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Infrared Convergence of Feynman Integrals for the Massless A^4-Model

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Abstract. For the massless A^4-model it is proved that renormalization can be formulated such that each Feynman diagram yields an ultraviolet and infrared convergent contribution to the Green's functions.

1. Introduction

In a previous paper a new renormalization scheme was proposed for theories with zero-mass propagators. The characteristic feature of this method is that subtraction terms involve massive denominators so that no new infrared infinities are introduced by making subtractions at zero external momenta. So far the method has been applied to the massive A^4-model, the Goldstone and the pre-Higgs model in Ref. [1], as well as the Higgs model by Clark [2]. Presently under consideration is the application [3, 4] to the pure Yang-Mills field as an extension of the work by Becchi, Rouet, and Stora [5] on non-Abelian gauge theories. For all models considered the new subtraction scheme yields ultraviolet and infrared convergent contributions for each Feynman diagram separately. This eliminates the need of discussing cancellations of infrared infinities by cumbersome limiting procedures.

The purpose of this paper is to present a complete and rigorous convergence proof for the massless A^4-model as an application of a general power counting theorem [6]. The extension to the other models treated in Refs. [1] and [2] is straightforward.

After some remarks on the general form of the renormalized integrands (Section 1) the convergence of Feynman integrals is proved for all diagrams which do not contain internal self-energy insertions. In Sections 3 and 4 the general case is reduced to the task of verifying dimensional rules for certain expressions involving massless propagators only. These rules are checked recursively in Section 5 and 6 using the method of propagator product expansions.

2. General Properties of the Renormalized Integral

For the definition of the renormalized integrand R_Γ of a Feynman diagram Γ we refer to Section IIB of Ref. [1]. We further define

$$\bar{R}_{\varrho s \varrho} = S_\varrho \sum_{U \in \mathscr{F}'_\varrho} \sum_{\gamma \in U} (-\tau_\gamma S_\gamma) I_\varrho(U) \tag{2.1}$$

for any subdiagram $\varrho \subseteqq \Gamma$ with $\mathscr{F}'_\varrho$ denoting the family of all forests $U \in \mathscr{F}_\varrho$ which do not contain ϱ. If ϱ is a self-energy part we also introduce the expression

$$\hat{R}_{\varrho s^\varrho} = (1 - t^2_{p^\varrho s^\varrho})\bar{R}_{\varrho s^\varrho} . \tag{2.2}$$

The values at $s^\varrho = 1$ are denoted by

$$\begin{aligned} &R_\varrho = R_{\varrho s^\varrho}|_{s^\varrho = 1}, \bar{R}_\varrho = \bar{R}_{\varrho s^\varrho}|_{s^\varrho = 1} , \\ &\hat{R}_\varrho = \hat{R}_{\varrho s^\varrho}|_{s^\varrho = 1} . \end{aligned} \tag{2.3}$$

The relations between $R_{\varrho s^\varrho}$ and $\bar{R}_{\varrho s^\varrho}$ or $\hat{R}_{\varrho s^\varrho}$ are

$$R_{\varrho s^\varrho} = \bar{R}_{\varrho s^\varrho} \tag{2.4}$$

if ϱ is not a renormalization part

$$R_{\varrho s^\varrho} = (1 - t^0_{\varrho^\varrho s^\varrho})\bar{R}_{\varrho s^\varrho} \tag{2.5}$$

if ϱ is a vertex part and

$$R_{\varrho s^\varrho} = (1 - \tau_\varrho)\bar{R}_{\varrho s^\varrho} \tag{2.6}$$

$$R_{\varrho s^\varrho} = \hat{R}_{\varrho s^\varrho} - t^1_{p^\varrho}\hat{R}_\varrho \tag{2.7}$$

if ϱ is a self-energy part. At $s^\varrho = 1$ we have

$$R_\varrho = (1 - t^1_{p^\varrho})\hat{R}_\varrho \tag{2.8}$$

for a self-energy part ϱ.

For a proper diagram Γ the renormalized integral is of the form

$$\int dk R_{\Gamma\varepsilon}(k, p) = \int dk \frac{P}{ABC} \tag{2.9}$$

with

$$\begin{aligned} &A = \textstyle\sum_j (l_j^2 + i\varepsilon \vec{l}_j^2) , \\ &B = \textstyle\prod_\gamma \prod_{j\gamma} (K_j^{\gamma 2} - M^2 + i\varepsilon_M(K_j^\gamma))^{n(\gamma j)} , \\ &C = \textstyle\prod_\sigma \prod_{j\sigma} (K_j^{\sigma 2} - M^2 + i\varepsilon_M(K_j^\sigma))^{n(\sigma j)} \\ &\qquad (K_j^{\sigma 2} + i\varepsilon \vec{K}_j^{\sigma 2})^{n'(\sigma j)} , \\ &\varepsilon_M(l) = \varepsilon(\vec{l}^2 + M^2),\ n(\gamma j) \geqq 0,\ n(\sigma j) \geqq 0,\ n'(\sigma j) \geqq 0 . \end{aligned} \tag{2.10}$$

P is a polynomial in k and p. $\prod_\gamma$ extends over the vertex insertions γ, $\prod_\sigma$ over the self-energy insertions σ of Γ. The internal lines of the diagram Γ are denoted by $L_1, \ldots, L_m$. The momenta l_j and K_j^γ, K_j^σ carry the same index j as the line L_j to which they belong. $\prod_{j\gamma}$, $\prod_{j\sigma}$ extend over the lines L_j of γ or σ respectively.

The power counting theorem of Ref. [6] applies to integrals of the form (2.5). The ultraviolet convergence conditions of the theorem are satisfied since the subtraction rules meet the criteria given in Ref. [7]. We may therefore restrict ourselves to checking the infrared convergence conditions.

3. Feynman Diagrams without Self-Energy Insertions

If Γ is not a self-energy diagram and does not contain self-energy insertions all subtractions are taken at $s=0$. Hence in each subtraction term some denominators $l_j^2+i\varepsilon\vec{l}_j^2$ are changed into $K_j^{\gamma 2}-M^2+i\varepsilon_M(K_j^\gamma)$ or a power thereof, while the other denominators remain the same. Therefore, the general form of J_Γ is

$$J_\Gamma=\int dk \frac{P}{\prod_j(l_j^2+i\varepsilon\vec{l}_j^2)\prod_\gamma\prod_{j\gamma}(K_j^{\gamma 2}-M^2+i\varepsilon_M(K_j^\gamma))^{n(\gamma j)}}, \tag{3.1}$$

where the first product $\prod_j(l_j^2+i\varepsilon\vec{l}_j^2)$ is the denominator of the unrenormalized integral. We now apply the Corollary of Ref. [6, p. 20]. With $Q=P$ and Δ_i being $(l_j^2+i\varepsilon\vec{l}_j^2)^{-1}$ or $(K_j^{\gamma 2}-M^2+i\varepsilon_M(K_j^\gamma))^{-n(j)}$ the integral (3.1) is of the form (4.13) of Ref. [5]. Then (4.14) of [5] becomes just the unrenormalized integral

$$J_\Gamma^{\text{unren}}=\int\frac{dk}{\prod_j(l_j^2+i\varepsilon\vec{l}_j^2)} \tag{3.2}$$

associated with the diagram Γ. To this integral Mack's infrared convergence conditions may be applied. According to the Corollary of Ref. [6] the integral (3.1) is absolutely convergent if any reduced integral of (3.2) with vanishing external momenta has positive dimension.

We will use Symanzik's concept of exceptional momenta in the Euclidean sense for the external momenta [9]. Accordingly a set of external momenta is called exceptional if any of the momenta or a partial sum of them vanishes. Exceptional momenta in the Minkowski sense become relevant for the singularities of Feynman integrals in the limit $\varepsilon\to+0$. We restrict ourselves to the case of non-exceptional momenta of a proper diagram Γ. The convergence conditions of the Corollary may then equivalently be stated as follows: Form the reduced diagrams Δ of Γ for which all external vertices of Γ are contracted to a single vertex of Δ. If the dimension of the unrenormalized integral of any such Δ is positive the integral (3.1) is absolutely convergent at non-exceptional momenta. Furthermore it can be shown that the limit $\varepsilon\to+0$ yields well-defined distributions in $p_1,\ldots,p_N$ [8].

Self-energy diagrams Σ may be treated similarly provided they do not contain internal self-energy insertions. In this case we form

$$\begin{aligned}\hat{J}_\Sigma&=\int dk\hat{R}_\Sigma\\ &=\int dk\{I_\Sigma-\text{subtractions at } s=0\}\end{aligned} \tag{3.3}$$

without taking the final postsubtraction $1-t_p^1$ [see (2.8)]. The total contribution from Σ to the function Π is then given by

$$\begin{aligned}\Pi_\Sigma(p^2)&=\hat{\Pi}_\Sigma(p^2)-\hat{\Pi}_\Sigma(0)\\ \hat{\Pi}_\Sigma(p^2)&=\lim_{\varepsilon\to+0}\hat{J}_\Sigma(p)\,.\end{aligned} \tag{3.4}$$

After carrying out the subtractions the integral (3.3) is again of the form (3.1) and the Corollary may be applied similarly. The exceptional momentum $p=0$ need not be excluded. It will be seen that the dimension of the renormalized integral of any reduced diagram Δ of Γ is positive. According to the Corollary this implies the absolute convergence of (3.3) even at $p=0$. It can further be shown that the limit $\varepsilon \to +0$ of the corresponding Minkowski integrals exists as distributions with finite values at $p=0$ [8]. With this result Π_Σ is well defined by (3.4).

For the proof of the above statements we have to show that the dimension of certain unrenormalized integrals is positive. Let Δ be a reduced diagram of the proper diagram Γ. Then the dimension δ of the unrenormalized integrand of Δ is

$$\delta=4+\sum_{\alpha\geqq 2}(\alpha-4)a_\alpha-b\,. \tag{3.5}$$

Here a_α is the number of reduced vertices at which α internal lines join. b is the number of external lines of Γ attached to external vertices which are not reduced in Δ. For non-exceptional momenta and a Δ without self-energy insertions we have

$$b=0 \quad \text{and} \quad a_2=0,1$$

since $\alpha=2$ is only possible for the one reduced external vertex of Δ. Therefore,

$$\begin{aligned}&\delta=\dim J_\Delta^{\mathrm{unren}}(p)\geqq 2\,,\\ &p=(p_1,\dots,p_N)\quad\text{non-exceptional}\,.\end{aligned} \tag{3.6}$$

This proves the absolute convergence of J_Γ if Γ is not a self-energy part, does not contain self-energy insertions and if the external momenta are non-exceptional.

A stronger result holds for self-energy parts Σ without internal self-energy parts. Δ may be any reduced diagram of Σ, including Σ itself. Then there are only three possibilities:

$$b=2,\alpha_2=0\,;\qquad b=0,\alpha_2=0\,;\qquad b=0,\alpha_2=1\,.$$

In each case

$$\delta=\dim J_\Sigma^{\mathrm{unren}}(p)\geqq 2,\ p\ \text{arbitrary}\,, \tag{3.7}$$

which implies the convergence of (3.3) if Σ does not contain self-energy insertions.

The criteria developed in this section are not sufficient to prove the convergence for diagrams with self-energy insertions. This is not surprising since the structure of the polynomial P has been ignored as far as the infrared properties are concerned. The following sections serve to extract some information about R_Γ which will be sufficient for the general proof of convergence.

4. Separation of Zero Mass Propagators

In this section Γ denotes a proper diagram (which may also be a self-energy diagram), $\gamma\subseteqq\Gamma$ denotes a renormalization part, $\Sigma\subseteqq\Gamma$ denotes a proper self-energy diagram. The following factorization formulae can be proved by induction

$$R_\Gamma=\sum_{K\in\mathscr{K}}E_{\Gamma\{\mu\nu\}}S_\Gamma\prod_{\tau\in K_V}(-\bar{R}_{\tau s})_{00}\prod_{\sigma\in K_S}(-\bar{R}_{\sigma s}^{\mu_\sigma\nu_\sigma})_{00}\,, \tag{4.1}$$

$$\bar{R}_\Sigma=\sum_{K\in\mathscr{K}'}\bar{E}_{\bar{\Sigma}\{\mu\nu\}}S_\Sigma\prod_{\tau\in K_V}(-\bar{R}_{\tau s})_{00}\prod_{\sigma\in K_S}(-\bar{R}_{\sigma s}^{\mu_\sigma\nu_\sigma})_{00}\,, \tag{4.2}$$

$$\prod_{\sigma_2\in L_{S2}}(-\bar{R}_{\sigma_2 s}^{\lambda\sigma_2})_{01}\prod_{\sigma_3\in L_{S3}}(-\bar{R}^{\mu_{\sigma_3}\nu_{\sigma_3}})_{00}\prod_{\sigma_4\in L_{S4}}\left(-\frac{\partial}{\partial s^{\sigma_4 2}}\bar{R}_{\sigma_4 s}\right)_{00}\,, \tag{4.3}$$

with

$$\bar{R}^{\lambda}_{\sigma s} = \frac{\partial \bar{R}_{\sigma s}}{\partial p^{\sigma\lambda}}, \qquad \bar{R}^{\mu\nu}_{\sigma s} = \frac{\partial^2 \bar{R}_{\sigma s}}{\partial p^{\sigma\mu} \partial p^{\sigma\nu}}.$$

$\mathscr{K}$ is the family of sets K of disjoint renormalization parts of Γ. $\mathscr{K}'$ is the family of sets K of renormalization parts of Σ not including Σ itself. $\mathscr{L}$ is the family of ordered sets $L = (L_V, L_{S_1}, \ldots, L_{S_4})$ where L_V is a set of proper vertex parts and $L_{S_1}, \ldots, L_{S_4}$ are disjoint sets of proper self-energy parts. Any two elements of

$$K = L_V \cup L_{S_1} \cup \ldots \cup L_{S_4}$$

should be disjoint, $\bar{\Gamma}$, $\bar{\Sigma}$ and $\bar{\gamma}$ denote the reduced diagrams obtained from Γ, Σ or γ by reducing the renormalization parts of K. K_V is the set of all vertex parts in K. K_S is the set of all self-energy parts in K. $(\,)_{00}$ indicates that the external momenta and s should be set equal to zero. In $(\,)_{01}$ the external momenta are zero while s is set equal to one. The function E_Γ is defined by

$$\begin{aligned} &E_{\bar{\Gamma}} = \textstyle\prod_{\tau_j} (1 - t^1_{p^\tau}) S_{\tau_j} I_{\bar{\Gamma}\{\mu\nu\}}(C) \\ &\{\mu\nu\} = \{(\mu_1\nu_1), \ldots, (\mu_b\nu_b)\}\,. \end{aligned} \tag{4.4}$$

The product extends over the set

$$C = (\tau_1, \ldots, \tau_b) \tag{4.5}$$

of all proper self-energy parts of the reduced diagram $\bar{\Gamma}$. $I_{\bar{\Gamma}}(C)$ is the unrenormalized integrand of $\bar{\Gamma}$ expressed in terms of the momentum variables pertaining to (4.5) and with all s-parameters set equal to one. Contraction of a self-energy part τ; with external momentum l leads to a 2-vertex to which the factor $\frac{1}{2} l_{\mu_j} l_{\nu_j}$ is assigned in $I_{\bar{\Gamma}\{\mu\nu\}}$. In Eq. (4.3) the function $I_{\bar{\gamma}L\{\lambda\}\{\mu\nu\}}$ is defined by

$$I_{\bar{\gamma}L\{\lambda\}\{\mu\nu\}} = I_{\bar{\gamma}} S_\gamma \textstyle\prod_{\sigma_2 \in L_{S_2}} p^{\sigma_2}_{\lambda_{\sigma_2}} \prod_{\sigma_3 \in L_{S_3}} \frac{1}{2} p^{\sigma_3}_{\mu_\sigma} \, p^{\sigma_3}_{\nu_{\sigma_3}} \prod_{\sigma_4 \in L_{S_4}} (1 - (S^{\sigma_4})^2)\,. \tag{4.6}$$

$\{\lambda\}$ is the set of indices λ_{σ_2} with $\sigma_2 \in L_{S_2}$, $\{\mu\nu\}$ is the set of index pairs $(\mu_{\sigma_3}\nu_{\sigma_3})$ with $\sigma_3 \in L_{S_3}$.

The function E_Λ (omitting Lorentz indices and setting $\Lambda = \bar{\Gamma} = \Gamma/K$ is determined by the recursion formulae

$$E_\sigma = p^{\sigma\mu} p^{\sigma\nu} F_{\sigma\mu\nu} = (1 - t^1_{p^\sigma}) \bar{E}_\sigma \tag{4.7}$$

$$\bar{E}_\sigma = I_{\bar{\sigma}\{\mu\nu\}} F^{\mu_1\nu_1}_{\varrho_1} \ldots F^{\mu_a\nu_a}_{\varrho_a} \tag{4.8}$$

$$\bar{\sigma} = \sigma/\varrho_1 \ldots \varrho_a$$

valid for the self-energy parts σ of Λ. $\varrho_1, \ldots, \varrho_a$ are the maximal self-energy insertions of σ. The reduced diagram $\bar{\sigma}$ does not contain any further self-energy insertions. $I_{\bar{\sigma}}$ is the unrenormalized integrand of $\bar{\sigma}$ constructed according to the rules below (4.3). In (4.8) the arguments of $F^{\mu\nu}_\varrho$ should be expressed in terms of the variables k^σ, p^σ.

For a self-energy part Σ the formulae (4.4) and (4.5) apply with $\sigma = \bar{\Sigma} = \Sigma/K$. If Γ is not a self-energy part we have

$$\begin{aligned} &E_\Lambda = \bar{E}_\Lambda = I_{\bar{\Lambda}\{\mu\nu\}} F^{\mu_1\nu_1}_{\varrho_1} \ldots F^{\mu_a\nu_a}_{\varrho_a} \\ &\Lambda = \bar{\Gamma} = \Gamma/K, \bar{\Lambda} = \Lambda/\varrho_1 \ldots \varrho_a\,. \end{aligned} \tag{4.9}$$

Our aim is to prove the absolute convergence of the Feynman integrals

$$J_\Gamma(p)=\int dk R_\Gamma(kp)\,, \qquad p \text{ non-exceptional}\,, \tag{4.10}$$

for proper diagrams Γ which are not self-energy parts and

$$\hat{J}_\Sigma(p)=\int dk \hat{R}_\Sigma(kp)\,, \qquad p \text{ arbitrary}\,, \tag{4.11}$$

for proper self-energy parts Σ. The convergence of (4.10) already provides enough information for self-energy parts since the contribution from Σ to the function Π is given by

$$\begin{aligned}\Pi_\Sigma(p^2)&=\hat{\Pi}_\Sigma(p^2)-\hat{\Pi}_\Sigma(0)\\ \hat{\Pi}_\Sigma&=\lim_{\varepsilon\to+0}\hat{J}_\Sigma\,.\end{aligned} \tag{4.12}$$

The infrared convergence conditions (3.2) and (3.3) of Ref. [6] for the integrals (4.10) and (4.11) are

$$\deg_u R_\Gamma(k,p)+4a>0\,, \qquad p \text{ non-exceptional}\,, \tag{4.13}$$

$$\deg_u \hat{R}_\Sigma(k,p)+4a>0\,, \qquad p \text{ arbitrary}\,. \tag{4.14}$$

The lower degree refers to a set

$$u=(u_1,\ldots,u_a) \tag{4.15}$$

of momentum vectors which are chosen as follows. Among the vectors

$$l_j,\, K_j^\gamma,\, K_j^\sigma \tag{4.16}$$

occurring in the denominators of R_Γ and $\hat{R}_\Sigma$ we select a basis

$$u_1,\ldots,u_a,v_1,\ldots,v_b \tag{4.17}$$

so that any vector of (4.16) is a linear combination of vectors (4.17) and $p_1,\ldots,p_N$. Moreover, according to (3.3) of [6] we require that the vectors $u_1,\ldots,u_a$ occur in massless denominators, i.e. be one of the vectors l_j or K_j^σ with $n'(\sigma_j)>0$. For any such basis the conditions (4.13) and (4.14) should hold.

A basis

$$u_1',\ldots,u_a',v_1',\ldots,v_b' \tag{4.18}$$

is called equivalent to (4.17) if it is related to (4.17) by a non-singular linear transformation which expresses the u_j' homogeneously by the u_j. The lower degree with respect to a set u does not change if u is replaced by the set

$$u'=(u_1',\ldots,u_a') \tag{4.19}$$

of an equivalent basis (4.18).

For the recursive derivation of the dimensional rules (4.13) and (4.14) it is useful to employ special sets (4.15) of momentum vectors which refer to a family K of disjoint subdiagrams of Γ. We can always find a basis equivalent to (4.17) which is of the form

$$u_1^{\bar{\Gamma}}, \ldots, u_{a(\bar{\Gamma})}^{\bar{\Gamma}}, v_1^{\bar{\Gamma}}, \ldots, v_{b(\bar{\Gamma})}^{\bar{\Gamma}}; \tag{4.20}$$

$$u_1^{\tau}, \ldots, u_{a(\tau)}^{\tau}, v_1^{\tau}, \ldots, v_{b(\tau)}^{\tau}; \qquad \tau \in K, \tag{4.21}$$

$$a = a(\bar{\Gamma}) + \sum_{\tau \in K} a(\tau), \qquad \bar{\Gamma} = \Gamma/K, \tag{4.22}$$

where the $u_j^{\bar{\Gamma}}$, $v_j^{\bar{\Gamma}}$ are momenta (4.16) affiliated with lines of the reduced diagram $\bar{\Gamma}$, the u_j^{τ}, v_j^{τ} are momenta (4.16) affiliated with lines of the diagram $\tau \in K$.

In this section the infrared conditions (4.13) and (4.14) will be established as a consequence of the inequalities

$$\underline{\deg}_{u^{\bar{\Gamma}}} E_{\bar{\Gamma}}(k, p) + 4a(\bar{\Gamma}) > 0 \tag{4.23}$$

(p non-exceptional)

$$\underline{\deg}_{u^{\bar{\Sigma}}} E_{\bar{\Sigma}}(k, p) + 4a(\bar{\Sigma}) \geqq 0 \tag{4.24}$$

(p arbitrary)

$$\underline{\deg}_{u^{\bar{\Sigma}}} \bar{E}_{\bar{\Sigma}}(k, p) + 4a(\bar{\Sigma}) > 0 \tag{4.25}$$

(p arbitrary)

$$u^{\bar{\Gamma}} = (u_1^{\bar{\Gamma}}, \ldots, u_{a(\bar{\Gamma})}^{\bar{\Gamma}}), \bar{\Gamma} = \Gamma/K, \bar{\Sigma} = \Sigma/K$$

which will be derived in the remainder of the paper. (4.24) and (4.25) further imply

$$\underline{\deg}_{u^{\bar{\Sigma}}} \left.\frac{\partial \bar{E}_{\bar{\Sigma}}}{\partial p^{\mu}}\right|_{p=0} + 4a(\bar{\Sigma}) \geqq 0 . \tag{4.26}$$

We now apply the rule (2.18) of Ref. [6] to the factorization formulae (4.2) and (4.3). With (4.24)–(4.26) we obtain

$$\underline{\deg}_{u^{\tau}} (\bar{R}_{\tau s})_{00} + 4a(\tau) \geqq 0 , \tag{4.27}$$

$$\underline{\deg}_{u^{\sigma}} (\bar{R}_{\sigma s})_{01} + 4a(\sigma) > 0 , \tag{4.28}$$

$$\underline{\deg}_{u^{\sigma}} (\bar{R}_{\sigma s}^{\lambda_\sigma})_{01} + 4a(\sigma) \geqq 0 , \tag{4.29}$$

$$\underline{\deg}_{u^{\sigma}} (\bar{R}_{\sigma s}^{\mu_\sigma \nu_\sigma})_{00} + 4a(\sigma) \geqq 0 , \tag{4.30}$$

by induction. In the recursive proof (4.2) is used for the factors $(\,)_{01}$ and (4.3) at $s=0$ for the factors $(\,)_{00}$.

With this result (4.1) and (4.2) yield the infrared convergence conditions (4.13) and (4.14). The inequalities (4.23)–(4.25), which we assumed for the functions $E_{\bar{\Sigma}}$ and $\bar{E}_{\bar{\Sigma}}$ will be derived in the work that follows.

5. Propagator Product Expansions

A useful tool for checking dimensional properties of renormalized Feynman integrals is the method of propagator product expansions which will be developed in this section. A representation

$$\begin{aligned} \bar{E}_{\sigma} &= \textstyle\sum_{\alpha} \bar{E}_{\sigma\alpha} \\ \bar{E}_{\sigma\alpha} &= \textstyle\prod_{j\sigma} \bar{e}_{\sigma\alpha j}(l_j) \prod_{\varrho} \prod_{j\varrho} \bar{e}_{\sigma\alpha\varrho j}(K_j^{\varrho}) \end{aligned} \tag{5.1}$$

or

$$F_\sigma = \sum_\alpha F_{\sigma\alpha}$$
$$F_{\sigma\alpha} = \prod_{j\sigma} f_{\sigma\alpha j}(l_j) \prod_\varrho \prod_{j\varrho} f_{\sigma\alpha\varrho j}(K_j^\varrho) \tag{5.2}$$

will be called a propagator product expansion of $\bar{E}_\sigma$ or F_σ if the factors

$$\Delta = \bar{e}_{\sigma\alpha j}, \bar{e}_{\sigma\alpha\varrho j} \quad \text{or} \quad f_{\sigma\alpha j}, f_{\sigma\alpha\varrho j}$$

of the argument $w = l_j$ or K_j^ϱ have the form

$$\Delta(w) = \frac{M}{(w^2 + i\varepsilon\vec{w}^2)^c} \tag{5.3}$$

with M being a monomial in the components of w. The momenta l_j and K_j^ϱ carry the index j of the line L_j of Γ to which they are assigned. $\bar{E}_{\sigma\alpha}$, $F_{\sigma\alpha}$ are called terms of the propagator product expansion. We will construct propagator product expansions of $\bar{E}_\sigma$, F_σ by using the recursion formulae (4.4) and (4.5). The propagator product expansions thus obtained will satisfy certain properties, in particular,

$$\deg\Delta \leqq 0 \tag{5.4}$$

for the factors of each term. For given decompositions $\sum_{\alpha j} F_{\varrho_j \alpha_j}$ of the factors F_{ϱ_j} the formula (4.5) induces a decomposition of $\bar{E}_\sigma$ by

$$\bar{E}_\sigma = \sum_\alpha \bar{E}_{\sigma\alpha},\ F_{\varrho j} = \sum_{\alpha_j} F_{\varrho_j \alpha_j},$$
$$\bar{E}_{\sigma\alpha} = I_{\bar{\sigma}} F_{\varrho_1\alpha_1} \dots F_{\varrho_a\alpha_a}, \tag{5.5}$$
$$\bar{\sigma} = \sigma/\varrho_1 \dots \varrho_a,\ \alpha = (\alpha_1, \dots, \alpha_n).$$

If (5.4) is satisfied for the $F_{\varrho_j\alpha_j}$ it will also hold for $\bar{E}_{\sigma\alpha}$. The non-trivial step is to construct the decomposition of a solution F_σ of (4.4) from a given propagator product expansion of E_σ. As hypothesis of induction we assume that the propagator expansion (5.1) of $\bar{E}_\sigma$ satisfies (5.4). Let the product $\prod_{j\sigma}$ in (5.1) extend over all internal lines L_j of σ for which the momentum l_j depends on p^σ, i.e.

$$l_j = x_j p^\sigma + K_j^\sigma, \qquad x_j \neq 0. \tag{5.6}$$

Then the product $\prod_\varrho$ is taken over all self-energy parts of σ, including σ itself. The product $\prod_{j\varrho}$ extends over some internal lines of ϱ with momentum K_j^ϱ. We construct a propagator product expansion of a solution to (4.4) in two steps. First we set up a propagator product expansion $\sum_\beta H_{\sigma\alpha\beta}$ which is a solution of

$$(1 - t_{p^\sigma}^0)\bar{E}_{\sigma\alpha} = \sum_\beta p_\mu^\sigma H_{\sigma\alpha\beta}^\mu. \tag{5.7}$$

Then we construct a propagator expansion $\sum_\gamma F_{\sigma\alpha\beta\gamma}$ as a solution of

$$(1 - t_{p^\sigma}^0) H_{\sigma\alpha\beta}^\mu = \sum_\gamma p_\nu^\sigma F_{\sigma\alpha\beta\gamma}^{\mu\nu}. \tag{5.8}$$

This implies

$$(1 - t_{p^\sigma}^1) \sum_\alpha \bar{E}_{\sigma\alpha} = \sum_{\alpha\beta\gamma} p_\mu^\sigma p_\nu^\sigma F_{\sigma\alpha\beta\gamma}^{\mu\nu} \tag{5.9}$$

yielding a solution

$$F_{\sigma}^{\mu\nu}=\sum_{\alpha\beta\gamma}F_{\sigma\alpha\beta\gamma}^{\mu\nu} \tag{5.10}$$

of (4.4) in the form of a propagator product expansion.

We begin with the construction of $H_{\alpha\beta}$ by applying $1-t_0^{p^\sigma}$ to the p^σ-dependent part

$$g=\prod_j \bar{e}_{\alpha j}(l_j) \tag{5.11}$$

of $\bar{E}_{\sigma\alpha}$. Each factor

$$\bar{e}_{\alpha j}=\frac{M_{\alpha j}}{(l_j^2+i\varepsilon\vec{l}_j^2)^{c(\alpha j)}}, \quad \deg\bar{e}_{\alpha j}\leqq 0,$$

may be written as a product of factors

$$\frac{1}{l_j^2+i\varepsilon\vec{l}_j^2},\ \frac{l_{j\mu_j}}{l_j^2+i\varepsilon\vec{l}_j^2},\quad \text{or}\quad \frac{l_{j\mu_j}l_{j\nu_j}}{l_j^2+i\varepsilon\vec{l}_j^2}. \tag{5.12}$$

Substituting these products for e_{α_j} into (5.11) we find

$$g=\prod_{i=1}^{A}g_i,$$

where each factor g_i is of one of the forms (5.12).

We now apply the formula

$$\begin{aligned}\Delta g&=\Delta g_1\cdot g_2\dots g_A+g_{10}\Delta g_2\cdot g_3\dots g_A\\&\quad+\dots+g_{10}\dots g_{A-1,0}\Delta g_A\\ g_{j0}&=t_{p^\sigma}^0 g_j,\ \Delta=(1-t_{p^\sigma}^0).\end{aligned} \tag{5.13}$$

For working out Δg_i we use (5.6), (5.12), and the identities

$$\begin{aligned}\Delta\frac{1}{ll_\varepsilon}&=-p^\lambda\frac{(l_\varepsilon+K_\varepsilon)_\lambda}{(ll_\varepsilon)(KK_\varepsilon)}\\ \Delta\frac{l_\mu}{ll_\varepsilon}&=p^\lambda\left[\frac{g_{\mu\lambda}}{ll_\varepsilon}-\frac{K_\mu(l_\varepsilon+K_\varepsilon)_\lambda}{(ll_\varepsilon)(KK_\varepsilon)}\right]\\ \Delta\frac{l_\mu l_\nu}{ll_\varepsilon}&=p^\lambda\left[\frac{g_{\mu\lambda}l_\nu}{ll_\varepsilon}+\frac{g_{\nu\lambda}K_\mu}{KK_\varepsilon}-\frac{K_\mu l_\nu(l_\varepsilon+K_\varepsilon)}{(ll_\varepsilon)(KK_\varepsilon)}\right]\end{aligned} \tag{5.14}$$

with the abbreviations

$$\begin{aligned}&r_\varepsilon=(r^0,(1-i\varepsilon)\vec{r})\quad\text{for a 4-vector}\quad r=(r^0,\vec{r}),\\&x=x_j,\ p=p^\sigma,\ K=K_j^\sigma,\ l=l_j=x_jp^\sigma+K_j^\sigma.\end{aligned} \tag{5.15}$$

Thus, in each of the three cases,

$$\Delta g_i=p_\lambda^\sigma\sum_\beta t_{i\beta}^\lambda(l_j,K_j^\sigma), \tag{5.16}$$

where each $t_{i\beta}^\lambda$ is a product of propagators of non-positive degree.

Inserting (5.16) into (5.13) we obtain a solution $\sum_\beta H_{\sigma\alpha\beta}^\mu$ of (5.7) in the form of a propagator product expansion

$$H_{\sigma\alpha\beta}=\prod_j h_{\sigma\alpha\beta j}(l_j)\prod_\varrho\prod_{j\varrho}h_{\sigma\alpha\beta\varrho j}(K_j^\varrho), \tag{5.17}$$

where again

$$\deg h_{\sigma\alpha\beta j} \leqq 0, \deg h_{\sigma\alpha\beta\varrho j} \leqq 0 . \tag{5.18}$$

Applying this construction once more we find a solution of (4.4) in the form of a propagator product expansion

$$\begin{aligned} F_\sigma &= \sum_{\alpha\beta\gamma} F_{\sigma\alpha\beta\gamma} \\ F_{\sigma\alpha\beta\gamma} &= \prod_j f_{\sigma\alpha\beta\gamma j}(l_j) \prod_\varrho \prod_{j\varrho} f_{\sigma\alpha\beta\gamma\varrho j}(K_j^\varrho) \end{aligned} \tag{5.19}$$

which again satisfies (5.4).

6. Recursive Derivation of Dimensional Rules

We begin by introducing some useful definitions. Let ω be a diagram obtained from Γ by forming subdiagrams and reduced diagrams. In particular, ω may be one of the diagrams σ which are self-energy parts of $\Lambda = \Gamma/K$. I_ω is the space of all linear forms

$$\begin{aligned} l &= \Sigma c_j K_j^\omega + \Sigma d_j p_j^\omega \\ K_j^\omega &= K_j^\omega(k_1, \ldots, k_m) \\ p_j^\omega &= p_j^\omega(k_1, \ldots, k_m, p_1, \ldots, p_N) . \end{aligned} \tag{6.1}$$

The notions of linear dependence, basis etc. in L_ω refer to the (in K_j^ω) homogeneous parts of the vectors (6.1) considered as linear forms in $k_1, \ldots, k_m$.

S is the set of all momenta l_j or K_j^ϱ which are linear combinations of the variables $u_1, \ldots, u_a$ which occur in the infrared convergence conditions. S_ω is the set of all internal momenta l_j or K_j^ϱ (ϱ self-energy part of ω) which are affiliated with ω and belong to S.

For any ω we choose a set of (in L_ω) linearly independent momenta

$$u^\omega = (u_1^\omega, \ldots, u_{a(\omega)}^\omega) \tag{6.2}$$

in S_ω such that any momentum of S_ω is a linear combination of them. By adding other elements

$$v^\omega = (v_1^\omega, \ldots, v_{b(\sigma)}^\omega)$$

of S_ω we extend (6.2) to a basis

$$u_1^\omega, \ldots, u_{a(\omega)}^\omega, v_1^\omega, \ldots, v_{b(\omega)}^\omega \tag{6.3}$$

of L_ω. We finally form the set S'_ω of all internal momenta affiliated with ω which are linear combinations of u_j^ω, p^ω only.

In this section the following dimensional rules will be derived:

$$\underline{\deg}_{u^\sigma p^\sigma} F_{\sigma\alpha} + 4a(\sigma) \geqq 0 , \tag{6.4}$$

$$\underline{\deg}_{u^\sigma p^\sigma} \bar{E}_{\sigma\alpha} + 4a(\sigma) \geqq 2 , \tag{6.5}$$

$$\underline{\deg}_{u^\sigma} F_{\sigma\alpha} + 4a(\sigma) \geqq 0 , \tag{6.6}$$

$$\underline{\deg}_{u^\sigma} \bar{E}_{\sigma\alpha} + 4a(\sigma) \geqq 2 . \tag{6.7}$$

The lower degrees are applied to the functions $F_{\sigma\alpha}$ and $\bar{E}_{\sigma\alpha}$ in which the momenta are expressed in terms of u^σ, v^σ, and p^σ. On account of the relation

$$l_j = x_j p^\sigma + K_j^\sigma \tag{6.8}$$

there are only two possibilities

$$K_j^\sigma \in S'_\sigma,\ l_j \in S'_\sigma$$

or

$$K_j^\sigma \notin S'_\sigma,\ l_j \notin S'_\sigma\,. \tag{6.9}$$

The relations (6.4)–(6.7) will be derived recursively. We first show that (6.4) implies (6.6). For this we need only verify

$$\underline{\deg}_{u^\sigma} F_{\sigma\alpha} \geqq \underline{\deg}_{u^\sigma p^\sigma} F_{\sigma\alpha}\,. \tag{6.10}$$

$F_{\sigma\alpha}$ is of the form

$$F_{\sigma\alpha} = \prod_j f_{\sigma\alpha j}(l_j) \prod_\varrho \prod_{j\varrho} f_{\sigma\alpha\varrho j}(K_j^\varrho)\,. \tag{6.11}$$

Let $\Delta(w)$ be any of the factors with w denoting l_j or K_j^ϱ.

$$w = x p^\sigma + U + V\,, \tag{6.12}$$

where U is a linear combination of $u_1^\sigma, \ldots$ and V a linear combination of $v_1^\sigma, \ldots$.
Then

$$\begin{aligned} \underline{\deg}_{u^\sigma} \Delta(w) &= \begin{cases} \underline{\deg}\Delta & \text{if} \quad x = V = 0 \\ 0 & \text{if} \quad x = 0,\ V \neq 0 \\ 0 & \text{if} \quad x \neq 0,\ V = 0\,, \end{cases} \\ \underline{\deg}_{u^\sigma p^\sigma} \Delta(w) &= \begin{cases} \underline{\deg}\Delta & \text{if} \quad x = V = 0 \\ 0 & \text{if} \quad x = 0,\ V \neq 0 \\ \underline{\deg}\Delta & \text{if} \quad x \neq 0,\ V = 0\,. \end{cases} \end{aligned} \tag{6.13}$$

Since $\underline{\deg}\Delta \leqq 0$ the inequality

$$\underline{\deg}_{u^\sigma} \Delta(w) \geqq \underline{\deg}_{u^\sigma p^\sigma} \Delta(w)$$

follows and thus (6.6). Similarly (6.5) implies (6.7).

As hypothesis of induction we now assume that (6.5) has been shown, and prove that (6.4) follows. To this end we show

$$\underline{\deg}_{u^\sigma p^\sigma} H_{\sigma\alpha\beta} \geqq \underline{\deg}_{u^\sigma p^\sigma} \bar{E}_{\sigma\alpha} - 1\,, \tag{6.14}$$

$$\underline{\deg}_{u^\sigma p^\sigma} F_{\sigma\alpha\beta\gamma} \geqq \underline{\deg}_{u^\sigma p^\sigma} H_{\sigma\alpha\beta} - 1\,. \tag{6.15}$$

We begin with (6.14). The factors $\bar{e}_{\sigma\varrho j\alpha}(K_j^\varrho)$ and some of the factors $g_i(l_j)$ of $\bar{E}_{\sigma\alpha}$ appear unchanged in $H_{\sigma\alpha\beta}$ and need not be checked. If the factor $g_i(l_j)$ appears as $g_i(K_j^\sigma)$ in $H_{\sigma\alpha\beta}$ we have [note (6.9)]

$$\underline{\deg}_{u^\sigma p^\sigma} g_i(l_j) = \underline{\deg}\, g_i = \underline{\deg}_{u^\sigma p^\sigma} g_i(K_j^\sigma)$$

if

$$l_j \in S'_\sigma, K_j^\sigma \in S'_\sigma$$

and

$$\underline{\deg}_{u^\sigma p^\sigma} g_i(l_j) = 0 = \underline{\deg}_{u^\sigma p^\sigma} g_i(K_j^\sigma)$$

if

$$l_j \notin S'_\sigma, K_j^\sigma \notin S'_\sigma .$$

If $g_i(l_j)$ becomes replaced by one of the terms $t_{\alpha\beta}$ in (5.16) we have

$$\underline{\deg}_{u^\sigma p^\sigma} t_{i\beta}(l_j K_j^\sigma) = \underline{\deg}_{u^\sigma p^\sigma} g_i(l_j) - 1$$

if

$$l_j, K_j^\sigma \in S'_\sigma$$

and

$$\underline{\deg}_{u^\sigma p^\sigma} g_i(l_j) = 0 = \underline{\deg}_{u^\sigma p^\sigma} t_{\alpha\beta}(l_j, K_j^\sigma)$$

if

$$l_j, K_j^\sigma \notin S'_\sigma .$$

Since the replacement $g_i \to t_{\alpha\beta}$ occurs once in going from $E_{\sigma\alpha}$ to $H_{\sigma\alpha\beta}$ we find (6.14). Similarly (6.15) is derived.

In order to complete the induction proof we have to show that (6.5) follows from (6.4). From (4.5) we get

$$\begin{aligned} \deg_{u^\sigma p^\sigma} \bar{E}_{\sigma\alpha} &= \deg_{u^{\bar\sigma} p^{\bar\sigma}} I_{\bar\sigma} \\ &\quad + \sum\nolimits_{\tau^1} \underline{\deg}_{u^\tau} F_{\tau\alpha_\tau} + \sum\nolimits_{\tau^2} \underline{\deg}_{u^\tau p^\tau} F_{\tau\alpha_\tau} . \end{aligned} \tag{6.16}$$

The sums extend over all maximal self-energy insertions τ of σ with the restriction $p^\tau \notin S'_\sigma$ for $\sum_{\tau^1}$ and $p^\tau \in S'_\sigma$ for $\sum_{\tau^2}$. $\bar\sigma$ denotes the diagram obtained from σ by reducing the maximal self-energy insertions. Using

$$a(\sigma) = a(\bar\sigma) + \sum\nolimits_{\tau^1} a(\tau) + \sum\nolimits_{\tau^2} a(\tau)$$

the relation

$$\begin{aligned} \underline{\deg}_{u^\sigma p^\sigma} \bar{E}_{\sigma\alpha} + 4a(\sigma) &= \underline{\deg}_{u^{\bar\sigma} p^{\bar\sigma}} I_{\bar\sigma} + 4a(\bar\sigma) \\ &\quad + \sum\nolimits_{\tau^1} (\underline{\deg}_{u^\tau} F_{\tau\alpha_\tau} + 4a(\tau)) \\ &\quad + \sum\nolimits_{\tau^2} (\underline{\deg}_{u^\tau p^\tau} F_{\tau\alpha_\tau} + 4a(\tau)) \end{aligned} \tag{6.17}$$

follows. According to the hypothesis of induction

$$\underline{\deg}_{u^\tau p^\tau} F_{\tau\alpha_\tau} + 4a(\tau) \geqq 0$$

for any τ. This implies

$$\underline{\deg}_{u^\tau} F_{\tau\alpha_\tau} + 4a(\tau) \geqq 0 .$$

Hence

$$\underline{\deg}_{u^\sigma p^\sigma} \bar{E}_{\sigma\alpha} + 4a(\sigma) \geqq \underline{\deg}_{u^{\bar\sigma} p^{\sigma}} I_{\bar\sigma} + 4a(\bar\sigma) .$$

Each factor of $I_{\bar{\sigma}}$ corresponding to an internal line of σ is of degree -2, therefore

$$\underline{\deg}_{u^{\bar{\sigma}}p^{\sigma}} I_{\bar{\sigma}} = -\deg \prod_{S_{\bar{\sigma}}^{1}} l_j^2$$

with the product extending over all momenta belonging to the set $S'_{\bar{\sigma}}$. Also $a(\bar{\sigma}) \geqq c$ where c is the number of linearly independent (in $L_{\bar{\sigma}}$) internal momenta of $S'_{\bar{\sigma}}$. Hence

$$\begin{aligned} &\underline{\deg}_{u^{\bar{\sigma}}p^{\sigma}} I_{\bar{\sigma}} + 4a(\bar{\sigma}) \\ &\geqq 4c - \deg \prod_{S_{\bar{\sigma}}^{1}} l_j^2 = \dim J_{\bar{\sigma}/Q}^{\text{unren}} . \end{aligned} \tag{6.18}$$

$\bar{\sigma}/Q$ is the diagram obtained from $\bar{\sigma}$ by reducing the set Q of all lines of $\bar{\sigma}$ which do not belong to $S'_{\bar{\sigma}}$. Combining (3.7) with (6.4), (6.17), and (6.18) we find the desired result (6.5). This completes the proof of the relations (6.4)–(6.7).

Using (2.18) of Ref. [6] we obtain

$$\underline{\deg}_{u^{\sigma}p^{\sigma}} F_{\sigma} + 4a(\sigma) \geqq 0 , \tag{6.19}$$

$$\underline{\deg}_{u^{\sigma}p^{\sigma}} \bar{E}_{\sigma} + 4a(\sigma) \geqq 2 , \tag{6.20}$$

$$\deg_{u^{\sigma}} F_{\sigma} + 4a(\sigma) \geqq 0 , \tag{6.21}$$

$$\underline{\deg}_{u^{\sigma}} \bar{E}_{\sigma} + 4a(\sigma) \geqq 2 . \tag{6.22}$$

With $\sigma = \Xi = \Sigma/K$ the inequality (6.20) implies condition (4.23) for self-energy diagrams. Condition (4.22) follows from (6.19). We finally derive (4.21) for $\Lambda = \Gamma/K$ where Γ is not a self-energy part. (4.6) implies

$$\underline{\deg}_u E_{\Lambda} = \underline{\deg}_u I_{\bar{\Lambda}} + \sum_{\tau^1} \underline{\deg}_{u^{\tau}} F_{\tau} + \sum_{\tau^2} \underline{\deg}_{u^{\tau}p^{\tau}} F_{\tau} .$$

The sums extend over all maximal self-energy parts τ of Λ with the restriction $p^{\tau} \notin S'_{\Lambda}$ for $\sum_{\tau^1}$ and $p^{\tau} \in S'_{\Lambda}$ for $\sum_{\tau^2}$. With

$$a = a(\bar{\Lambda}) + \sum_{\tau^1} a(\tau) + \sum_{\tau^2} a(\tau)$$

we find

$$\begin{aligned} \underline{\deg}_u E_{\bar{\Lambda}} + 4a = {} & \underline{\deg}_u I_{\Lambda} + 4a(\bar{\Lambda}) \\ & + \sum_{\tau^1} (\underline{\deg}_{u^{\tau}} F_{\tau} + 4a(\tau)) \\ & + \sum_{\tau^2} (\underline{\deg}_{u^{\tau}p^{\tau}} F_{\tau} + 4a(\tau)) \\ \geqq {} & 4a(\bar{\Lambda}) + \underline{\deg}_u I_{\bar{\Lambda}} . \end{aligned} \tag{6.23}$$

Here

$$\deg_u I_{\bar{\Lambda}} = -\deg \prod_{S_{\bar{\Lambda}}} l_j^2$$

and $a(\bar{\Lambda}) \geqq c$ where c is the number of linearly independent internal momenta of $S_{\bar{\Lambda}}$. Hence

$$\begin{aligned} &4a(\bar{\Lambda}) + \underline{\deg}_u I_{\bar{\Lambda}} \\ &\geqq 4c - \deg \prod_{S_{\bar{\Lambda}}} l_j^2 = \dim J_{\bar{\Lambda}/Q}^{\text{unren}} . \end{aligned} \tag{6.24}$$

Q is the set of all lines of $\bar{\Lambda}$ which do not belong to $S_{\bar{\Lambda}}$. Since all elements of $S_{\bar{\Lambda}}$ are linear combinations of $u_1, \ldots, u_a$ the external momenta of $\bar{\Lambda}/Q$ vanish. Further assuming non-exceptional momenta $p_1, \ldots, p_N$ we obtain (4.21) by combining (6.23), (6.24), with (3.6). This completes the check of the infrared conditions.

References

1. Lowenstein, J., Zimmermann, W.: Nucl. Phys. B**86**, 77 (1975)
2. Clark, T.: To be published in Nucl. Phys.
3. Becchi, C.: To be published
4. Lowenstein, J.: To be published in Nucl. Phys.
5. Becchi, C., Rouet, A., Stora, R.: To be published
6. Lowenstein, J., Zimmermann, W.: To be published in Commun. math. Phys.
7. Gomes, M., Lowenstein, J., Zimmermann, W.: Commun. math. Phys. **39**, 1 (1974)
8. Mack, G.: In: Gatto, R. (Ed.): Scale and conformed symmetry in hadron physics. New York: J. Wiley 1973
9. Symanzik, K.: Springer Tracts in Modern Physics **57**, 222. Berlin-Heidelberg-New York: Springer 1971
10. Lowenstein, J., Speer, E.: To be published

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