{(3,6)}$ and $I'_{(1,6)}$ denote the "intervals" on $I_{(3,6)}$ and $I_{(1,6)}$ with endpoints $\alpha^{(6)}$, $\alpha^{(9)}$, and $\alpha^{(8)}$, $\alpha^{(10)}$, respectively, then $I'_{(3,6)}, I'_{(1,6)} \subset U_0$;

(iv) If α, α' are in $\Gamma \cup \Gamma'$ such that the relations $\alpha_i = \alpha'_i$ $(i = 1, 2)$ hold, then $\alpha = \alpha'$.

Set

$$\Gamma_1 = I_{(2,6)} \cup I'_{(3,6)} \cup I'_{(1,6)} \cup \Gamma'$$

$$\Gamma_2 = I_{(4,6)} \cup (I_{(3,6)} \backslash I'_{(3,6)}) \cup (I_{(1,6)} \backslash I'_{(1,6)}).$$

We then may find a smooth 2-dimensional manifold $\mathcal{N}_7 \subset U_0$ having Γ_1 as boundary. Therefore it will be sufficient to find a 2-dimensional C^0-manifold $\mathcal{N}_8$ with boundary Γ_2 such that

$$y_1 > \tfrac{5}{4} y_1^0 \quad \text{for} \quad y \in T(\mathcal{N}_8). \tag{3.38}$$

For then we may obtain a smooth 2-dimensional manifold $\mathcal{N}_6$ by smoothing out $\mathcal{N}'_6 = \mathcal{N}_7 \cup \mathcal{N}_8$ while keeping the boundary Γ fixed and such that

$$y_1 > y_1^0 \quad \text{for} \quad y \in T(\mathcal{N}_6)$$

is valid. Now to find $\mathcal{N}_8$, let

$$2t_4 = \inf_{\alpha \in \Gamma_2} \alpha_1 > 0$$

$$t_5 = 2 \sup_{\alpha \in \Gamma_2} \alpha_1 < \infty$$

$$s_4 = 0$$

$$s_5 = 2 \sup_{\alpha \in \Gamma_2} \alpha_2 > 0$$

$$Q = \{ = (s,t) \subset \mathbb{R}^2 | t_4 \leqq t \leqq t_5, s_4 \leqq s \leqq s_5\}.$$

Using Griffiths second inequality, continuity and standard compactness arguments again, there is $a_6 > 0$ such that $y = T(\alpha)$ satisfies $y_1 > \frac{5}{4} y_1^0$ for all $\alpha \in Q \times (0, a_6) \subset (\mathbb{R}^+)^3$. In particular for $\Gamma_3 = \Gamma_2 + (0, 0, a_6)$ we may easily construct a 2-dimensional manifold $\mathcal{N}_9 \subset Q \times (0, 2a_6)$ having Γ_3 as boundary.

Also we may assume, by possibly enlarging a_6, that

$$\mathcal{N}_9 \cap \Gamma_2 = \emptyset . \tag{3.39}$$

Next let

$$\mathcal{N}_{10} = \{\alpha = \alpha' + t(0, 0, a_6) | 0 \leqq t \leqq 1, \alpha' \in \Gamma_2\} .$$

Since $y_1 > \frac{5}{4} y_1^0$ for $y \in T(\Gamma_2)$, by Griffiths second inequality again, we have

$$y_1 > \tfrac{5}{4} y_1^0 \quad \text{for} \quad y \in T(\mathcal{N}_{10}) .$$

Now by (3.37 iv) and (3.39)

$$\mathcal{N}_8 = \mathcal{N}_{10} \cup \mathcal{N}_9$$

is a C^0-manifold with boundary Γ_2 and satisfying (3.38). By the preceeding discussion, this concludes the proof of Theorem 8.

Appendix A

The proof of Lemma 4, which we present here is due to Th. Bröcker and K. Jänich [3]:

As a preparation we define a partial ordering on $\mathbb{R}$ by setting $x < x'$ if $x_i < x_i'$ $(i = 1, 2)$.

Lemma A1. *Let S be a C^∞-mapping of an open convex subset B of $\mathbb{R}^2$ into $\mathbb{R}^2$ such that with $y = y(x) = S(x)$ the relations*

$$\frac{\partial y_i}{\partial x_j} \geqq 0 \ (i, j = 1, 2) ; \qquad x \in B \tag{A1}$$

and

$$\frac{\partial y_i}{\partial x_1} + \frac{\partial y_i}{\partial x_2} > 0 \ (i = 1, 2) ; \qquad x \in B \tag{A2}$$

are valid.

Then S preserves the ordering $<$.

Proof. Assume $x < x'$ and let

$$x(t) = x + t(x' - x) \qquad 0 \leqq t \leqq 1 .$$

Then

$$y_i(x') - y_i(x) = \sum_{j=1}^{2} \int_0^1 (x' - x)_j \left(\frac{\partial}{\partial x_j} y_i \right) (x(t)) dt > 0$$

q.e.d.

Remark. Assume S has maximal rank 2 everywhere, i.e.

$$\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} \neq 0 \qquad x \in B . \tag{A3}$$

Then (A2) is a consequence of (A1) and (A3).

To prove Lemma 4, assume there is $x^1 \neq x^2$ with $(S(x^1)=S(x^2))$. Then x^1 and x^2 are not comparable with respect to $<$. In particular we may find x^3 and x^4 with

$$x^3 < x^1 < x^4;\ x^3 < x^2 < x^4$$

such that the x^i $(i=1 \ldots 4)$ form a parallelogram P. Now

$$L = \{y \in \mathbb{R}^2 \mid (y - S(x^1))_2 = -(y - S(x^1))_1\}$$

is a straight line passing through $S(x^1)$. Thus $S^{-1}(L)$ is a 1-dimensional closed manifold, because S has maximal rank 2 everywhere. Write $S^{-1}(L) = \bigcup_i \gamma_i$ where the γ_i are disjoint curves without boundary. Choose i_0 such that $x^1 \in \gamma_{i_0}$. We claim $x^2 \in \gamma_{i_0}$. Indeed, since γ_{i_0} is without boundary, γ_{i_0} which has to cross ∂P at x^1 due to Lemma A1 and the form of L, also has to leave P again. It cannot leave on on $\partial P \backslash \{x^1, x^2\}$, again due to Lemma A1 and the form of L. It cannot leave at x^1 since then γ_{i_0} would intersect itself, which is easily seen to contradict local injectivity. As a consequence $S|_{\gamma_{i_0}}$ is not injective on γ_{i_0}. On the other hand, an immersion of a curve into a straight line is injective and we have arrived at a contradiction. This proves Lemma 4.

Appendix B

In this appendix we present a heuristic discussion of Conjecture 3 on the basis of the behaviour of the Ising model near its critical point $J = J_c$. We make assumptions similar to those given in [7]. Let $\varepsilon = (J_c - J)J_c^{-1}$ $(\varepsilon > 0)$. Then asymptotically

$$\langle \sigma_i \sigma_j \rangle \sim \frac{c}{|i-j|^{d+\eta-2}} \cdot e^{-L(\varepsilon)^{-1}|i-j|} \tag{B1}$$

$$-\langle \sigma_i; \sigma_j; \sigma_k; \sigma_l \rangle = \chi_4(0, j-k, j-l, l-k) \cdot e^{-L(\varepsilon)^{-1}\Delta(i,j,k,l)}. \tag{B2}$$

Here

$$L(\varepsilon) = l_0 \varepsilon^{-\bar{w}}(1 + 0(\varepsilon)) \qquad (\varepsilon > 0) \tag{B3}$$

for some $l_0, \bar{w} > 0$.

$\Delta(i,j,k,l) =$ length of the shortest graph which connects all the points i, j, k, l.

χ_4 is assumed to be non zero and homogeneous

$$\chi_4\left(0, \frac{\xi_1}{\sigma}, \frac{\xi_2}{\sigma}, \frac{\xi_3}{\sigma}\right) = \sigma^{\omega_4} \chi_4(0, \xi_1, \xi_2, \xi_3) \tag{B4}$$

ω_4 may then be calculated [7] to be

$$\omega_4 = 3d - \frac{(3\delta - 1)(2 - \eta)}{(\delta - 1)} \tag{B5}$$

in the standard notation of critical exponents (see e.g. [5]). Then we obtain with

$L = |\mathcal{T}|^{\frac{1}{d}}$, $\Theta = L(\varepsilon)^{-1}$. L (L large):

$$\left\langle \sum_{i,j} \sigma_i \sigma_j \right\rangle \sim L^{d+2-\eta} \int_0^1 d\xi e^{-\Theta\xi} \xi^{1-\eta}$$

$$\left\langle \sum_{i,j} (i-j)^2 \sigma_i \sigma_j \right\rangle \sim L^{d+4-\eta} \int_0^1 d\xi e^{-\Theta\xi} \xi^{3-\eta} \tag{B6}$$

$$-\sum_{i,j,k,l} \langle \sigma_i ; \sigma_j ; \sigma_k ; \sigma_l \rangle \sim L^{4d-\omega_4} \int_{|\xi_i| \leq 1} d\xi_1 d\xi_2 d\xi_3 e^{-\Theta \Delta(0,\xi_1,\xi_2,\xi_3)} \chi_4(0, \xi_1 \xi_2 \xi_3) .$$

We insert this into (2.19), the parametric description of $\mathcal{M}_4$, and take $\tilde{y}_1$ and Θ (instead of γ_3 and J) as new variables:

$$\tilde{y}_2 \sim \tilde{y}_1 D(\mathcal{T}, a) \frac{\int_0^1 d\xi e^{-\Theta\xi} \xi^{3-\eta}}{\int_0^1 d\xi e^{-\Theta\xi} \xi^{1-\eta}}$$

$$\tilde{y}_3 \sim (\tilde{y}_1)^2 V(\mathcal{T}, a) L^f \frac{\int_{|\xi_i| \leq 1} d\xi_1 d\xi_2 d\xi_3 e^{-\Theta \Delta(0,\xi_1,\xi_2,\xi_3)} \chi_4(0, \xi_1, \xi_2, \xi_3)}{\left(\int_0^1 d\xi e^{-\Theta\xi} \xi^{1-\eta} \right)^2} \tag{B7}$$

with

$$f = \frac{(\delta+1)}{(\delta-1)} (2-\eta) - d . \tag{B8}$$

Relations (B7) give a parametric form of $\mathcal{M}_4$ with $J \leqq J_c$. The relations (B7) already lead us to expect that Conjecture 3 can only be true if $f \geqq 0$. Now the Buckingham-Gunton inequality ([4, 6]) states that $f \leqq 0$. Hence $f = 0$, which is a scaling hypothesis, is the best we can hope for. (For $d = 2$, this seems well established. For $d = 3$, however, see the discussion in [5].) Indeed if $f < 0$, that part of $\mathcal{M}_4$ we just have described should move into $\mathcal{M}_2$ which is a Gaussian theory. This conforms with the result in [7]. To see all this more clearly, we consider that part of $\mathcal{M}_4$ corresponding to $J \approx 0$. Then Θ is large and we obtain

$$\begin{aligned} \tilde{y}_2 &\sim \tilde{y}_1 D(\mathcal{T}, a) \Theta^{-2} \\ \tilde{y}_3 &\sim (\tilde{y}_1)^2 V(\mathcal{T}, a) L^f \Theta^{-(f+d)} \end{aligned} \tag{B9}$$

(Θ large).

Eliminating Θ gives

$$\tilde{y}_3 \sim (\tilde{y}_1)^{2 - \frac{(f+d)}{2}} (\tilde{y}_2)^{\frac{f+d}{2}} a^{-f} . \tag{B10}$$

Hence, if $f < 0$, the short distance behaviour (i.e. the behaviour for $a \to 0$) would be responsible for $\mathcal{M}_4$ moving into $\mathcal{M}_2$.

For $f = 0$, (B10) reduces to

$$\tilde{y}_3 \sim (\tilde{y}_1)^{2 - \frac{d}{2}} (\tilde{y}_2)^{\frac{d}{2}} \tag{B11}$$

and the $(\mathcal{T}, a)$-dependence has dropped out in first approximation. We note that (B11) is in agreement with the result that given normalization of the two-point function, the renormalized coupling constant is bounded in absolute value [9].

From (B11) we also see that $\mathcal{M}_4$ moves closer to $\mathcal{M}_2$ with increasing d. Note also the singular $\tilde{y}_1$ dependence of $\tilde{y}_3$ for $d>4$, indicating that then this approach should fail. It would be interesting to know whether this is related to the non-renormalizeability of $\phi_d^4(d>4)$.

We summarize our discussion as follows: Any nontrivial ϕ_4^4 theory with not too singular (euclidean) short distance behaviour should be obtainable by lattice approximations. (Take e.g. the lattice field to be the euclidean field averaged out over cubes.) Conversely a euclidean construction of a nontrivial ϕ_4^4 theory through lattice approximations can only succeed if $\mathcal{M}_4$ asymptotically does not fall completely into $\mathcal{M}_2$. For this to be the case it is necessary that the Buckingham-Gunton inequality is actually an equality.

For $d=4$, one should be rather optimistic. According to standard folklore (see e.g. [23]), the critical exponents of the Ising model should already for $d=4$ take the values of mean field theory: $\delta=3$, $\eta=0$.

Thus it will be necessary to check the influence of logarithmic corrections to scaling.

An alternative discussion of Conjecture 3 has been given in [20]: The limiting Ising model surface (i.e. the case $\mathcal{T}\to\mathbb{Z}^d$, $a\to 0$) takes the precise form

$$\tilde{y}_3 = C(d)\tilde{y}_1^2 \left(\frac{\tilde{y}_2}{\tilde{y}_1}\right)^{\frac{d}{2}} .$$

Here

$$C(d) = \lim_{T\to T_c} -\frac{\tilde{u}_4(T)}{\xi^d(T)\chi^2(T)} \geqq 0$$

with

$$\chi(T) = \sum_{i\in\mathbb{Z}^d} \langle\sigma_0\sigma_i\rangle$$

$$\chi(T)\xi^2(T) = \sum_{i\in\mathbb{Z}^d} |i|^2 \langle\sigma_0\sigma_i\rangle$$

$$\tilde{u}_4(T) = \sum_{j,k,l\in\mathbb{Z}^d} \langle\sigma_0;\sigma_i;\sigma_j;\sigma_k\rangle \qquad (T = \text{Temperature}).$$

Hence the relation $C(d)>0$ is equivalent to Conjecture 3. It would be interesting to try to determine the number $C(d)$ numerically, say by high temperature expansions.

The renormalization group techniques give a partial answer to what this number is: Assume there is a fixed point g^* in the coupling of the renormalization group transformation which describes the critical behaviour of the Ising model. Then $C(d)$ is of the form $g^* + o(g^*)$. Not surprisingly therefore $C(d)$ is zero if this fixed point is the Gaussian fixed point $g^*=0$ [16].

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