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On Invariant and Covariant Schwartz Distributions in the Case of a Compact Linear Group

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Abstract. A proof is given for the representations of invariant and covariant (Schwartz) distributions on $\boldsymbol{R}^n$, which are often used in theoretical physics. We express invariant distributions as distributions of standard polynomial invariants and decompose covariant distributions in standard polynomial covariants. Our consideration is restricted to compact groups acting linearly on $\boldsymbol{R}^n$. The representation for invariant distributions is obtained provided the standard invariants form an algebraically independent generating set in the ring of invariant polynomials. As for the standard covariants we assume that in the class of covariant polynomials they provide a unique decomposition into a sum of the standard covariants multiplied with invariant polynomials.

Introduction

It is part of the folklore of mathematical physics that distributions on the Euclidean space $\boldsymbol{R}^n$, invariant with respect to a classical group (acting linearly on $\boldsymbol{R}^n$), can be represented as "distributions" of a fixed (finite) family of standard polynomial invariants (provided that the invariants separate orbits, at least those of some kind of "regularity"). Similarly, it is often believed that covariant distributions can be decomposed into a sum of a fixed (finite) family of standard polynomial covariants multiplied with invariant distributions.

Special results in this direction are provided by the descriptions of rotation invariant distributions of one [1] or two [2] vectors and Lorentz invariant distributions of one vector [3]. (Concerning Lorentz covariant distributions of one vector see e.g. [4], Section 3.) Prior to such a description, one needs a choice of standard polynomial invariants (or covariants) which yield resolution of the corresponding algebraic problem on invariant (or covariant) polynomials. Of interest are more general situations which exhibit a close relationship between the distribution theoretic and the algebraic problem.

The purpose of this paper is to show that in the case of a compact linear group the desired representation for invariant (resp. covariant) distributions on $\boldsymbol{R}^n$ exists and is unique (in a certain sense), provided that the corresponding representation in the class of polynomials exists and is unique with respect to a given family of standard polynomial invariants (resp. covariants). By this means our consideration starts from certain algebraic assumptions (see Conditions I and II in Sections 1 and 2, respectively).

Needless to say, the compactness hypothesis is crucial for our consideration. However it should be mentioned that in specific cases the results can be applied also to the description of distributions (on subdomains of $\boldsymbol{R}^n$) invariant or covariant with respect to a non-compact Lie group, which may appear sufficient for practical purposes. (Such a possibility occurs e.g. in the case of Lorentz invariant [2] or Lorentz covariant [5] distributions of several vectors in a domain where at least one of the vectors is time-like; here the compactness of isotropy groups plays an important part.)

As for the role of the uniqueness assumption in the algebraic problem (on invariant or covariant polynomials), it is essentially the same as in the problem on invariant or covariant analytic functions [6]. In the problem on invariant distributions one should introduce a suitable space of distributions on the manifold of values of standard invariants, and this manifold appears to be the closure of an open set in Euclidean space. Further generalizations which dispose of this restriction would be of interest. On the other hand, in the problem on covariant distributions the uniqueness assumption seems to be indispensable for an effective treatment (otherwise the decomposition into standard covariants would be highly non-unique).

1. Invariant Functions and Distributions

In both Sections 1 and 2, G is a compact Lie group acting linearly on n-dimensional Euclidean space $\boldsymbol{R}^n = \boldsymbol{R} \times \ldots \times \boldsymbol{R}$ (n factors, $\boldsymbol{R}$ being the field of reals). μ is the normalized Haar measure on G:

$$\int d\mu(g) = 1 .$$

The value of an element $g \in G$ at $x \in \boldsymbol{R}^n$ is denoted by gx. Since a G-invariant (positive definite) inner product always exists in $\boldsymbol{R}^n$, we assume without essential loss of generality that the norm $\|x\| = ((x_1)^2 + \ldots + (x_n)^2)^{1/2}$ is G-invariant.

We look for representations of G-invariant functions and distributions on $\boldsymbol{R}^n$. We consider the commonly used spaces of complex testing functions of three types, $\mathscr{E}$, $\mathscr{D}$, and $\mathscr{S}$ ([7]; see also Appendix A). In order to treat them in parallel, the letter $\mathscr{A}$ is introduced which stands for either of the symbols $\mathscr{E}$, $\mathscr{D}$, $\mathscr{S}$ [thus e.g. $\mathscr{A}(\boldsymbol{R}^n)$ denotes either of the spaces $\mathscr{E}(\boldsymbol{R}^n)$, $\mathscr{D}(\boldsymbol{R}^n)$, $\mathscr{S}(\boldsymbol{R}^n)$]. $\mathscr{A}'(\boldsymbol{R}^n)$ is the strong topological dual of the space $\mathscr{A}(\boldsymbol{R}^n)$, i.e. the space of distributions on $\boldsymbol{R}^n$ of type $\mathscr{A}'(=\mathscr{E}', \mathscr{D}'$ or $\mathscr{S}')$. The value of $T \in \mathscr{A}'(\boldsymbol{R}^n)$ at $f \in \mathscr{A}(\boldsymbol{R}^n)$ is denoted by (T, f) or $(T(x), f(x))_x$.

The (right) action of G on $\mathscr{A}(\boldsymbol{R}^n)$ and $\mathscr{A}'(\boldsymbol{R}^n)$ is defined in the standard way. Namely, the value of $g \in G$ at $f \in \mathscr{A}(\boldsymbol{R}^n)$ is the function $f \circ g \in \mathscr{A}(\boldsymbol{R}^n)$ such that $(f \circ g)(x) = f(gx)$, $\forall x \in \boldsymbol{R}^n$. Also the value of $g \in G$ at $T \in A'(\boldsymbol{R}^n)$ is the distribution $T \circ g \in A'(\boldsymbol{R}^n)$ defined by $(T \circ g, f) = (T, f \circ g^{-1})$, $\forall f \in \mathscr{A}(\boldsymbol{R}^n)$. A function (or a distribution) f is called *G-invariant* if $f \circ g = f$ for all $g \in G$. The totality of G-invariant functions of $\mathscr{A}(\boldsymbol{R}^n)$ [resp. G-invariant distributions of $\mathscr{A}'(\boldsymbol{R}^n)$] is denoted by $\mathscr{A}(\boldsymbol{R}^n)^G$ [resp. $\mathscr{A}'(\boldsymbol{R}^n)^G$]. It is as a (closed) subspace of $\mathscr{A}(\boldsymbol{R}^n)$ [resp. $\mathscr{A}'(\boldsymbol{R}^n)$]; it is endowed with the induced topology.

Our description of G-invariant functions and distributions is based on an assumption concerning the algebraic problem on G-invariant polynomials. By $\mathscr{P}(\boldsymbol{R}^n)$ [resp. $\mathscr{P}(\boldsymbol{R}^n)^G$] we denote the totality of complex polynomial functions on $\boldsymbol{R}^n$ [resp. of G-invariant elements of $\mathscr{P}(\boldsymbol{R}^n)$]. We assume that there exist real G-invariant polynomials $I_1,\dots,I_m$ on $\boldsymbol{R}^n$ such that each polynomial $P\in\mathscr{P}(\boldsymbol{R}^n)^G$ can be represented uniquely in the form $P(x)=p(I_1(x),\dots,I_m(x))\equiv(p\circ I)(x)$ with $p\in\mathscr{P}(\boldsymbol{R}^m)$; here I is the polynomial mapping

$$I:\boldsymbol{R}^n\ni x\mapsto(I_1(x),\dots,I_m(x))\in\boldsymbol{R}^m\,. \tag{1.1}$$

Therefore the basic hypothesis which is assumed throughout Section 1 is the validity of the following

Condition I. The mapping $p\mapsto p\circ I$ is a bijection of $\mathscr{P}(\boldsymbol{R}^m)$ onto $\mathscr{P}(\boldsymbol{R}^n)^G$.

There are a few important properties of the mapping I and the image of $\boldsymbol{R}^n$ under I, denoted by $\mathfrak{M}$:

$$\mathfrak{M}=I(\boldsymbol{R}^n)\,. \tag{1.2}$$

First of all, the uniqueness property of the representation $P=p\circ I$ for all $P\in\mathscr{P}(\boldsymbol{R}^n)^G$ can be reformulated in several equivalent forms according to the lemma.

1.1. Lemma. *For a polynomial mapping $J:\boldsymbol{R}^n\to\boldsymbol{R}^m$, the following properties are equivalent:*

a) *the mapping $\mathscr{P}(\boldsymbol{R}^m)\ni p\mapsto p\circ J\in\mathscr{P}(\boldsymbol{R}^n)$ is injective;*

b) *there exists at least one point $x\in\boldsymbol{R}^n$ such that rank of the Jacobian matrix $DJ(x)=(\partial I_\varkappa(x)/\partial x_\lambda\,|\,\lambda=1,\dots,n;\,\varkappa=1,\dots,m)$ equals m;*

c) *the set $J(\boldsymbol{R}^n)$ possesses a non-void interior (in $\boldsymbol{R}^m$).*

Proof. The implications b)$\Rightarrow$c) and c)$\Rightarrow$a) are trivial. There remains the implication a)$\Rightarrow$b) to be proved. Suppose that rank of $DJ(x)$ is $<m$ for all x. Then, by the Sard theorem, the set $\mathfrak{B}=J(\boldsymbol{R}^n)$ is of the Lebesgue measure zero. Furthermore, by the Seidenberg-Tarski theorem ([8]), $\mathfrak{B}$ is a semi-algebraical set, i.e. $\mathfrak{B}$ is the union of finitely many non-void sets $\mathfrak{B}_\sigma$ in $\boldsymbol{R}^m$ of the form

$$\mathfrak{B}_\sigma=\{y\in\boldsymbol{R}^m|a_\sigma^{(j)}(y)=0,\ b_\sigma^{(k)}(y)>0,\ \forall j=1,\dots,s_\sigma,\ \forall k=1,\dots,t_\sigma\}\,,$$

where $s_\sigma,\,t_\sigma\in\boldsymbol{Z}_+$, $\boldsymbol{Z}_+$ being the totality of non-negative integers; $a_\sigma^{(j)}$ and $b_\sigma^{(k)}$ are real polynomials. We set

$$a_\sigma(y)=\textstyle\sum_{j=1}^{s_\sigma}(a_\sigma^{(j)}(y))^2\,.$$

Then $a_\sigma\not\equiv0$. (Otherwise $\mathfrak{B}_\sigma$ would be defined only by strict inequalities and hence it would be a non-void open set in $\boldsymbol{R}^m$; this would contradict to the statement that $\mathfrak{B}$ is of measure zero.) Define $a(y)=\prod_\sigma a_\sigma(y)$. Then $\mathfrak{B}=\bigcup_\sigma\mathfrak{B}_\sigma$ is contained in the set of zeros of the polynomial $a\not\equiv0$. Hence $a\circ J=0$, and the mapping $p\mapsto p\circ J$ is not injective. Q.E.D.

We denote by $\mathfrak{R}$ the totality of points $x\in\boldsymbol{R}^n$, called *regular* for I, such that rank of the Jacobian matrix $DI(x)$ equals m:

$$\mathfrak{R}=\{x\in\boldsymbol{R}^n|\operatorname{rank}DI(x)=m\}\,. \tag{1.3}$$

It is an open set in $\boldsymbol{R}^n$, which is non-void (according to Lemma 1.1) and dense in $\boldsymbol{R}^n$ [due to the polynomial dependence of $I(x)$ on x]. By the rank theorem, the restriction of I to $\mathfrak{N}$ is an open mapping, hence $I(\mathfrak{N})$ is an open subset in $\boldsymbol{R}^m$, which is dense in $\mathfrak{M}$. It is useful to note that the set $\mathfrak{N}$ is representable in the form $Q^{-1}(\boldsymbol{R}\backslash\{0\})$ with some non-negative polynomial $Q \not\equiv 0$ in $\boldsymbol{R}^n$, e.g. $Q=\sum_j |Q_j|^2$, where $\{Q_j\}$ is the totality of $m \times m$ minors of DI. (We shall utilize this in order to apply Construction A.8 of Appendix A.) Note that $\mathfrak{N}$ is I-saturated:

$$\mathfrak{N}=I^{-1}(I(\mathfrak{N}))\,. \tag{1.4}$$

Indeed, the standard argument (based on the Weierstrass approximation theorem) shows that the totality of G-invariant polynomials separates G-orbits, hence $GS=I^{-1}(I(S))$ for all sets $S \subset \boldsymbol{R}^n$.

Further, $I^{-1}(B)$ is compact for each compact B in $\boldsymbol{R}^m$, since (by Condition I)

$$\|x\|^2=(\pi \circ I)(x) \tag{1.5}$$

for some $\pi \in \mathscr{P}(\boldsymbol{R}^m)$. [Hence $I(V)$ is closed in $\boldsymbol{R}^m$ for any closed V in $\boldsymbol{R}^n$.] Moreover, $I(U)$ is (relatively) open in $\mathfrak{M}$ for each open $U \subset \boldsymbol{R}^n$ [since $\mathfrak{M}\backslash I(U)=I(\boldsymbol{R}^n\backslash GU)$ is closed in $\mathfrak{M}$]. Now Lemma A.7 of Appendix A implies:

1.2. Lemma. *$\mathfrak{M}$ is the closure of the open set $I(\mathfrak{N})$ in $\boldsymbol{R}^m$. Moreover, $\mathfrak{M}$ possesses the regularity property* (R)′ (see Definition A.4).

We now turn to the representation of functions $f \in \mathscr{A}(\boldsymbol{R}^n)^G$ in the form $f=\varphi \circ I$. For this purpose we must introduce an appropriate space of testing functions φ on the set $\mathfrak{M}$. The discussion in Appendix A (especially Construction A.8 and Proposition A.9) puts forward the following definition.

1.3. Definition. $\mathscr{A}(\mathfrak{M})$ stands for either of the spaces $\mathscr{E}(\mathfrak{M})$, $\mathscr{D}(\mathfrak{M})$, $\mathscr{S}(\mathfrak{M})$, defined as follows. $\mathscr{E}(\mathfrak{M})$ is the totality of complex functions φ on $\mathfrak{M}$ such that their restrictions to $I(\mathfrak{N})$ are $\mathscr{C}^\infty$ and all derivatives $D^\beta\varphi \restriction I(\mathfrak{N})$ admit (unique) continuous extensions (denoted by $D^\beta\varphi$) into $\mathfrak{M}$. $\mathscr{E}(\mathfrak{M})$ is endowed with the topology of a Frechét space with the seminorms[1]

$$|\varphi|_l^K=\sup_{|\beta|\leqq l}\ \sup_{y\in K}|\partial^\beta\varphi(y)|\,,$$

where $l \in \boldsymbol{Z}_+$, and K is an arbitrary compact in $\mathfrak{M}$. $\mathscr{D}(\mathfrak{M})$ is the inductive limit of subspaces of $\mathscr{E}(\mathfrak{M})$ indexed by compacts $K \subset \mathfrak{M}$ and formed by the totality of functions $\varphi \in \mathscr{E}(\mathfrak{M})$ with support in K. $\mathscr{S}(\mathfrak{M})$ is the totality of functions $\varphi \in \mathscr{E}(\mathfrak{M})$ for which all the seminorms

$$|\varphi|_{r,s}=\sup_{|\beta|\leqq r}\ \sup_{y\in\mathfrak{M}}(1+\|y\|)^s|\partial^\beta\varphi(y)|\,(r,s\in\boldsymbol{Z}_+)$$

are finite. It is a Frechét space with these seminorms. By $\mathscr{A}'(\mathfrak{M})$ we denote the strong topological dual of $\mathscr{A}(\mathfrak{M})$ (i.e. the space of distributions on $\mathfrak{M}$ of type $\mathscr{A}'$). According to Appendix A (Propositions A.2 and A.9), $\mathscr{A}'(\mathfrak{M})$ can be identified with the subspace of $\mathscr{A}'(\boldsymbol{R}^m,\mathfrak{M})$ of distributions of $\mathscr{A}'(\boldsymbol{R}^m)$ with support in $\mathfrak{M}$.

[1] We use the abbreviations

$$D^\alpha=(\partial/\partial x_1)^{\alpha_1}\dots(\partial/\partial x_n)^{\alpha_n} \quad \text{for} \quad \alpha\in\boldsymbol{Z}_+^n \quad \text{and} \quad \partial^\beta=(\partial/\partial y_1)^{\beta_1}\dots(\partial/\partial y_m)^{\beta_m} \quad \text{for} \quad \beta\in\boldsymbol{Z}_+^m\,,$$

where $\boldsymbol{Z}_+^k=\boldsymbol{Z}_+\times\dots\times\boldsymbol{Z}_+$ (k factors).

It is very important that (by virtue of Proposition A.9) each $\varphi \in \mathscr{A}(\mathfrak{M})$ is the restriction to $\mathfrak{M}$ of a complex $\mathscr{C}^\infty$ function on $\boldsymbol{R}^m$. Consequently $\varphi \circ I$ is $\mathscr{C}^\infty$ and G-invariant for any $\varphi \in \mathscr{A}(\mathfrak{M})$. Moreover, it is easily seen that a linear continuous operator $\mathscr{I}$ is defined by

$$\mathscr{I}: \mathscr{A}(\mathfrak{M}) \ni \varphi \mapsto \varphi \circ I \in \mathscr{A}(\boldsymbol{R}^n)^G . \tag{1.6}$$

We claim (Theorem 1.6 below) that $\mathscr{I}$ is a topological isomorphism (onto).

It is clear that $\mathscr{I}$ is injective, consequently there exists the inverse operator $\mathscr{I}^{-1}: \mathrm{im}\,\mathscr{I} \to \mathscr{A}(\mathfrak{M})$, where $\mathrm{im}\,\mathscr{I}$ denotes the image of $\mathscr{A}(\mathfrak{M})$ under $\mathscr{I}$. We are now concerned with proof of the continuity[2] of $\mathscr{I}^{-1}$. We note that, for $\varphi \in \mathscr{A}(\mathfrak{M})$, $\varphi \circ I = 0$ implies $\varphi = 0$ and hence $(\partial^\beta \varphi) \circ I = 0$ for all $\beta \in \boldsymbol{Z}_+^m$. Thus linear operators $\varphi \circ I \mapsto (\partial^\beta \varphi) \circ I$ are well defined on $\mathrm{im}\,\mathscr{I}$. We have the following lemma.

1.4. Lemma. *For any $\beta \in \boldsymbol{Z}_+^m$, the operator $\varphi \circ I \mapsto (\partial^\beta \varphi) \circ I$ in $\mathrm{im}\,\mathscr{I}$ is continuous.*

Proof. We proceed by induction on $|\beta|$. For $|\beta| = 0$, the statement is trivial. Assuming the validity of the statement for $|\beta| \leqq k-1$, we will show it for $|\beta| = k$. We apply the differential operators $D_{\mu_1} \dots D_{\mu_k}$ to $\varphi \circ I$ (where $\mu_1, \dots, \mu_k \in \{1, \dots, n\}$). Due to the induction hypothesis, all the expressions

$$\textstyle\sum_{\varkappa_1, \dots, \varkappa_k \in \{1, \dots, m\}} \left(\prod_{j=1}^k D_{\mu_j} I_{\varkappa_j}(x)\right) (\partial_{\varkappa_1} \dots \partial_{\varkappa_k} \varphi) \circ I$$

are obtained by applying certain continuous operators [from $\mathrm{im}\,\mathscr{I}$ into $\mathscr{A}(\boldsymbol{R}^n)$] to $\varphi \circ I$. Let us consider $\prod_j D_{\mu_j} I_{\varkappa_j}(x)$ here as elements of the complex $kn \times km$ matrix $(DI(x))^{\otimes k}$ with a polynomial dependence on x. By Lemma 1.1, its rank equals km at least for one x. Now Theorem in Appendix B implies that the functions $(\partial_{\varkappa_1} \dots \partial_{\varkappa_k} \varphi) \circ I$ depend continuously on $\varphi \circ I$. Q.E.D.

Now it is straightforward to prove the claimed statement on the mapping $\mathscr{I}$ [see below Theorem 1.6, part (i)] and thus to obtain a description of $\mathscr{A}(\boldsymbol{R}^n)^G$. Also we will present an alternative description of $\mathscr{A}(\boldsymbol{R}^n)^G$ which will be necessary for studying G-invariant distributions. In fact, an analogue of the representation $f = \varphi \circ I$ for $f \in \mathscr{A}(\boldsymbol{R}^n)^G$ would be the representation of $T \in \mathscr{A}'(\boldsymbol{R}^n)^G$ as the composite $\tau \circ I$ of the mapping I with certain "distribution" τ on $\mathfrak{M}$. It is natural to interpret $\tau \circ I$ according to the formula (with δ the Dirac m-dimensional distribution)

$$(\tau \circ I, f) = (\tau(y), (\delta(y - I(x)), f(x))_x)_y, \qquad f \in \mathscr{A}(\boldsymbol{R}^n) . \tag{1.7}$$

To give a precise meaning to this formula, one should introduce the totality of functions ψ on $\mathfrak{M}$ of the form

$$\psi(y) = (\delta(y - I(x)), f(x))_x \quad \text{with} \quad f \in \mathscr{A}(\boldsymbol{R}^n) \tag{1.8}$$

and then consider the "distribution" τ in (1.7) as a functional defined on the ψ's.

We shall not try to define ψ as a function on the whole $\mathfrak{M}$. Instead we show that ψ can be defined as a complex $\mathscr{C}^\infty$ function on $I(\mathfrak{R})$ [$I(\mathfrak{R})$ being dense in $\mathfrak{M}$]. Indeed, for $y \in I(\boldsymbol{R})$, $\|y\|^2 < b$, the heuristic formula (1.8) can be rewritten (with

[2] Throughout, any subspace of a (Hausdorff) locally convex space X is endowed with the relative topology (unless otherwise indicated). X' stands for the strong topological dual of X.

the use of formal manipulations) in the following well defined form:

$$\psi(y)=\left(\prod_{j=1}^{m} \partial/\partial y_j\right) \int \left(\prod_{j=1}^{m} \theta(y_j-I_j(x))\right)\theta(b-\|I(x)\|^2) f(x)d^n x\,, \tag{1.9}$$

where $\theta(\lambda)=1$ for $\lambda \geqq 0$ and $\theta(\lambda)=0$ for $\lambda<0$. [For $\|y\|^2<b$, the right hand side of (1.9) does not depend on the cut-off $\theta(b-\|I(x)\|^2)$ which may be necessary for $f\in\mathscr{E}(\boldsymbol{R}^n)$.] Since I is regular in the I-saturated open set $\mathfrak{R}$, it is easily seen that a $\mathscr{C}^\infty$ function ψ on $I(\mathfrak{R})$ is defined according to (1.9). In this way the precise meaning is given to formula (1.8).

In particular, if we put $f=\varphi\circ I$ with $\varphi\in\mathscr{A}(\mathfrak{M})$ into (1.8), then the resulting ψ is

$$\psi=h\cdot\varphi\restriction I(R)\,, \tag{1.10}$$

where h is a picked positive $\mathscr{C}^\infty$ function on $I(\mathfrak{R})$,

$$h(y)=(\delta(y-I(x)),\, 1(x))_x=\int \delta(y-I(x))d^n x \tag{1.11}$$

[corresponding to the function $1(x)\equiv 1$ of $\mathscr{E}(\boldsymbol{R}^n)$]. Further, for a general element $f\in\mathscr{A}(\boldsymbol{R}^n)$, the right hand side of (1.8) is unchanged after the replacement of f by the G-invariant function $f'=Ef$, where $E:\mathscr{A}(\boldsymbol{R}^n)\to\mathscr{A}(\boldsymbol{R}^n)^G$ is the averaging over G:

$$(Ef)(x)=\int f(gx)d\mu(g)\,. \tag{1.12}$$

Thus assuming the validity of the claimed statement about $\mathscr{I}$ motivates introducing the following candidate for the totality of ψ's.

1.5. Definition. For $\mathscr{A}=\mathscr{E}$, $\mathscr{D}$, or $\mathscr{S}$, we define $\mathscr{A}_h(\mathfrak{M})$ as the space of all $\mathscr{C}^\infty$ functions ψ on $I(\mathfrak{R})$ of the form (1.10) with h the picked function (1.11) and φ an arbitrary function of $\mathscr{A}(\mathfrak{M})$. The topology of $\mathscr{A}(\mathfrak{M})$ induces a topology on $\mathscr{A}_h(\mathfrak{M})$ via the isomorphism $\mathscr{A}(\mathfrak{M})\ni\varphi\mapsto h\cdot\varphi\restriction I(\mathfrak{R})$. Since the multiplication operator $\psi\mapsto(1/h)\cdot\psi$ is a topological isomorphism of $\mathscr{A}_h(\mathfrak{M})$ onto $\mathscr{A}(\mathfrak{M})$, its dual operator, called also the multiplication by $(1/h)$, maps isomorphically $\mathscr{A}'(\mathfrak{M})$ onto $(\mathscr{A}_h(\mathfrak{M}))'$, the strong dual of $\mathscr{A}_h(\mathfrak{M})$. Therefore it is natural to denote $(\mathscr{A}_h(\mathfrak{M}))'$ also by $\mathscr{A}'_{1/h}(\mathfrak{M})$. Thus we have the identity:

$$(\varrho,\varphi)=((1/h)\varrho,\, h\varphi) \quad \text{for all} \quad \varrho\in\mathscr{A}'(\mathfrak{M}),\ \varphi\in\mathscr{A}(\mathfrak{M})\,. \tag{1.13}$$

After these preliminaries we are in a position to describe $\mathscr{A}(\boldsymbol{R}^n)^G$ in two equivalent forms. (Remind that Condition I is assumed throughout this Section.)

1.6. Theorem. (i) *The operator* $\mathscr{I}:\varphi\mapsto\varphi\circ I$ *is a topological isomorphism of* $\mathscr{A}(\mathfrak{M})$ *onto* $\mathscr{A}(\boldsymbol{R}^n)^G$.

(ii) *The operator* $\mathscr{J}$,

$$(\mathscr{J}f)(y)=(\delta(y-I(x)), f(x))_x \quad \text{for all} \quad f\in\mathscr{A}(\boldsymbol{R}^n), \quad y\in I(\mathfrak{R})\,, \tag{1.14}$$

is a topological homomorphism of $\mathscr{A}(\boldsymbol{R}^n)$ *onto* $\mathscr{A}_h(\mathfrak{M})$, *whose restriction to* $\mathscr{A}(\boldsymbol{R}^n)^G$ *is a topological isomorphism of* $\mathscr{A}(\boldsymbol{R}^n)^G$ *onto* $\mathscr{A}_h(\mathfrak{M})$. *Its kernel,* $\ker\mathscr{J}=\mathscr{J}^{-1}\{0\}$, *coincides with the kernel,* $\ker E$, *of the averaging operator* E (1.12).

Proof. (i) First, $\mathscr{I}$ maps isomorphically $\mathscr{A}(\mathfrak{M})$ onto $\operatorname{im}\mathscr{I}\subset\mathscr{A}(\boldsymbol{R}^n)^G$. It suffices to consider only $\mathscr{A}=\mathscr{E}$ or $\mathscr{S}$; in these cases $\mathscr{I}^{-1}$ is continuous since any defining seminorm [of $\mathscr{A}(\mathfrak{M})$] at $\varphi\in\mathscr{A}(\mathfrak{M})$ is obviously majorized by seminorms [of $\mathscr{A}(\boldsymbol{R}^n)$] of finitely many functions $(\partial^\beta\varphi)\circ I$, which in turn are majorized (according to Lemma 1.4) by a seminorm [of $\mathscr{A}(\boldsymbol{R}^n)$] of the function $\varphi\circ I$. Now the completeness of $\mathscr{A}(\mathfrak{M})$ implies the completeness of $\operatorname{im}\mathscr{I}$ [in the topology induced from $\mathscr{A}(\boldsymbol{R}^n)^G$], consequently $\operatorname{im}\mathscr{I}$ is a closed subspace of $\mathscr{A}(\boldsymbol{R}^n)^G$.

There remains to show that $\operatorname{im}\mathscr{I}$ contains certain dense subset of $\mathscr{A}(\boldsymbol{R}^n)^G$. For this purpose we introduce the totality $\mathscr{L}$ of functions $f\in\mathscr{A}(\boldsymbol{R}^n)^G$ of the form $f(x)=P(x)\cdot u(\|x\|^2)$, where $P\in\mathscr{P}(\boldsymbol{R}^n)^G$, $u\in\mathscr{D}(\boldsymbol{R})$. It is clear that $\mathscr{L}$ is dense in $\mathscr{A}(\boldsymbol{R}^n)^G$. [Indeed, by the Weierstrass approximation theorem, each function $f\in\mathscr{A}(\boldsymbol{R}^n)^G$ can be approximated with an arbitrary accuracy in the topology of $\mathscr{A}(\boldsymbol{R}^n)$ by functions of the form $Q(x)\cdot u(\|x\|^2)$ with $Q\in\mathscr{P}(\boldsymbol{R}^n)$, $u\in\mathscr{D}(\boldsymbol{R})$; replacing here Q with $P=EQ$, we approximate f by functions of $\mathscr{L}$.] Now Condition I and formula (1.5) imply $\mathscr{L}\subset\operatorname{im}\mathscr{I}$, hence $\operatorname{im}\mathscr{I}$ is dense in $\mathscr{A}(\boldsymbol{R}^n)^G$. This proves (i).

(ii) It is clear that E maps continuously $\mathscr{A}(\boldsymbol{R}^n)$ onto its subspace $\mathscr{A}(\boldsymbol{R}^n)^G$ and is the identity on this subspace, therefore it is a topological homomorphism. We have seen [Eq. (1.10)] that, for $f=\varphi\circ I$ with $\varphi\in\mathscr{A}(\mathfrak{M})$, $\mathscr{J}f=h\cdot\varphi\upharpoonright I(\mathfrak{R})$. Consequently,

$$\mathscr{J}f=\mathscr{J}Ef=h\cdot(\mathscr{I}^{-1}Ef)\upharpoonright I(\mathfrak{R})\,.$$

By the definition of $\mathscr{A}_h(\mathfrak{M})$, the operator $h\cdot$ (of multiplication by h) is a topological isomorphism of $\mathscr{A}(\mathfrak{M})$ onto $\mathscr{A}_h(\mathfrak{M})$. Hence the formula $\mathscr{J}=(h\cdot)\circ\mathscr{I}^{-1}\circ E$ completes the proof. Q.E.D.

We now turn to G-invariant distributions. By analogy with G-invariant functions, we will represent any $T\in\mathscr{A}'(\boldsymbol{R}^n)^G$ in the form $\tau\circ I$. Here $\tau\circ I$ is the composite of I with $\tau\in\mathscr{A}'_{1/h}(\mathfrak{M})$; it is defined according to formula (1.7):

$$(\tau\circ I,f)=(\tau,\mathscr{J}f)\quad\text{for all}\quad f\in\mathscr{A}(\boldsymbol{R}^n)\,. \tag{1.15}$$

That is, $\tau\circ I=\mathscr{I}'\tau$, where $\mathscr{I}':\mathscr{A}'_{1/h}(\mathfrak{M})\to\mathscr{A}'(\boldsymbol{R}^n)$ is the dual of $\mathscr{I}$. In this way we will obtain an analogue of part (i) of Theorem 1.6. To have an analogue of part (ii), we assign to each distribution $T\in\mathscr{A}'(\boldsymbol{R}^n)$ a distribution ϱ of $\mathscr{A}'(\mathfrak{M})$ defined by

$$(\varrho,\varphi)=(T,\varphi\circ I)\quad\text{for all}\quad\varphi\in\mathscr{A}(\mathfrak{M}) \tag{1.16}$$

or, symbolically,

$$\varrho(y)=(T(x),\delta(y-I(x)))_x\,. \tag{1.17}$$

Equivalently,

$$\varrho=(\omega\mathscr{I})'T\,, \tag{1.18}$$

where $\omega:\mathscr{A}(\boldsymbol{R}^n)^G\to\mathscr{A}(\boldsymbol{R}^n)$ is the embedding operator.

As in the case of G-invariant functions, we claim that $\mathscr{I}'$ [resp. $(\omega\mathscr{I})'$] yields a topological isomorphism between $\mathscr{A}'_{1/h}(\mathfrak{M})$ and $\mathscr{A}'(\boldsymbol{R}^n)^G$ [resp. between $\mathscr{A}'(\boldsymbol{R}^n)^G$ and $\mathscr{A}'(\mathfrak{M})$].

We need the following general fact (which makes use only of the compactness of G but not Condition I): There exists a canonical isomorphism between $\mathscr{A}'(\boldsymbol{R}^n)^G$ and $(\mathscr{A}(\boldsymbol{R}^n)^G)'$ given by restricting distributions of $\mathscr{A}'(\boldsymbol{R}^n)^G$ to $\mathscr{A}(\boldsymbol{R}^n)^G$ (cf. [1]). In other words, the operator $\omega':\mathscr{A}'(\boldsymbol{R}^n)\to(\mathscr{A}(\boldsymbol{R}^n)^G)'$, the dual of ω, maps isomorphically $\mathscr{A}'(\boldsymbol{R}^n)^G$ onto $(\mathscr{A}(\boldsymbol{R}^n)^G)'$ [and thus defines a canonical duality between $\mathscr{A}'(\boldsymbol{R}^n)^G$ and $\mathscr{A}(\boldsymbol{R}^n)^G$]. This follows[3] from the two next properties of the averaging

[3] We are using the following general fact. Let X and Y be (Hausdorff) locally convex spaces with the strong topological duals X' and Y'. Let $A:X\to Y$ be a linear continuous operator and $B:Y\to X$ a continuous cross-section (i.e. a linear continuous right inverse) of A. Then A is a topological homomorphism (onto); its dual operator $A':Y'\to X'$ is a topological isomorphism of Y' onto the subspace $(\ker A)^\circ$ of X' orthogonal to the kernel of A, $\ker A=A^{-1}\{0\}$, and $B':X'\to Y'$ is a topological homomorphism of X' onto Y' which maps isomorphically $(\ker A)^\circ$ onto Y'.

operator E. First, $E\omega$ is the identity operator in $\mathscr{A}(\boldsymbol{R}^n)^G$. Consequently, E' maps isomorphically $(\mathscr{A}(\boldsymbol{R}^n)^G)'$ onto $(\ker E)^\circ$. Second, $(\ker E)^\circ$ coincides with $\mathscr{A}'(\boldsymbol{R}^n)^G$. Indeed, if $T\in(\ker E)^\circ$ then $T=E'\xi$ for some $\xi\in(\mathscr{A}(\boldsymbol{R}^n)^G)'$; this implies G-invariance of T. Conversely, if $T\in\mathscr{A}'(\boldsymbol{R}^n)^G$ then $(T,f)=(T,\sum_{g\in G}\lambda_g\cdot f\circ g)$, where $g\mapsto\lambda_g$ is an arbitrary function on G which is 0 at all $g\in G$ but a finite number and $\sum_g\lambda_g=1$. Approximating Ef [in the topology of $\mathscr{A}(\boldsymbol{R}^n)$] by the functions $\sum_g\lambda_g\cdot f\circ g$, we obtain $(T,f)=(T,Ef)$ which implies $T\in(\ker E)^\circ$.

Thus we proved the following lemma.

1.7. Lemma. *The dual $E':(\mathscr{A}(\boldsymbol{R}^n)^G)'\to\mathscr{A}'(\boldsymbol{R}^n)$ of the averaging operator E establishes a topological isomorphism of $(\mathscr{A}(\boldsymbol{R}^n)^G)'$ onto $\mathscr{A}'(\boldsymbol{R}^n)^G$.*

The central result of Section 1 is the following theorem.

1.8. Theorem. (i) *The operator $\mathscr{I}':\mathscr{A}'_{1/h}(\mathfrak{M})\ni\tau\mapsto\tau\circ I\in\mathscr{A}'(\boldsymbol{R}^n)$ establishes a topological isomorphism between $\mathscr{A}'_{1/h}(\mathfrak{M})$ and $\mathscr{A}'(\boldsymbol{R}^n)^G$.*

(ii) *The operator $(\omega\mathscr{I})':\mathscr{A}'(\boldsymbol{R}^n)\to\mathscr{A}'(\mathfrak{M})$* [defined by (1.16)–(1.18)] *is a topological homomorphism which maps isomorphically $\mathscr{A}'(\boldsymbol{R}^n)^G$ onto $\mathscr{A}'(\mathfrak{M})$.*

Proof. (i) By Theorem 1.6, part (ii), $\mathscr{I}'$ maps isomorphically $\mathscr{A}'_{1/h}(\mathfrak{M})$ onto the subspace $(\ker\mathscr{I})^\circ=(\ker E)^\circ$ in $\mathscr{A}'(\boldsymbol{R}^n)$. There remains the coincidence of $(\ker E)^\circ$ with $\mathscr{A}'(\boldsymbol{R}^n)^G$ to be shown. But $(\ker E)^\circ$ is the image of $(\mathscr{A}(\boldsymbol{R}^n)^G)'$ under E' (cf. Footnote [3]) which coincides with $\mathscr{A}'(\boldsymbol{R}^n)^G$ (by Lemma 1.7).

(ii) It suffices to show that there exists a continuous right inverse operator for $(\omega\mathscr{I})'$ which maps $\mathscr{A}'(\mathfrak{M})$ onto $\mathscr{A}'(\boldsymbol{R}^n)^G$. We claim that $E'(\mathscr{I}^{-1})'$ is such an operator. In fact, $E\omega=1$ implies $(\omega\mathscr{I})'E'(\mathscr{I}^{-1})'=\mathscr{I}'(E\omega)'(\mathscr{I}')^{-1}=1$. Moreover, Theorem 1.6, part (i) implies that $(\mathscr{I}^{-1})':\mathscr{A}'(\mathfrak{M})\to(\mathscr{A}(\boldsymbol{R}^n)^G)'$ is a topological isomorphism. Hence, by virtue of Lemma 1.7, $E'(\mathscr{I}^{-1})'$ maps isomorphically $\mathscr{A}'(\mathfrak{M})$ onto $\mathscr{A}'(\boldsymbol{R}^n)^G$. Q.E.D.

2. Covariant Functions and Distributions

We go on denoting by G a compact Lie group acting linearly on $\boldsymbol{R}^n$. Now we need to fix a linear representation $r:g\mapsto r_g$ of G in a finite dimensional complex vector space X, $d=\dim X$. Let X' be the dual of X and $\langle\xi',\xi\rangle$ the value of $\xi'\in X'$ at $\xi\in X$. Then G acts on X' according to r', the representation adjoint of r (i.e. $\langle r'_g\xi',\xi\rangle=\langle\xi',r_g^{-1}\xi\rangle$).

By $\mathscr{A}(\boldsymbol{R}^n;X)$ we denote the space of X-valued function on $\boldsymbol{R}^n$ of type $\mathscr{A}$ $(=\mathscr{E},\mathscr{D}$, or $\mathscr{S})$. If $\{b_1,\dots,b_d\}$ is a basis in X then each $f\in\mathscr{A}(\boldsymbol{R}^n;X)$ is representable uniquely in the form

$$f=\textstyle\sum_{\delta=1}^d b_\delta\cdot f_\delta \quad\text{with}\quad f_\delta\in\mathscr{A}(\boldsymbol{R}^n).$$

In particular, if $X=\boldsymbol{C}^d$, the d-dimensional complex Euclidean space, then $\mathscr{A}(\boldsymbol{R}^n;X)$ is nothing but $\bigoplus^d\mathscr{A}(\boldsymbol{R}^n)$. The correspondence $(f_1,\dots,f_d)\mapsto\sum_\delta b_\delta\cdot f_\delta$ is a linear isomorphism of the direct sum $\bigoplus^d\mathscr{A}(\boldsymbol{R}^n)$ onto $\mathscr{A}(\boldsymbol{R}^n;X)$. The direct sum topology of $\bigoplus^d\mathscr{A}(\boldsymbol{R}^n)$ induces (via this isomorphism) a topology on $\mathscr{A}(\boldsymbol{R}^n;X)$ (independent of a choice of the basis in X).

Similarly, $\mathscr{A}'(\boldsymbol{R}^n;X')$ is the space of X'-valued distributions on $\boldsymbol{R}^n$ of type $\mathscr{A}'$, so that, for a basis $\{b'_1,\dots,b'_d\}$ in X', the assignment to each $(T_1,\dots,T_d)\in\bigoplus^d\mathscr{A}'(\boldsymbol{R}^n)$

of the element $T=\sum_\delta b'_\delta\cdot T_\delta$ is a linear isomorphism of $\bigoplus^d \mathscr{A}'(\boldsymbol{R}^n)$ onto $\mathscr{A}'(\boldsymbol{R}^n;X')$. The direct sum topology of $\bigoplus^d \mathscr{A}'(\boldsymbol{R}^n)$ induces a topology on $\mathscr{A}'(\boldsymbol{R}^n;X')$. Since $\bigoplus^d \mathscr{A}'(\boldsymbol{R}^n)$ is canonically isomorphic with $(\bigoplus^d \mathscr{A}(\boldsymbol{R}^n))'$, we may consider $\mathscr{A}'(\boldsymbol{R}^n;X')$ as a strong topological dual of $\mathscr{A}(\boldsymbol{R}^n;X)$. The (basis independent) duality of $\mathscr{A}'(\boldsymbol{R}^n;X')$ and $\mathscr{A}(\boldsymbol{R}^n;X)$ is defined by the bilinear form

$$(T,f)\equiv(\textstyle\sum_\delta b'_\delta\cdot T_\delta, \sum_\varepsilon b_\varepsilon f_\varepsilon)=\sum_{\delta,\varepsilon}\langle b'_\delta, b_\varepsilon\rangle\cdot(T_\delta,f_\varepsilon)$$

for $T\in\mathscr{A}'(\boldsymbol{R}^n;X')$, $f\in\mathscr{A}(\boldsymbol{R}^n;X)$.

For a linear non-degenerate transformation L in $\boldsymbol{R}^n$, a linear operator A in X (resp. in X') and each element f of $\mathscr{A}(\boldsymbol{R}^n;X)$ [resp. of $\mathscr{A}'(\boldsymbol{R}^n;X')$], the composition $A\circ f\circ L\in\mathscr{A}(\boldsymbol{R}^n;X)$ [resp. $\mathscr{A}'(\boldsymbol{R}^n;X')$] is defined naturally. In particular, the (right) actions of G on X-valued functions and X'-valued distributions are defined:

$$(g,f)\mapsto r_g^{-1}\circ f\circ g;\,(g,T)\mapsto r_g'^{-1}\circ T\circ g\quad\text{for}\quad g\in G, f\in\mathscr{A}(\boldsymbol{R}^n;X), T\in\mathscr{A}'(\boldsymbol{R}^n;X')$$

[so that the form (T,f) is invariant]. By $\mathscr{A}(\boldsymbol{R}^n;X)^G$ [resp. $\mathscr{A}'(\boldsymbol{R}^n;X')^G$] we denote the subspace of all $f\in\mathscr{A}(\boldsymbol{R}^n;X)$ [resp. $T\in\mathscr{A}'(\boldsymbol{R}^n;X')$], called *G-covariant*, which satisfy $r_g^{-1}\circ f\circ g=f$ [resp. $r_g'^{-1}\circ T\circ g=T$] for all $g\in G$. Similarly the totality $\mathscr{P}(\boldsymbol{R}^n;X)^G$ [resp. $\mathscr{P}(\boldsymbol{R}^n;X')^G$] of X-valued (resp. X'-valued) G-covariant polynomials on $\boldsymbol{R}^n$ is defined.

Our treatment of G-covariant functions and distributions is based on the following condition which is assumed throughout Section 2.

Condition II. There exists a family $\{Q_1,\dots,Q_N\}$ of G-covariant polynomials of $\mathscr{P}(\boldsymbol{R}^n;X)$ such that an arbitrary element $P\in\mathscr{P}(\boldsymbol{R}^n;X)^G$ can be represented uniquely in the form

$$P=\textstyle\sum_{\nu=1}^N Q_\nu\cdot p_\nu\,, \tag{2.1}$$

where $(p_1,\dots,p_N)$ is an N-tuple of polynomials of $\mathscr{P}(\boldsymbol{R}^n)^G$.

In other words, the mapping

$$\textstyle\bigoplus^N \mathscr{P}(\boldsymbol{R}^n)^G\ni(p_1,\dots,p_N)\mapsto\sum_\nu Q_\nu\cdot p_\nu\in\mathscr{P}(\boldsymbol{R}^n;X)^G \tag{2.2}$$

is a linear isomorphism (onto).

Condition II has its counterpart in terms of $\mathscr{P}(\boldsymbol{R}^n;X')^G$. Indeed, there exists a G-invariant (positive definite) inner product $X\times X\ni(\xi,\eta)\mapsto\langle\!\langle\xi,\eta\rangle\!\rangle$ (anti-linear with respect to η) on X. It defines a (canonical) G-invariant antilinear isomorphism $\Theta:X\to X'$ by $\langle\Theta\eta,\xi\rangle=\langle\!\langle\xi,\eta\rangle\!\rangle$ for $\xi,\eta\in X$. Therefore Q'_ν defined by

$$Q'_\nu(x)=\Theta Q_\nu(x) \tag{2.3}$$

are G-covariant polynomials of $\mathscr{P}(\boldsymbol{R}^n;X')^G$. Now, if we replace X by X' and Q_ν by Q'_ν, Condition II remains to hold. This remark gains a symmetry between X and X' in Condition II.

The uniqueness aspect of the representation (2.1) admits several equivalent formulations. We have

2.1. Lemma. *For an arbitrary N-tuple* $(P_1,\dots,P_N)$ *of elements of* $\mathscr{P}(\boldsymbol{R}^n;X)^G$, *the following properties are equivalent:*

a) *the mapping* $\bigoplus^N \mathscr{P}(\boldsymbol{R}^n)^G\ni(p_1,\dots,p_N)\mapsto\sum_\nu P_\nu\cdot p_\nu\in\mathscr{P}(\boldsymbol{R}^n;X)^G$ *is injective;*

b) *the mapping* $\bigoplus^N \mathscr{P}(\boldsymbol{R}^n)\ni(p_1,\dots,p_N)\mapsto\sum_\nu P_\nu\cdot p_\nu\in\mathscr{P}(\boldsymbol{R}^n;X)$ *is injective;*

c) $P_1(x), \ldots, P_N(x)$ *are linearly independent (as vectors of* X*) at least at one point* $x \in \boldsymbol{R}^n$.

Proof. Assuming a), let us prove b). We must show that $\sum_\nu P_\nu \cdot p_\nu = 0$ with $p_1, \ldots, p_N \in \mathscr{P}(\boldsymbol{R}^n)$ implies $p_\lambda = 0$ for all λ. For this purpose we multiply the equation $\sum_\nu P_\nu \cdot (p_\nu \circ g) = 0$ (which holds since P_ν are G-covariant) with $\bar{p}_\lambda \circ g$ and integrate over G. Then [by a)] the coefficient at P_λ is 0, that is,

$$\int |p_\lambda|^2 \circ g \, d\mu(g) = 0\,,$$

which implies $p_\lambda = 0$. This proves a)$\Rightarrow$b). Further, assuming b), let us prove c). We consider the totality $\mathscr{R}(\boldsymbol{R}^n; X)$ of X-valued rational functions on $\boldsymbol{R}^n$ as a N-dimensional vector space over the field $\mathscr{R}(\boldsymbol{R}^n)$ of complex rational functions on $\boldsymbol{R}^n$. Then condition b) means that $P_1, \ldots, P_N$ [as elements of $\mathscr{R}(\boldsymbol{R}^n; X)$] are linearly independent over the field $\mathscr{R}(\boldsymbol{R}^n)$. Introducing the polynomials $P_{\nu\delta}$ by $P_\nu = \sum_{\delta=1}^d b_\delta \cdot P_{\nu\delta}$ with $\{b_1, \ldots, b_d\}$ a basis in X, we obtain that at least one of the $N \times N$ minors of the matrix $(P_{\nu\delta})$ as a non-zero element of $\mathscr{R}(\boldsymbol{R}^n)$, hence it is a polynomial on $\boldsymbol{R}^n$ which does not equal identically 0. This proves b)$\Rightarrow$c). At last, if c) holds then $P_1(x), \ldots, P_N(x)$ are linearly independent for all x in an open dense subset of $\boldsymbol{R}^n$. Therefore c) implies a). Q.E.D.

First of all, we will extend the representation of type (2.1) for covariant functions and thus obtain a description of $\mathscr{A}(\boldsymbol{R}^n; X)^G$ [see below Theorem 2.2, part (i)]. In order to develop an analogous decomposition for covariant distributions, we need an alternative description of $\mathscr{A}(\boldsymbol{R}^n; X)^G$. Let us assign to each function $f \in \mathscr{A}(\boldsymbol{R}^n; X)$ the N-tuple $u \equiv (u_1, \ldots, u_N)$ of functions $u_\nu \in \mathscr{A}(\boldsymbol{R}^n)$, $u_\nu(x) = \langle Q'_\nu(x), f(x) \rangle$. Thus, a linear continuous operator

$$\mathscr{Q}\colon \mathscr{A}(\boldsymbol{R}^n; X) \ni f \mapsto (\langle Q'_1, f \rangle, \ldots, \langle Q'_N, f \rangle) \in \boldsymbol{A} \tag{2.4}$$

from $\mathscr{A}(\boldsymbol{R}^n; X)$ onto a certain subspace $\boldsymbol{A}$ ($= \boldsymbol{E}$, $\boldsymbol{D}$ or $\boldsymbol{S}$) of $\bigoplus^N \mathscr{A}(\boldsymbol{R}^n)$ is defined. Assuming that G acts on $\boldsymbol{A}$ by $(u \circ g)_\nu = u_\nu \circ g$, we denote by $\boldsymbol{A}^G$ the subspace of G-invariant elements of $\boldsymbol{A}$. Now we have

2.2. Theorem. (i) *The mapping*

$$\bigoplus^N \mathscr{A}(\boldsymbol{R}^n)^G \ni (f_1, \ldots, f_N) \mapsto \sum_\nu Q_\nu \cdot f_\nu \in \mathscr{A}(\boldsymbol{R}^n; X)^G \tag{2.5}$$

is a topological isomorphism (onto).

(ii) *The operator* $\mathscr{Q}$ (2.4) *maps isomorphically* $\mathscr{A}(\boldsymbol{R}^n; X)^G$ *onto the closed subspace* $\boldsymbol{A}^G$ *of* $\bigoplus^N \mathscr{A}(\boldsymbol{R}^n)^G$.

Proof. (i) Denote the operator (2.5) by $\mathscr{K}$. First, we will show that im $\mathscr{K}$, the image of $\bigoplus^N \mathscr{A}(\boldsymbol{R}^n)^G$ under $\mathscr{K}$, is dense in $\mathscr{A}(\boldsymbol{R}^n; X)^G$. Consider the totality $\mathscr{N}$ of functions of $\mathscr{A}(\boldsymbol{R}^n; X)^G$ of the form $P(x)v(\|x\|^2)$ with $P \in \mathscr{P}(\boldsymbol{R}^n; X)^G$ and $v \in \mathscr{D}(\boldsymbol{R})$. By Condition II, $\mathscr{N} \subset \operatorname{im} \mathscr{K}$, therefore it suffices to show that $\mathscr{N}$ is dense in $\mathscr{A}(\boldsymbol{R}^n; X)^G$. By the Weierstrass approximation theorem, each $f \in \mathscr{A}(\boldsymbol{R}^n; X)^G$ can be approximated with an arbitrary accuracy [in the topology of $\mathscr{A}(\boldsymbol{R}^n; X)$] by functions of the form $Q(x)v(\|x\|^2)$ with $Q \in \mathscr{P}(\boldsymbol{R}^n; X)$, $v \in \mathscr{D}(\boldsymbol{R})$. Replacing here Q by $P = \int r_g^{-1} \circ Q \circ g \, d\mu(g)$, we obtain that f belongs to the closure of $\mathscr{N}$.

Second, we will prove that im $\mathscr{K}$ is closed in $\mathscr{A}(\boldsymbol{R}^n; X)^G$. We introduce the $N \times d$ polynomial matrix $(Q_{\nu\delta})$ by $Q_\nu(x) = \sum_\delta b_\delta \cdot Q_{\nu\delta}(x)$, where $\{b_1, \ldots, b_d\}$ is a basis in X. Its rank equals N at least at one point of $\boldsymbol{R}^n$ (by virtue of Lemma 2.1). Applying Theorem of Appendix B, we obtain that the operator

$$\bigoplus^N \mathscr{A}(\boldsymbol{R}^n) \ni (f_1, \dots, f_N) \mapsto \left(\sum_{\nu=1}^N Q_{\nu 1} f_\nu, \dots, \sum_{\nu=1}^N Q_{\nu d} \cdot f_\nu\right) \in \bigoplus^d \mathscr{A}(\boldsymbol{R}^n)$$

maps isomorphically $\bigoplus^N \mathscr{A}(\boldsymbol{R}^n)^G$ onto a closed subspace of $\bigoplus^d \mathscr{A}(\boldsymbol{R}^n)$. This proves (i).

(ii) We introduce the projection operator $E_r : \mathscr{A}(\boldsymbol{R}^n; X) \to \mathscr{A}(\boldsymbol{R}^n; X)^G$:

$$E_r f = \int r_g^{-1} \circ f \circ g \, d\mu(g), \qquad f \in \mathscr{A}(\boldsymbol{R}^n; X) \tag{2.6}$$

(which generalizes the averaging operator E of Section 1). It is easily seen that $\mathscr{Q}$ satisfies the relation $\mathscr{Q} E_r = (\bigoplus^N E)\mathscr{Q}$, which implies $\mathscr{Q}\mathscr{A}(\boldsymbol{R}^n; X)^G = (\bigoplus^N E)\boldsymbol{A}$. Since the operator $\bigoplus^N E$ (the direct sum of N copies of the operator E) maps $\boldsymbol{A}$ into $\boldsymbol{A}^G$ and equals the identity on $\boldsymbol{A}^G \subset \boldsymbol{A}$, it follows that $(\bigoplus^N E)\boldsymbol{A} = \boldsymbol{A}^G$. Consequently $\mathscr{Q}$ maps $\mathscr{A}(\boldsymbol{R}^n; X)^G$ onto $\boldsymbol{A}^G$. Recalling part (i) of the theorem, it suffices to show that the operator $\mathscr{Q}\mathscr{K}$ is a topological isomorphism of $\bigoplus^N \mathscr{A}(\boldsymbol{R}^n)^G$ onto a closed subspace of $\bigoplus^N \mathscr{A}(\boldsymbol{R}^n)^G$ (in fact, onto $\boldsymbol{A}^G$). This operator is expressed by

$$(f_1, \dots, f_N) \mapsto \left(\sum_{\lambda=1}^N q_{1\lambda} \cdot f_\lambda, \dots, \sum_{\lambda=1}^N q_{N\lambda} \cdot f_\lambda\right)$$

with $q_{\nu\lambda}(x) = \langle Q'_\nu(x), Q_\lambda(x) \rangle$ which form a $N \times N$ matrix of polynomial functions. By virtue of (2.3), $q_{\nu\lambda}(x) = \langle\!\langle Q_\lambda(x), Q_\nu(x) \rangle\!\rangle$. Since the vectors $Q_1(x), \dots, Q_N(x)$ are linearly independent at least at one point x, the matrix $(q_{\nu\lambda}(x))$ is non-degenerate at such x. Now theorem of Appendix B yields the desired result. Q.E.D.

In order to apply duals of the isomorphisms in Theorem 2.2 to G-covariant distributions, we need to identify $(\mathscr{A}(\boldsymbol{R}^n; X)^G)'$ and $(\boldsymbol{A}^G)'$ with certain subspaces of $\mathscr{A}'(\boldsymbol{R}^n; X')$ and $\boldsymbol{A}'$, respectively. Along the line of Section 1, we introduce the embedding $\omega_r : \mathscr{A}(\boldsymbol{R}^n; X)^G \to \mathscr{A}(\boldsymbol{R}^n; X)$ (which is a continuous cross-section of E_r). Then we have the following analogue of Lemma 1.7 (the proof being essentially the same).

2.3. Lemma. *$E'_r : (\mathscr{A}(\boldsymbol{R}^n; X)^G)' \to \mathscr{A}'(\boldsymbol{R}^n; X')$ establishes a topological isomorphism of $(\mathscr{A}(\boldsymbol{R}^n; X)^G)'$ onto $\mathscr{A}'(\boldsymbol{R}^n; X')^G$, and $\omega'_r : \mathscr{A}'(\boldsymbol{R}^n; X') \to (\mathscr{A}(\boldsymbol{R}^n; X)^G)'$ maps isomorphically $\mathscr{A}'(\boldsymbol{R}^n; X')^G$ onto $(\mathscr{A}(\boldsymbol{R}^n; X)^G)'$.*

Similarly, let

$$e : \boldsymbol{A} \to \boldsymbol{A}^G \tag{2.7}$$

be the operator obtained by restricting $\bigoplus^N E : \bigoplus^N \mathscr{A}(\boldsymbol{R}^n) \to \bigoplus^N \mathscr{A}(\boldsymbol{R}^n)^G$ [E being defined by (1.12)]. There exists a continuous cross-section of e, e.g. the embedding $\varepsilon : \boldsymbol{A}^G \to \boldsymbol{A}$. Consequently, e' maps isomorphically $(\boldsymbol{A}^G)'$ onto $(\ker e)^\circ$ (cf. Footnote [3]). As in the proof of Lemma 1.7, one deduces easily that $(\ker e)^\circ = \boldsymbol{A}'^G$. Thus, we obtain

2.4. Lemma. *$e' : (\boldsymbol{A}^G)' \to \boldsymbol{A}'$ maps isomorphically $(\boldsymbol{A}^G)'$ onto $\boldsymbol{A}'^G$, and $\varepsilon' : \boldsymbol{A}' \to (\boldsymbol{A}^G)'$ maps isomorphically $\boldsymbol{A}'^G$ onto $(\boldsymbol{A}^G)'$.*

We now turn to G-covariant distributions. It is obvious that $\mathscr{Q}'$, the dual of $\mathscr{Q}$ (2.4), assigns to each $t \in \boldsymbol{A}'^G$ a G-covariant distribution $\mathscr{Q}' t = t \circ \mathscr{Q} \in \mathscr{A}'(\boldsymbol{R}^n; X')^G$ (the composite of the operator $\mathscr{Q}$ with the linear form t). We will show that this correspondence is an isomorphism. The representation of an arbitrary $T \in \mathscr{A}'(\boldsymbol{R}^n; X')^G$,

$$T = \mathscr{Q}'t \quad \text{with} \quad t \in A'^G, \tag{2.8}$$

so obtained, is an analogue of the above decomposition of G-covariant functions in standard covariants.

In an alternative description analogous of Theorem 2.2, part (ii), we assign to each $T \in \mathscr{A}'(\boldsymbol{R}^n; X')^G$ an N-tuple $s \equiv (s_1, \ldots, s_N)$ of distributions $s_\nu = \langle T, Q_\nu \rangle \in \mathscr{A}'(\boldsymbol{R}^n)^G$ defined by

$$(s_\nu, v) = (T, Q_\nu \cdot v) \quad \text{for all} \quad v \in \mathscr{A}(\boldsymbol{R}^n).$$

By this means we obtain the following characterizations of G-covariant distributions.

2.5. Theorem. (i) *The assignment* (2.8) *(to each $t \in A'^G$ of a distribution T) is a topological isomorphism of A'^G onto $\mathscr{A}'(\boldsymbol{R}^n; X')^G$.*

(ii) *The correspondence*

$$T \mapsto (\langle T, Q_1 \rangle, \ldots, \langle T, Q_N \rangle) \tag{2.9}$$

is a topological isomorphism of $\mathscr{A}'(\boldsymbol{R}^n; X')^G$ onto $\bigoplus^N \mathscr{A}'(\boldsymbol{R}^n)^G$.

Proof. (i) According to Theorem 2.2, part (ii), the restriction $\hat{\mathscr{Q}}: \mathscr{A}(\boldsymbol{R}^n; X)^G \to A^G$ of $\mathscr{Q}$ is an isomorphism. Recalling Lemmas 2.3 and 2.4, $E'_r \hat{\mathscr{Q}}' \varepsilon'$ maps isomorphically A'^G onto $\mathscr{A}'(\boldsymbol{R}^n; X')^G$. Let T be the image of some $t \in A'^G$ under this isomorphism. It suffices to show that $T = \mathscr{Q}'t$. In fact, it is obvious that the two distributions T and $\mathscr{Q}'t$ of $\mathscr{A}'(\boldsymbol{R}^n; X')^G$ coincide on $\mathscr{A}(\boldsymbol{R}^n; X)^G$. Since ω'_r maps isomorphically $\mathscr{A}'(\boldsymbol{R}^n; X')^G$ onto $(\mathscr{A}(\boldsymbol{R}^n; X)^G)'$ (according to Lemma 2.3), this implies that T and $\mathscr{Q}'t$ are equal.

(ii) Let $\mathscr{M}: \mathscr{A}'(\boldsymbol{R}^n; X')^G \to \bigoplus^N \mathscr{A}'(\boldsymbol{R}^n)^G$ denote the operator (2.9). It suffices to show that $(\bigoplus^N \omega') \mathscr{M} E'_r$ maps isomorphically $(\mathscr{A}(\boldsymbol{R}^n; X)^G)'$ onto $\bigoplus^N (\mathscr{A}(\boldsymbol{R}^n)^G)'$ [since ω', the dual of the embedding $\omega: \mathscr{A}(\boldsymbol{R}^n)^G \to \mathscr{A}(\boldsymbol{R}^n)$, maps isomorphically $\mathscr{A}'(\boldsymbol{R}^n)^G$ onto $(\mathscr{A}(\boldsymbol{R}^n)^G)'$, and E'_r maps isomorphically $(\mathscr{A}(\boldsymbol{R}^n; X)^G)'$ onto $\mathscr{A}'(\boldsymbol{R}^n; X')^G$]. But $(\bigoplus^N \omega') \mathscr{M} E'_r \Phi = \mathscr{K}' \Phi$ for all $\Phi \in (\mathscr{A}(\boldsymbol{R}^n; X)^G)'$, where $\mathscr{K}': (\mathscr{A}(\boldsymbol{R}^n; X)^G)' \to (\bigoplus^N \mathscr{A}(\boldsymbol{R}^n)^G)' = \bigoplus^N (\mathscr{A}(\boldsymbol{R}^n)^G)'$ is the dual of the isomorphism $\mathscr{K}$ (2.5). This completes the proof. Q.E.D.

2.6. Corollary. *Each $T \in A'(\boldsymbol{R}^n; X')^G$ is representable in the form*

$$T = \textstyle\sum_{\nu=1}^N Q'_\nu \cdot T_\nu \tag{2.10}$$

with G-invariant distributions $T_1, \ldots, T_N \in \mathscr{A}'(\boldsymbol{R}^n)^G$.

Proof. We represent T in the form (2.8). The functional $\varepsilon' t$ (see Lemma 2.4) can be extended (by the Hahn-Banach theorem) to a functional $(\tilde{T}_1, \ldots, \tilde{T}_N)$ of $(\bigoplus^N \mathscr{A}(\boldsymbol{R}^n)^G)' = \bigoplus^N (\mathscr{A}(\boldsymbol{R}^n)^G)'$. Applying $\bigoplus^N E'$ to $(\tilde{T}_1, \ldots, \tilde{T}_N)$, we obtain (by virtue of Lemma 1.7) an N-tuple $(T_1, \ldots, T_N) \in \bigoplus^N \mathscr{A}'(\boldsymbol{R}^n)^G$. With such $T_1, \ldots, T_N$ both sides of (2.10) are G-covariant distributions of $\mathscr{A}'(\boldsymbol{R}^n; X')^G$, which coincide on $\mathscr{A}(\boldsymbol{R}^n; X)^G$ (by construction of $T_1, \ldots, T_N$). By Lemma 2.3, they coincide on the whole $\mathscr{A}(\boldsymbol{R}^n; X)$. Q.E.D.

It should be noted however that in general the decomposition (2.10) is non-unique (i.e. $T_1, \ldots, T_N$ are not defined by T and $Q'_1, \ldots, Q'_N$). Hence, in contradistinction with the representation (2.8), it does not provide an adequate description of G-covariant distributions. Only formula (2.8) may serve as an adequate version for decomposition of covariant distributions in standard covariants.

The last problem we discuss here is to express the G-invariant functionals, entering into covariant decompositions of distributions, in terms of invariants. Assuming the validity of Condition I (Section 1), any $T \in \mathscr{A}'(\boldsymbol{R}^n; X')^G$ admits the representation (which follows by combining Corollary 2.6 with Theorem 1.8):

$$T = \sum_{\nu=1}^{N} Q'_\nu \cdot (\tau_\nu \circ I) \quad \text{with} \quad \tau_1, \ldots, \tau_N \in \mathscr{A}'_{1/h}(\mathfrak{M}). \tag{2.11}$$

Just as (2.10), in general such a representation is non-unique. For an adequate version, we will find a representation of each $t \in A'^G$ in the form

$$t = \mathrm{t} \circ I, \tag{2.12}$$

or, equivalently, a representation of $T \in \mathscr{A}'(\boldsymbol{R}^n; X')^G$ of the type

$$(T, f) = (\mathrm{t} \circ I, \mathscr{Q} f) \quad \text{for all} \quad f \in \mathscr{A}(\boldsymbol{R}^n; X). \tag{2.13}$$

A precise meaning to formula (2.13) can be given along the line of formulas (1.7), (1.8). Namely, we rewrite (2.13) in the form

$$(T, f) = (\mathrm{t}, (\textstyle\bigoplus^N \mathscr{J}) \mathscr{Q} f) \quad \text{for all} \quad f \in \mathscr{A}(\boldsymbol{R}^n; X) \tag{2.14}$$

with $\mathscr{J} : \mathscr{A}(\boldsymbol{R}^n) \to \mathscr{A}_h(\mathfrak{M})$ defined by (1.14). Let $\mathfrak{A}$ ($=\mathfrak{E}$, $\mathfrak{D}$ or $\mathfrak{S}$) denote the totality of N-tuples $(\chi_1, \ldots, \chi_N)$ of functions $\chi_\nu = \mathscr{J}\langle Q'_\nu, f\rangle \in \mathscr{A}_h(\mathfrak{M})$ with $f \in \mathscr{A}(\boldsymbol{R}^n; X)$; in other terms, $\mathfrak{A}$ is the image of A (or, equivalently, of A^G) under the operator $\bigoplus^N \mathscr{J} : \bigoplus^N \mathscr{A}(\boldsymbol{R}^n) \to \bigoplus^N \mathscr{A}_h(\mathfrak{M})$. Then formulas (2.13) and (2.14) [and also (2.12)] are meaningful provided that t is assumed to be a linear continuous functional on $\mathfrak{A}$. Let $\hat{\mathscr{Q}} : \mathscr{A}(\boldsymbol{R}^n; X)^G \to A^G$ and $\hat{\mathscr{J}} : A^G \to \mathfrak{A}$ denote the restrictions of the operators $\mathscr{Q}$ and $\bigoplus^N \mathscr{J}$, respectively. They are isomorphisms (according to Theorems 2.2 and 1.6). In view of the formula $(\bigoplus^N \mathscr{J}) \mathscr{Q} = \hat{\mathscr{J}} \hat{\mathscr{Q}} E_r$, we can rewrite (2.14) as

$$T = (\hat{\mathscr{J}} \hat{\mathscr{Q}} E_r)' \mathrm{t} = E'_r \hat{\mathscr{Q}}' \hat{\mathscr{J}}' \mathrm{t}. \tag{2.15}$$

Recalling Lemma 2.3, this implies the desired result:

2.7. Theorem. *Let both Conditions* I *and* II *be fulfilled. Then the assignment* (2.13) *(to each* $\mathrm{t} \in \mathfrak{A}'$ *of a distribution* T*) is a topological isomorphism of* $\mathfrak{A}'$ *onto* $\mathscr{A}'(\boldsymbol{R}^n; X')^G$.

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Appendix A. Testing Functions and Distributions on Closed Regular Sets in $\boldsymbol{R}^k$

Normally the Schwartz spaces of functions and distributions ([7]) are defined on open sets in $\boldsymbol{R}^k$. Here we present an exposition on testing functions and distributions on closed sets in $\boldsymbol{R}^k$. We obtain several characterizations of spaces of testing functions on the sets satisfying certain regularity conditions. This result provides a deeper insight into the construction of the spaces appearing in Section 1.

In what follows Ω stands for a closed set in $\boldsymbol{R}^k$. We make use of the standard notations

$$D^\alpha = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_k)^{\alpha_k}, \qquad x^\alpha = x_1^{\alpha_1} \dots x_k^{\alpha_k},$$
$$|\alpha| = \alpha_1 + \dots + \alpha_k, \qquad \alpha! = \alpha_1! \dots \alpha_k!$$

for any $\alpha \in \boldsymbol{Z}_+^k$, $\boldsymbol{Z}_+^k$ being the totality of k-tuples $\alpha \equiv (\alpha_1, \dots, \alpha_k)$ of non-negative integers. By $\mathscr{T}(\Omega)$ we denote the totality of ∞-jets on Ω, i.e. the totality of families $f \equiv (f_\alpha)_{\alpha \in \boldsymbol{Z}_+^k}$ of complex continuous functions f_α on Ω. $\mathscr{T}(\Omega)$ is a Frechét space in the topology of seminorms

$$|f|_l^K = \sup_{|\alpha| \leq l} \sup_{x \in K} |f_\alpha(x)|,$$

where $l \in \boldsymbol{Z}_+$, and K is an arbitrary compact subset in Ω. For $s \in \boldsymbol{Z}_+$, $\alpha \in \boldsymbol{Z}_+^k$, $f \in \mathscr{T}(\Omega)$, we define the functions $R_\alpha^s f$ on $\Omega \times \Omega$ by setting $(R_\alpha^s f)(x, x) = 0$ and

$$(R_\alpha^s f)(x, y) = \|x - y\|^{-s}\left(f_\alpha(x) - \textstyle\sum_{|\beta| \leq s} \beta!^{-1} f_{\alpha+\beta}(y)(x-y)^\beta\right) \quad \text{for} \quad x \neq y.$$

Now we define spaces of the type $\mathscr{E}$, $\mathscr{D}$, and $\mathscr{S}$ on Ω.

A.1. Definition (cf. [9], Chapter I). A jet $f \in \mathscr{T}(\Omega)$ is said to be a $\mathscr{C}^\infty$ function on Ω in the sense of Whitney if all the functions $R_\alpha^s f$ are continuous[4]. The totality of such "functions", denoted by $\mathscr{E}(\Omega)$, forms a Frechét space with the seminorms

$$\|f\|_l^K = |f|_l^K + \sup_{|\alpha| \leq l} \sup_{s \leq l} \sup_{x, y \in K} |(R_\alpha^s f)(x, y)|;$$

here $l \in \boldsymbol{Z}_+$, and K is an arbitrary compact in Ω. $\mathscr{D}(\Omega)$ is defined to be the inductive limit of the (closed) subspaces in $\mathscr{E}(\Omega)$ indexed by compact sets $K \subset \Omega$ and formed by all elements $f \in \mathscr{E}(\Omega)$ with support in K. Further, $\mathscr{S}(\Omega)$ is the totality of $f \in \mathscr{E}(\Omega)$ for which all the seminorms (with $l, m \in \boldsymbol{Z}_+$)

$$\|f\|_{l,m} = \sup_{|\alpha| \leq l} \sup_{x \in \Omega} (1 + \|x\|)^m |f_\alpha(x)| + \sup_{\substack{|\alpha| \leq l \\ s \leq l}} \sup_{\substack{x, y \in \Omega \\ \|x - y\| \leq 1}} (1 + \|x\|)^m |(R_\alpha^s f)(x, y)|$$

are finite. $\mathscr{S}(\Omega)$ is a Frechét space in the topology of these seminorms.

In what follows we use the notation $\mathscr{A}(\Omega)$ for denoting either of the spaces $\mathscr{E}(\Omega)$, $\mathscr{D}(\Omega)$, $\mathscr{S}(\Omega)$. It is clear that $\mathscr{E}(\boldsymbol{R}^k)$ can be naturally identified with the space of all complex $\mathscr{C}^\infty$ functions on $\boldsymbol{R}^k$. In this sense $\mathscr{A}(\Omega)$ generalizes the Schwartz space $\mathscr{A}(\boldsymbol{R}^k)$. By analogy, we define the space $\mathscr{A}'(\Omega)$ (of distributions on Ω of type $\mathscr{A}' = \mathscr{E}'$, $\mathscr{D}'$ or $\mathscr{S}'$) as the strong topological dual of $\mathscr{A}(\Omega)$.

Relations from the spaces so defined and the Schwartz spaces on $\boldsymbol{R}^k$ are evident from the following implication of the Whitney continuation theorem [9].

A.2. Proposition. *The restriction operator* $j: \mathscr{A}(\boldsymbol{R}^k) \to \mathscr{A}(\Omega)$ *is a (surjective) topological homomorphism; its dual,* j', *is a topological isomorphism of* $\mathscr{A}'(\Omega)$ *onto the subspace* $\mathscr{A}'(\boldsymbol{R}^k, \Omega)$ *of distributions of* $\mathscr{A}'(\boldsymbol{R}^k)$ *with support in* Ω.

We identify $\mathscr{A}'(\Omega)$ and $\mathscr{A}'(\boldsymbol{R}^k, \Omega)$ via the isomorphism j'.

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[4] This is equivalent to the condition that each $R_\alpha^s f$ is locally bounded in $\Omega \times \Omega$. (This can be seen by expressing $R_\alpha^s f$ in terms of $R_\alpha^{s+1} f$ and the $f_{\alpha+\beta}$ with $|\beta| = s + 1$.)

Proof. According to the $\mathscr{C}^\infty$ version of the Whitney extension theorem ([9], Theorem 4.1 of Chapter I[5]) j is a continuous surjection and hence (by virtue of the open mapping theorem) is a topological homomorphism. Hence $j':\mathscr{A}'(\Omega)\to\mathscr{A}'(\boldsymbol{R}^k)$ is a continuous bijection of $\mathscr{A}'(\Omega)$ onto the closed subspace $(\ker j)^\circ$ of $\mathscr{A}'(\boldsymbol{R}^k)$ orthogonal to $\ker j=j^{-1}\{0\}\subset\mathscr{A}(\boldsymbol{R}^k)$. The open mapping theorem is applicable to continuous operators from $\mathscr{A}'(\Omega)$ onto $(\ker j)^\circ$ (see [11]), which implies that j' maps isomorphically $\mathscr{A}'(\Omega)$ onto $(\ker j)^\circ$. There remains the equality $(\ker j)^\circ=\mathscr{A}'(\boldsymbol{R}^k,\Omega)$ to be proved. The inclusion $(\ker j)^\circ\subset\mathscr{A}'(\boldsymbol{R}^k,\Omega)$ is trivial. Conversely, by definition of support, $\mathscr{A}'(\boldsymbol{R}^k,\Omega)$ is orthogonal to functions of $\mathscr{A}(\boldsymbol{R}^k)$ which equal zero in a neighborhood of Ω. The totality of such functions is dense in $\ker j$ (see e.g. [9], proof of Lemma 4.3 of Chapter I), which implies $\mathscr{A}'(\boldsymbol{R}^k,\Omega)\subset(\ker j)^\circ$. Q.E.D.

By definition, $\mathscr{E}(\Omega)$ is a linear subspace in $\mathscr{T}(\Omega)$, and its topology is in general finer than that induced from $\mathscr{T}(\Omega)$. Of interest are the sets Ω for which the two topologies coincide. Certain regularity property of Ω is sufficient for this purpose. Below we present two forms, (R) and $(R)'$, of the regularity condition (cf. [12], § 3.5 and 3.8); the strengthened form, $(R)'$, is adapted also to formulating an analogue for $\mathscr{S}(\Omega)$.

A.3. Definition. A closed set Ω in $\boldsymbol{R}^k$ is said to possess the *regularity property* (R) if 1) each compact subset of Ω intersects only finitely many connected components of Ω, and 2) for each compact subset $K\subset\Omega$ contained in a connected component of Ω there exist numbers $a,\lambda>0$ such that any two points $x,y\in K$ can be connected in Ω by a continuous rectifiable curve of length $\leqq a\|x-y\|^\lambda$.

A.4. Definition. A closed subset Ω in $\boldsymbol{R}^k$ is said to possess the *regularity property* (R)′ if there exist numbers $a,p,d,\lambda,\varepsilon>0$ such that 1) for all $\varrho>0$, the set $K_\varrho=\{x\in\boldsymbol{R}^k\,|\,\|x\|\leqq\varrho\}$ intersects only finitely many connected components of Ω, and distances between the parts of K_ϱ contained in different components of Ω are $\geqq d(1+\varrho)^{-\varepsilon}$, 2) any two points x,y in the same component of Ω can be connected in Ω by a continuous rectifiable curve of length $\leqq a(1+\|x\|+\|y\|)^p\|x-y\|^\lambda$.

It is obvious that (R)′ implies (R).

A.5. Proposition. (i) *Let Ω possess the regularity property* (R), *then the topology of $\mathscr{E}(\Omega)$ coincides with that induced from $\mathscr{T}(\Omega)$, that is, then the systems $\{\|\cdot\|_l^K\}$ and $\{|\cdot|_l^K\}$ of seminorms on $\mathscr{E}(\Omega)$ are equivalent.*

(ii) *Let Ω possess the regularity property* (R)′, *then the topology of $\mathscr{S}(\Omega)$ coincides with the topology of the seminorms (with $l,m\in\boldsymbol{Z}_+$)*

$$|f|_{l,m}=\sup_{|\alpha|\leqq l}\ \sup_{x\in\Omega}(1+\|x\|)^m|f_\alpha(x)|\,.$$

Proof. We restrict ourselves with the proof of (i) [(ii) being similar]. It suffices to show that, for all $s\in\boldsymbol{Z}_+$, $\alpha\in\boldsymbol{Z}_+^k$ and a compact $K\subset\Omega$, there exist $c\geqq0$, $r\in\boldsymbol{Z}_+$ and a compact $K'\subset\Omega$ such that, for all $f\in\mathscr{E}(\Omega)$, $x,y\in K$,

$$|(R_\alpha^s f)(x,y)|\leqq c\cdot|f|_r^{K'}\,.$$

Since distances between the parts $K_1,\dots,K_n$ of K contained in different components of Ω are non-zero, it suffices to assume that x and y belong to the same

[5] The case $\mathscr{A}=\mathscr{S}$ needs a slight modification of the argument on the basis of the extension theorem in the form of Hörmander [10].

part (say K_1) of K. Given $f \in \mathscr{E}(\Omega)$, we choose $F \in \mathscr{E}(\boldsymbol{R}^k)$ such that $D^\beta F = f_\beta$ for all $\beta \in \boldsymbol{Z}_+^k$. For a rectifiable curve in $\boldsymbol{R}^k$, $\gamma:[0,1] \to \boldsymbol{R}^k$, and $G \in \mathscr{E}(\boldsymbol{R}^k)$, we have:

$$G(\gamma(1)) - \sum_{|\beta| \leq l} \beta!^{-1} (D^\beta G)(\gamma(0)) \cdot (\gamma(1) - \gamma(0))^\beta$$
$$= \sum_{\mu_1, \ldots, \mu_{l+1}} \int_0^1 d\gamma_{\mu_1}(t_1) \int_0^{t_1} d\gamma_{\mu_2}(t_2) \ldots \int_0^{t_l} d\gamma_{\mu_{l+1}}(t_{l+1}) (D_{\mu_1} \ldots D_{\mu_{l+1}} G)(\gamma(t_{l+1})).$$

[This formula is evident for sufficiently smooth γ and for functions G satisfying at least one of the conditions: 1) $(D^\beta G)(\gamma(0)) = 0$ for all β with $|\beta| \leq l$, 2) G is a polynomial of degree $\leq l$. The general case follows easily.] Setting $G = D^\alpha F$ and γ a curve in Ω of length $\leq a\|x-y\|^\lambda$ such that $\gamma(0) = y$ and $\gamma(1) = x$, we obtain

$$\|x-y\|^l |(R_\alpha^l f)(x,y)| \leq c' |f|_r^{K'} \|x-y\|^{\lambda \cdot l},$$

where K' is the totality of points of Ω with distance from K $\leq a \cdot (\operatorname{diam} K)^\lambda$, and $r = |\alpha| + l + 1$. Consequently,

$$\|x-y\|^s |(R_\alpha^s f)(x,y)| \leq \|x-y\|^l |(R_\alpha^l f)(x,y)| + c'' \|x-y\|^s |f|_r^K$$
$$\leq c \|x-y\|^s |f|_r^{K'},$$

provided that $l \geq s$ and $\lambda \cdot l \geq s$. Q.E.D.

A.6. Corollary. (i) *Let $\Omega \subset \boldsymbol{R}^k$ possess the regularity property* (R) *and coincide with the closure of its interior* ($\operatorname{int}\Omega$) *in $\boldsymbol{R}^k$. Let $\check{\mathscr{E}}(\Omega)$ (resp. $\check{\mathscr{D}}(\Omega)$) be the space of restrictions $\varphi = jf$ to Ω of all complex $\mathscr{C}^\infty$ functions f (resp. with compact support) on $\boldsymbol{R}^k$, the topology of $\check{\mathscr{E}}(\Omega)$ being defined by the seminorms*

$$|\varphi|_l^K = \sup_{|\alpha| \leq l} \sup_{x \in K \cap \operatorname{int}\Omega} |D^\alpha \varphi(x)|$$

with arbitrary $l \in \boldsymbol{Z}_+$ and a compact $K \subset \Omega$. (The topology of $\check{\mathscr{D}}(\Omega)$ is defined through the subspaces of $\check{\mathscr{E}}(\Omega)$ in line with Definition A.1.) Then the mapping $jf \mapsto (D^\alpha f \upharpoonright \Omega)_{\alpha \in \boldsymbol{Z}_+^k}$ is a topological isomorphism of $\check{\mathscr{E}}(\Omega)$ (resp. $\check{\mathscr{D}}(\Omega)$) onto $\mathscr{E}(\Omega)$ (resp. $\mathscr{D}(\Omega)$).

(ii) *Let $\Omega \subset \boldsymbol{R}^k$ possess the regularity property* (R)$'$ *and coincide with the closure of its interior in $\boldsymbol{R}^k$. Let $\check{\mathscr{S}}(\Omega)$ be the space of restrictions $\varphi = jf$ to Ω of all complex functions $f \in \mathscr{S}(\boldsymbol{R}^k)$, the topology of $\check{\mathscr{S}}(\Omega)$ being defined by the seminorms (with $l, m \in \boldsymbol{Z}_+$)*

$$|\varphi|_{l,m} = \sup_{|\alpha| \leq l} \sup_{x \in \operatorname{int}\Omega} (1 + \|x\|)^m |D^\alpha \varphi(x)|.$$

Then the mapping $jf \mapsto (D^\alpha f \upharpoonright \Omega)_{\alpha \in \boldsymbol{Z}_+^k}$ is a topological isomorphism of $\check{\mathscr{S}}(\Omega)$ onto $\mathscr{S}(\Omega)$.

We close this exposition with a special class of regular sets Ω used in Section 1. Remind that a continuous mapping $J: \boldsymbol{R}^n \to \boldsymbol{R}^k$ is proper if the inverse image of any compact in $\boldsymbol{R}^k$ is a compact.

A.7. Lemma. *Let $J: \boldsymbol{R}^n \to \boldsymbol{R}^k$ be a polynomial proper mapping such that the interior (in $\boldsymbol{R}^k$) of the set $\Omega = J(\boldsymbol{R}^n)$ is non-void. Moreover, assume that, for each $x \in \Omega$, there exists $\xi \in J^{-1}(x)$ such that $J(U)$ is a (relative) neighborhood of x in Ω for any neighborhood U of ξ in $\boldsymbol{R}^n$. Then the set Ω is the closure (in $\boldsymbol{R}^k$) of its interior and posses the regularity property* (R)$'$.

Proof. For any compact $K \subset \boldsymbol{R}^k$, the set $K \cap \Omega = J(J^{-1}(K))$ is compact and hence closed. This implies that Ω is closed. The implication c)$\Rightarrow$b) of Lemma 1.1 (Section 1) shows that the rank of the Jacobian matrix DJ of J equals k at least at one point of $\boldsymbol{R}^n$ and hence on a dense open subset of $\boldsymbol{R}^n$. The image of this subset under J is an open subset in $\boldsymbol{R}^k$ which is dense in Ω. This proves that Ω is the closure of $\operatorname{int}\Omega$. There remains the property (R)′ to be verified. Define the function ϱ on $\Omega\times\Omega$ by setting

$$\varrho(x,y)=\inf\{\|\xi-\eta\|\cdot(1+\|\xi\|^2+\|\eta\|^2)^{-1}\,|\xi,\eta\in\boldsymbol{R}^n, J(\xi)=x, J(\eta)=y\}.$$

The last hypothesis of Lemma implies the continuity of ϱ at each "diagonal" point (x,x) (i.e. $x_j\to x$, $y_j\to x$ for sequences in Ω imply $\varrho(x_j,y_j)\to 0$). We claim the bound $\varrho(x,y)\leqq A\cdot\|x-y\|^\lambda$ for some $A,\lambda>0$. To prove that, it suffices, in line with the proof of Lemma 1 of [10], to define two semi-algebraic sets (below $\boldsymbol{R}_+$ is the totality of positive reals):

$$Q=\{(x,y,\delta)\in\Omega\times\Omega\times\boldsymbol{R}_+|\varrho(x,y)^2\geqq\delta^2\},$$

$$S=\{(\tau,\delta)\in\boldsymbol{R}_+\times\boldsymbol{R}_+|\exists(x,y,\delta)\in Q\quad\text{with}\quad\|x-y\|^2=\tau^2\}.$$

Then the piece-wise algebraic function $T(\delta)=\inf\{\tau|(\tau,\delta)\in S\}$, which is defined on an interval $0<\delta<\delta_0$ and positive [due to the continuity of ϱ at the points (x,x)], possesses the required estimate $T(\delta)\geqq A\cdot\delta^\lambda$. Now the proven bound on ϱ allows to assign to any given pair $x,y\in\Omega$ points $\xi\in J^{-1}(x),\eta\in J^{-1}(y)$ such that

$$\|\xi-\eta\|\leqq A\cdot(1+\|\xi\|^2+\|\eta\|^2)\cdot\|x-y\|^\lambda.$$

Let $\sigma:[0,1]\to\boldsymbol{R}^n$ be the segment connecting ξ and η in $\boldsymbol{R}^n$, then $\gamma=J\circ\sigma$ is a curve in Ω connecting y and x with length $\leqq A'(1+\|\xi\|+\|\eta\|)^{\mu'}\|x-y\|^\lambda$ (with $A',\mu'>0$ independent of x,y). It remains to note that there exist $B,\nu>0$ such that $\|\xi\|^2\leqq B(1+\|J(\xi)\|^2)^\nu$ for all $\xi\in\boldsymbol{R}^n$. This can be shown by considering the function $\varphi(\delta)=\inf\{\tau\,|\,\exists\,\xi\in\boldsymbol{R}^n$ such that $\tau\cdot\|\xi\|^2\geqq 1$ and $\delta\cdot(1+\|J(\xi)\|^2)\leqq 1\}$ (defined and positive on an interval $0<\delta<\delta_0$, since J is proper). Then the argument used in the proof of Lemma 1 of Ref. [10] gives an estimate of the type $\varphi(\delta)\geqq B^{-1}\cdot\delta^\nu$ with some $B,\nu>0$. Q.E.D.

A.8. Construction. In the situation considered in Lemma A.7, the spaces $\mathscr{A}(\Omega)$ admit a particularly simple characterization. Assume that the hypothesis of Lemma A.7 holds. Let, in addition, Q be a real polynomial $\not\equiv 0$ on $\boldsymbol{R}^n$ such that $\mathfrak{O}=J(Q^{-1}(\boldsymbol{R}\setminus\{0\}))$ is an open set in $\boldsymbol{R}^k$. Since $Q^{-1}(\boldsymbol{R}\setminus\{0\})$ is dense in $\boldsymbol{R}^n$, Ω is the closure of $\mathfrak{O}$ in $\boldsymbol{R}^k$. For a minute, we denote by $\hat{\mathscr{E}}(\Omega)$ the totality of complex functions φ on Ω such that their restrictions to $\mathfrak{O}$ are $\mathscr{C}^\infty$ and all derivatives $D^\alpha(\varphi\restriction\mathfrak{O})$ admit (unique) continuous extensions (denoted by $D^\alpha\varphi$) into Ω. Then $\hat{\mathscr{E}}(\Omega)$ is a Frechét space with the seminorms

$$|\varphi|_l^K=\sup_{|\alpha|\leqq l}\ \sup_{x\in K}|D^\alpha\varphi(x)|,$$

where $l\in\boldsymbol{Z}_+$, K is a compact in Ω. In line with Definition A.1, we define also the spaces $\hat{\mathscr{D}}(\Omega)$ and $\hat{\mathscr{S}}(\Omega)$ [formed by certain functions of $\hat{\mathscr{E}}(\Omega)$]. Now the topology of $\hat{\mathscr{S}}(\Omega)$ is defined by the seminorms

$$|\varphi|_{l,m}=\sup_{|\alpha|\leqq l}\ \sup_{x\in\Omega}(1+\|x\|)^m|D^\alpha\varphi(x)|.$$

Then we have

A.9. Proposition. *Under the hypotheses of Construction A.8, the mapping* $\varphi\mapsto(D^\alpha\varphi)_{\alpha\in\boldsymbol{Z}_+^K}$ *is a topological isomorphism of* $\hat{\mathscr{A}}(\Omega)$ *onto* $\mathscr{A}(\Omega)$. *(Hence* $\hat{\mathscr{A}}(\Omega)$ *coincides with its subspace* $\check{\mathscr{A}}(\Omega)$ *defined in Corollary A.6.)*

We identify $\mathscr{A}(\Omega)$ with $\hat{\mathscr{A}}(\Omega)$ via this mapping.

Proof. By virtue of Proposition A.5, it suffices to show that, for an arbitrary $\varphi \in \hat{\mathscr{A}}(\Omega)$, the jet $f=(D^\alpha \varphi)_{\alpha \in \mathbf{Z}_+^k}$ belongs to $\mathscr{E}(\Omega)$. In fact it suffices to prove that all the functions $R_\alpha^s f$ are locally bounded in $\Omega \times \Omega$ (see Footnote [4]). Since $\mathfrak{D}$ is dense in Ω and the functions $f_\alpha = D^\alpha \varphi$ are continuous in Ω, the problem reduces to an estimate of the type

$$|(R_\alpha^s f)(x, y)| \leqq c$$

for all $x, y \in \mathfrak{D}$ with $\|x\| \leqq \varrho$, $\|y\| \leqq \varrho$. (Here c depends on $\varphi, s, \alpha, \varrho$.) As in the proof of Lemma A.7, we choose a curve $\gamma:[0,1] \to \mathbf{R}^k$ with length $\leqq a(1+\|x\|+\|y\|)^p \|x-y\|^\lambda$ which depends polynomially on $t \in [0,1]$ and connects y and x in Ω. Since $Q(\gamma(t))$ is polynomial in t and $Q(\gamma(0)) \neq 0$, then $Q(\gamma(t))=0$ is possible only for finite number of points, say $t_1, \ldots, t_\nu \in [0,1]$. Hence $\gamma(t) \in \mathfrak{D}$ for all $t \in L = [0,1] \setminus \{t_1, \ldots, t_\nu\}$, consequently, for $G = D^\alpha \varphi$, $t \mapsto G(\gamma(t))$ is $\mathscr{C}^\infty$ on L and all its derivatives have continuous extensions into $[0,1]$. This implies that $G \circ \gamma$ is $\mathscr{C}^\infty$ on $[0,1]$. Now the formula (used in the proof of Proposition A.5) expressing

$$G(\gamma(1)) - \textstyle\sum_{|\beta| \leqq l} \beta!^{-1} (D^\beta G)(\gamma(0)) \cdot (\gamma(1)-\gamma(0))^\beta$$

is applicable, thus the estimates in the proof of Proposition A.5 hold. Q.E.D.

Appendix B. Auxiliary Theorem

We present here a (slightly generalized) version of the Hörmander division theorem [10]. It lies at the basis of the arguments both in Sections 1 and 2.

Throughout, $\mathscr{A}(\mathbf{R}^n)$ stands for either of the Schwartz spaces $\mathscr{E}(\mathbf{R}^n)$, $\mathscr{D}(\mathbf{R}^n)$, $\mathscr{S}(\mathbf{R}^n)$. By $\bigoplus^k \mathscr{A}(\mathbf{R}^n)$ we denote the direct sum of k $(=1,2,3\ldots)$ copies of the space $\mathscr{A}(\mathbf{R}^n)$.

Theorem. *For $k \leqq l$, let $U: \bigoplus^k \mathscr{A}(\mathbf{R}^n) \to \bigoplus^l \mathscr{A}(\mathbf{R}^n)$ be a linear operator defined by*

$$(Uf)_i(x) = \textstyle\sum_{j=1}^k u_{ij}(x) f_j(x), \qquad i=1, \ldots, l,$$

where $u \equiv (u_{ij})$ is a $l \times k$ matrix of complex polynomial functions u_{ij} on $\mathbf{R}^n$ such that the rank of $u(x)$ equals k at least for one $x \in \mathbf{R}^n$. Then the image of $\bigoplus^k \mathscr{A}(\mathbf{R}^n)$, $\operatorname{im} U$, *is a closed linear subspace of $\bigoplus^l \mathscr{A}(\mathbf{R}^n)$, and U is a topological isomorphism of $\bigoplus^k \mathscr{A}(\mathbf{R}^n)$ onto* $\operatorname{im} U$.

Proof. For $k=l=1$, the statement is just Hörmander's division theorem[6]. We will reduce the general case to the special one. It is clear that U is continuous. It suffices to construct a continuous left inverse $V: \operatorname{im} U \to \bigoplus^k \mathscr{A}(\mathbf{R}^n)$ of U. [Indeed, then U is a topological isomorphism of $\bigoplus^k \mathscr{A}(\mathbf{R}^n)$ onto $\operatorname{im} U$, hence $\operatorname{im} U$ is complete in the topology induced from $\bigoplus^l \mathscr{A}(\mathbf{R}^n)$ and thus closed in $\bigoplus^l \mathscr{A}(\mathbf{R}^n)$.]

Let us construct V. For definiteness, we assume that $\det u' \not\equiv 0$, where $u' = (u_{ij})_{i,j=1,\ldots,k}$. Then the equality $Uf=g$ can be rewritten in the form $Af=Bg$, where $A: \bigoplus^k \mathscr{A}(\mathbf{R}^n) \to \bigoplus^k \mathscr{A}(\mathbf{R}^n)$ and $B: \operatorname{im} U \to \bigoplus^k \mathscr{A}(\mathbf{R}^n)$ are the continuous

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[6] In Ref. [10] the division theorem was formulated for the space $\mathscr{S}(\mathbf{R}^n)$; its proof remains true also for $\mathscr{E}(\mathbf{R}^n)$ and $\mathscr{D}(\mathbf{R}^n)$.

operators,

$$(Af)_i(x) = \det u'(x) \cdot f_i(x), \qquad (Bg)_i(x) = \sum_{j=1}^{k} b_{ij}(x) \cdot g_j(x),$$

and $b = (b_{ij})$ is the matrix of cofactors of u' (so that $b \cdot u' = \det u' \cdot \mathbf{1}$). A is the multiplication by a polynomial $\neq 0$, hence (by Hörmander's division theorem) there exists a continuous left inverse operator $C: \operatorname{im} A = \operatorname{im} B \to \bigoplus^k \mathscr{A}(\boldsymbol{R}^n)$ of A. Now $V = C \cdot B$ is the desired operator. Q.E.D.

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