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The Decay of the Bethe-Salpeter Kernel in $P(\varphi)_2$ Quantum Field Models*

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Abstract. We extend methods of high temperature expansions to show that for even weakly coupled $P(\varphi)_2$ quantum field models the Bethe-Salpeter kernel has 4 particle decay. More precisely if K denotes the Euclidean Bethe-Salpeter kernel

$$|K(x_1, x_2, x_3, x_4)| \leqq O e^{-m_0(1-\varepsilon)d_2},$$

where $x=(x^0, x^1)$, $d_2 = 2|x_1^0 + x_2^0 - x_3^0 - x_4^0| + |x_1^0 - x_2^0| + |x_3^0 - x_4^0|$ and $\varepsilon(\lambda) \to 0$ as $\lambda \to 0$. Our methods apply to other r particle irreducible kernels.

Introduction

In this paper we estimate the decay of r-particle irreducible kernels ($r \leqq 3$) for weakly coupled $\lambda P(\varphi)_2$ quantum field models. To obtain our estimates we extend the techniques of Glimm, Jaffe, and the author [1] which are related to high temperature expansions in statistical mechanics. See also [2]. A separate paper with Zirilli will use the decay of the two particle irreducible Bethe-Salpeter kernel to investigate the energy momentum spectrum of even $\lambda P(\varphi)_2$ models. For weak coupling we shall establish discreteness of the mass spectrum below $2m$ and (for $\lambda\varphi^4$) asymptotic completeness for states of mass less that $4m-\varepsilon$. Here m is the mass gap and $\varepsilon \to 0$ as $\lambda \to 0$. The detailed decay estimates of [3] also yield important information about the energy momentum spectrum such as the existence of single particle states. However such estimates do not seem to be formulated to give sufficient decay of the Bethe-Salpeter kernel. In statistical mechanics Minlos and Sinai [4] have made a detailed investigation of the spectral structure of the transfer matrix for Ising type models. Their techniques are vaguely related to ours.

The free Gaussian measure for the Euclidean field $\Phi(x)$ is denoted by $d\Phi(C) = d\Phi$ where the covariance is $C = (-\Delta + m_0^2)^{-1}$. Here Δ is the two dimensional Laplacian and m_0 is the bare mass. The action $V(\Lambda)$ in a region $\Lambda \subset R^2$ is defined by

$$V(\Lambda) = \lambda \int_\Lambda :P(\Phi(x)): dx, \qquad x = (x^0, x^1), \tag{1.1}$$

where P is a positive polynomial. The Wick order is always defined with respect to $d\Phi$. The spatially cutoff expectation

$$\langle Q \rangle_\Lambda = \frac{\int e^{-V(\Lambda)} Q \, d\Phi}{\int e^{-V(\Lambda)} d\Phi} \tag{1.2}$$

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exists for $Q\in L^p(d\Phi) p>1$. Moreover when P is even[1] [5, 6] or $\lambda/m_0^2 \ll 1$ [1, 3] the spatially cutoff Schwinger functions

$$\lim_{\Lambda\uparrow R^2} S_\Lambda(x_1,\ldots,x_n) \equiv \lim_{\Lambda\uparrow R^2} \langle \textstyle\prod_i^n \Phi(x_i)\rangle_\Lambda = S(x_1,\ldots,x_n)$$

converge and are Euclidean invariant.

For Q_i polynomials in Φ we define the truncated or partially connected expectation

$$\langle Q_1,\ldots,Q_n\rangle^c = \textstyle\sum_\varrho (-1)^{|\varrho|}(|\varrho|-1)!\prod_{\sigma\in\varrho}\langle\prod_{i\in\sigma}Q_i\rangle\,, \tag{1.3}$$

where ϱ ranges over partitions of $\{1,\ldots,n\}$. When $Q_i=\prod_{y\in X_i}\Phi(y)$ we identify (1.3) with $S^c(X_1;X_2;\ldots;X_n)$. We use the semicolon to indicate how the truncation is made. In the case of weak coupling we have an exponential cluster property so that $S^c(X_1;\ldots;X_n)$ decays exponentially fast with the distance between X_i and X_j. This is a result of [1, 3].

For the purpose of defining our expansion we consider the lattice $Z^2=Z\times Z$ as a subset of the plane R^2. Here Z denotes the integers and R denotes the reals. Let Δ_j, $j=(j^0,j^1)\in Z^2$ be the closed unit square centered at $(j^0+\frac{1}{2}, j^1+\frac{1}{2})$. For a subset $X\subset R^2$ let X_* be the set of all line segments b (bonds) joining nearest neighbor lattice sites in the interior of X. Let $U(X)$ [resp. $U_*(X)$] be the family of sets formed of unions of lattice squares $\Delta\subset X$ (resp. of bonds in X_*). If $I\subset R$ let $I_*=I\cap Z$ correspond to the lines $l_i=\{x\in R^2|x^0=i\}$ in $I\times R$.

The coupling of the field $\Phi(x)$ at different points [in the expectation (1.2)] comes entirely from the Gaussian measure $d\Phi$. The idea behind our expansion is to perturb $d\Phi$ about a decoupled measure. We break the coupling across a line l_i by introducing (zero) Dirichlet boundary conditions along the line l_i in the covariance. For a set $\gamma\subset R^2$ let Δ_γ be the Laplacian with Dirichlet boundary conditions on γ and define

$$C_\gamma=(-\Delta_\gamma+m_0^2)^{-1}\,.$$

In general we still take $\gamma\in U_*=U_*(R^2)$.

If we express $R^2\sim\gamma$ as a union of components $X_1\cup\ldots\cup X_n$, then C_γ is a direct sum over the spaces $L_2(X_1)\oplus\ldots\oplus L_2(X_n)$. Hence the corresponding measure $d\Phi(C_\gamma)$ factors over the regions X_i. Since the interaction is local the expectation (1.2) also factors.

To interpolate between covariances C and C_γ we introduce two families of parameters $t=(t_i)_{i\in Z}$ and $s=(s_b)_{b\in R^2_*}$. Qur interpolating covariances $C(t,s)$ are convex combinations of C_γ hence they are positive as bilinear forms. We shall define and estimate these covariances using the Wiener process $w(\cdot)$. Let $dP^T_{xy}(w)$ be the probability that a path starting at x at time zero will end at time T at y. We regard this expression as a density in y.

Let $J^T_\Gamma(w)=0$ if $w(\sigma)\in\Gamma$ for some σ $0\leqq\sigma\leqq T$ and let $J^T_\Gamma(w)=1$ otherwise. We define

$$C(t,s,x,y)=\textstyle\int_0^\infty e^{-m_0^2T}dT\int dP^T_{x,y}(w)\prod_i(t_i+(1-t_i)J^T_{l_i})\prod_b(s_b+(1-s_b)J^T_b)\,. \tag{1.4}$$

[1] For this case we replace $d\Phi(C)$ by $d\Phi(C_{\partial\Lambda})$ where $C_{\partial\Lambda}$ indicates that the Laplacian has Dirichlet conditions on $\partial\Lambda$.

From this representation we see that

$$0 \leqq C(t, s, x, y) \leqq C(x, y),$$

and for $(t_i)=(s_b)=(1)$, $C(s,t)=C=(-\Delta+m_0^2)^{-1}$. When $t_i=0$ $(s_b=0)$ we use Dirichlet boundary conditions on l_i (on b). If l_i separates x and y $C(t,x,y)=0$ at $t_i=0$. Hence t_i and s_b measure the coupling across l_i and b respectively. Let $d\Phi(t,s)$ be the Gaussian measure of mean zero and covariance $C(t,s)$. The expectation (1.2) defined with respect to $d\Phi(t,s)$ is denoted by

$$\langle\,\rangle(t,s) \quad \text{or} \quad S_{t,s}.$$

To illustrate a key aspect of our expansion consider a two particle irreducible kernel $k(t,x,y)$ formed with covariance $C(t)$. A precise definition of $k(t)$ appears in the next section. We expand $k(t,x,y)$ in a Taylor series in each t_i to third order about $t_i=0$. In lowest order perturbation theory (for $\lambda\varphi^4$) $k(t,x,y)$ equals

$$x \ominus y = \lambda^2 C(t,x,y)^3 .$$

Notice if l_i separates x and y

$$\frac{d^r}{dt_i^r} C(t,x,y)^3 \bigg|_{t_i=0} = 0 \qquad 0 \leqq r \leqq 2 . \tag{1.5}$$

As we shall see in Section 2 this identity holds with $k(t)$ replacing $C(t)^3$. Hence we can write

$$k(x,y) = k(t,x,y)\bigg|_{t=1} = \int_0^1 \left[\prod_i \frac{t_i^2}{2!}\frac{\partial^3}{\partial t_i^3}\right] k(t)dt ,$$

where i ranges over the interval (x^0, y^0). This representation enables us to obtain a 3 particle decay for $k(x,y)$. Similar methods are used to estimate the Bethe-Salpeter kernel.

The remainder of this paper is organized as follows. Section 2 defines the r-particle irreducible kernels and shows that their first r derivatives in t_i vanish at $t_i=0$ as in (1.5). In the next section we extend analyticity methods of Fröhlich [7] to obtain bounds on the t_i derivatives. We shall obtain the desired analyticity via an expansion in the s parameters. This expansion is called the cluster expansion [1]. We shall review it in Section 4. The final section establishes the convergence of the cluster expansion by obtaining bounds on the s derivatives.

2. The Bethe-Salpeter Kernel

We express our r-particle irreducible kernels [e.g. $k(t,x,y)$] as a Neumann series of Schwinger functions. This will enable us to reduce estimates on irreducible kernels to estimates on Schwinger functions.

We shall define $\Gamma_t(x,y)$ to be the inverse of $S_t^c(x;y)$ so that

$$\int \Gamma_t(x,y') S_t^c(y';y)dy' = \delta(x-y) . \tag{2.1}$$

Using integration by parts (see [1, 8, 9]) we have

$$C(t)_x^{-1}\langle\Phi(x);\Phi(y)\rangle^c(t)=\delta(x-y)-\lambda\langle :P'(\Phi(x)):;\Phi(y)\rangle^c(t)$$
$$\equiv I-\lambda A_1(t)\,, \tag{2.2}$$

where the second equality identifies the kernel with the corresponding operator. Another application of integration by parts yields

$$(A_1(t)C^{-1}(t))(x,y)=\langle :P''(\Phi(x)):\rangle(t)\delta(x-y)$$
$$-\lambda\langle :P'(\Phi(x):;:P'(\Phi(y)):\rangle^c(t)$$
$$\equiv B_1(t)\,. \tag{2.3}$$

P' and P'' denote the first and second derivatives respectively of the polynomial P.

For weak coupling A_1 is a bounded operator on L^2 since its kernel decays exponentially for large $|x-y|$. Local regularity is assured by the fact that Schwinger functions for $\lambda P(\varphi)_2$ models have at worst log singularities at coinciding arguments. Hence for small λ we can define

$$\Gamma_t=(1-\lambda A_1(t))^{-1}C(t)^{-1}$$
$$=C(t)^{-1}+\lambda(1-\lambda A_1)^{-1}B_1\,. \tag{2.4}$$

The one particle irreducible kernel is then

$$k(t,x,y)=(\Gamma(t)-C(t)^{-1})(x,y)$$
$$=\lambda(1-\lambda A_1(t))^{-1}B_1(t)\,. \tag{2.5}$$

Let Q_1,Q_2 be polynomials in Φ. We define the one particle irreducible expectation by

$$\langle Q_1;Q_2\rangle^1(t)=\langle Q_1;Q_2\rangle^c-\int\langle Q_1;\Phi(z_1)\rangle^c(t)\Gamma_t(z_1,z_2)\langle\Phi(z_2);Q_2\rangle^c(t)dz\,. \tag{2.6}$$

To define the Bethe-Salpeter kernel we restrict our attention to the case where P is even and define

$$D_t(x_1,x_2;x_3,x_4)=S_t^c(x_1,x_2;x_3,x_4)$$
$$D_{0t}(x_1,x_2;x_3,x_4)=S_t(x_1,x_3)S_t(x_2,x_4)+S_t(x_1,x_4)S_t(x_2,x_3)$$
$$G_t(x)=(D_t-D_{0t})(x)\,.$$

The Bethe-Salpeter kernel K is defined to be the solution of the equation

$$D_t=D_{0t}+D_{0t}K_tD_t\,. \tag{2.7}$$

We regard D_0 and D as operators on $L^2(R^2)\otimes_s L^2(R^2)$ where $\otimes_s$ is the symmetric tensor product. Hence formally

$$K_t=D_{0t}^{-1}-D_t^{-1}=\tfrac{1}{2}\Gamma_t\Gamma_t-D_t^{-1}\,. \tag{2.8}$$

Let $H_{\pm 1}$ be the Sobolev space defined by the inner product

$$\langle f,g\rangle=\langle f,(-\Delta+m_0^2)^{\pm 1}g\rangle_{L^2(R^2)}\,.$$

Glimm and Jaffe [9] have shown that in any $P(\varphi)_2$ theory with a positive mass, the Bethe-Salpeter kernel ($t=(1)$) is defined and is a bounded operator from $\otimes_s^2 H_{+1}$ to $\otimes_s^2 H_{-1}$. We shall define D_t^{-1} for weak coupling by a Neumann series

$$D_t^{-1}=(D_{0t}+G_t)^{-1}=(1+D_{0t}^{-1}G_t)^{-1}D_{0t}^{-1}.$$

Hence

$$K=(1+D_{0t}^{-1}G_t)^{-1}(D_{0t}^{-1}G_tD_{0t}^{-1}). \tag{2.9}$$

We isolate the singularities of $D_{0t}^{-1}G_t$ using integration by parts

$$C(t)_{x_1}^{-1}G(x_1,x_2,x_3,x_4)=-\lambda\langle :P'(\Phi(x_1)):;\Phi(x_2);\Phi(x_3);\Phi(x_4)\rangle_t^c$$

and

$$C(t)_{x_1}^{-1}C(t)_{x_2}^{-1}G(x)=\lambda^2\langle :P'\Phi(x_1):;:P'(\Phi(x_2)):;\Phi(x_3);\Phi(x_4)\rangle^c(t)$$
$$-\lambda\delta(x_1-x_2)\langle :P''(\Phi(x_1)):;\Phi(x_3);\Phi(x_4)\rangle^c(t).$$

See [1] and [9] for similar calculations. Using (2.5) we can write

$$D_{0t}^{-1}G_t=\lambda A_2(t)=\lambda\delta(x_1-x_2)A'_{2t}(x_1,x_3,x_4)+\lambda^2A''_{2t}(x_1,x_2,x_3,x_4)$$

also

$$D_{0t}^{-1}G_tD_{0t}^{-1}\equiv\lambda B_2(t)\equiv\sum\lambda\delta^{(k)}B_2^{(k)}(t),$$

where $\delta^{(k)}$ is a product of δ functions of (x_i-x_j). The kernels A'_{2t}, A''_{2t}, $B_{2t}^{(k)}$ are of the form

$$\int\prod_i^n\sigma_i(x_i,y_i)\langle\prod_i^n:\Phi(y_i)^{p_i}:\rangle^c(t)dy, \tag{2.10}$$

where $\sigma_i(x_i,x_i)=k(t,x_i,y_i)$ or $\delta(x_i-y_i)$. By the cluster expansion [1] the function (2.10) is exponentially decreasing in $|x_i-x_j|$, $i\neq j$ and is locally in L^4. We say that (2.10) is in DL^4. By definition we have

$$A_{2t}(x_1,\dots x_4)=2\int B_{2t}(x_1,x_2,y_3,y_4)S_t(y_3,x_3)S_t(y_4,x_4)dy$$

so that $A'_{2t}(x_1,x_3,x_4)$ and $A''_{2t}(x_1,x_2,x_3,x_3)$ are well defined functions in DL^2. Thus for small λ we can define

$$K(t)=\lambda(1+\lambda A_2(t))^{-1}B_2(t) \tag{2.11}$$

as a distribution by a convergent Neumann series. When $t=(1)$ it is easy to see that K is a bounded function in momentum space [apart from a factor $\delta(\sum^4 p_i)$].

In the case of $\lambda\varphi^4$ we can write B_2 as

$$\lambda B_2=24\lambda\delta(x_1-x_2)\delta(x_2-x_3)\delta(x_3-x_4)+\lambda^2\sum{}'\delta^{(k)}B_2^{(k)}(t)$$

so that

$$K(t)=-24\lambda\delta+\lambda^2K_1, \tag{2.12}$$

where $B_2^{(k)}$ is again of the form (2.10)

If Q_1, Q_2 are Polynomials in Φ and P is even we define the two particle irreducible expectation by

$$\langle Q_1;Q_2\rangle^2(t)=\langle Q_1;Q_2\rangle^c(t)-\int\langle Q_1,\Phi(x)\rangle\Gamma(x,y)\langle\Phi(y),Q_2\rangle dxdy$$
$$-\int\langle Q_1;\Phi(x_1)\Phi(x_2)\rangle^c(t)D_t^{-1}(x_1,x_2,x_3,x_4)\langle\Phi(x_1)\Phi(x_2);Q_2\rangle^c(t)dx\,. \tag{2.13}$$

We can again use previous methods to show the last term is well defined but we omit details.

Next we state our main theorems in momentum space. We let q, p, z be the (Euclidean) momentum conjugate to the variables

$$\frac{x_1-x_2}{2},\ \frac{x_4-x_3}{2},\ \frac{x_3+x_4}{2}-\frac{x_1+x_2}{2}$$

respectively and we express K in these variables. The one particle irreducible kernel $k(x-y)$ is expressed using u as the momentum conjugate to $x-y$.

Theorem 1. *Let k, K_1, and K be defined as above [with $t=(1)$] and suppose P is even and $\varepsilon>0$ is given. Then for m_0 sufficiently large and $\lambda>0$ sufficiently small $k(u)$, $K_1(p,q,z)$ and $K(p,q,z)$ are analytic and bounded by a constant (depending on m_0) in the region*

$$\begin{aligned}&|\mathrm{Im}\,u^0|\leqq 3m_0(1-\varepsilon),\ |\mathrm{Im}\,q^0|,\ |\mathrm{Im}\,p^0|\leqq 2m_0(1-\varepsilon)\\&|\mathrm{Im}\,q^1|,\ |\mathrm{Im}\,p^1|\leqq\varepsilon,\ |\mathrm{Im}\,z^0|\leqq 4m_0(1-\varepsilon)\,.\end{aligned} \tag{2.14}$$

Remark. The analyticity properties of K follow formally from a bound of the form

$$|K(x)|\leqq O(x)e^{-m_0(1-\varepsilon)(d_2-\mathrm{const.})},$$

where

$$d_2(x^0)=2|x_1^0+x_2^0-x_3^0-x_4^0|+|x_1^0-x_2^0|+|x_3^0-x_4^0|\,,$$

and $O(x)$ is the kernel of a finite positive measure.

For our next theorem suppose Q_1 and Q_2 are polynomials in Φ localized at times $\leqq -T/2$ and at times $\geqq T/2$ respectively.

Theorem 2. *Let $\langle\,;\rangle^i$ be the i particle irreducible expectation defined above, $i=1,2$. Suppose P is even and $\varepsilon>0$ is given. Then for m_0 sufficiently large and $\lambda>0$ sufficiently small (independent of Q_i)*

$$|\langle Q_1,Q_2\rangle^i|\leqq M(Q_1,Q_2,m_0)e^{-(1+i)m_0(1-\varepsilon)T}\,. \tag{2.15}$$

Now we show that for the r-particle irreducible kernels defined above the first r derivatives in t_i vanish at $t_i=0$. We remark that this result is obviously valid in every order of perturbation theory. Moreover using Theorem B of [2] it is possible to show perturbation theory actually establishes these identities in the case of $\lambda\varphi^4$ for weak coupling.

For $i\in Z$ let $R_\pm^2=\{x\in R^2:\pm x^0\pm i\}$. Let $P_\pm$ (resp. $P^\pm$) be the projection onto $L^2(R_\pm^2)$ (resp. onto $L^2(R_\pm^2\times R_\pm^2)$. To compute the derivatives in t we shall use the formula

$$\dot{S}_t^c(X_1;\ldots;X_n)=-\tfrac{1}{2}\int S_t^c(X_1;\ldots;X_n;z_1,z_2)\dot{C}^{-1}(z_1,z_2)dz\,, \tag{2.16}$$

where the superscript denotes a derivative with respect to t_i. Recall that the semicolon indicates which subtractions are made in the truncation (1.3). The right hand side of (2.16) is well defined using integration by parts as above and the identity

$$\dot{C}^{-1} = -C_t^{-1}\dot{C}_t C_t^{-1}.$$

See [1, 8].

The proofs of the following lemmas are complete except for minor technicalities about the domains of various kernels. These technicalities may be dealt with either by using a lattice approximation or by a more indirect method using integration by parts.

Lemma 2.1. *Let l_i separate x and y. Then*

$$\left.\frac{d^r}{dt_i^r} k(t,x,y)\right|_{t_i=0} = 0 \qquad r=0,1. \tag{2.17}$$

When P is even (2.17) *also holds for* $r=2$.

Proof. First note that since $S_t^c(x;y)=0$ when $t_i=0$ S_t^c commutes with P_+. Hence Γ_t commutes with P_+ and therefore (2.17) holds when $r=0$. By (2.16) we have

$$\dot{S}_t^c(x_1;x_2) = -\tfrac{1}{2}\int S_t^c(x_1;x_2;z_1,z_2)\dot{C}^{-1}(z_1,z_2)dz.$$

When $t_i=0$ and $x_1^0, z_1^0 \leqq i \leqq x_2^0, z_2^0$

$$S_t^c(x_1;x_2;z_1,z_2) = S_t^c(x_1;z_1)S_t^c(z_2;x_2). \tag{2.18}$$

Combining the above identities we obtain

$$\begin{aligned}\left.\frac{d}{dt_i}k(t,x,y)\right|_{t_i=0} &= \left.\frac{d}{dt_i}(\Gamma(t)-C^{-1}(t))\right|_{t_i=0}\\ &= (-\Gamma_t\dot{S}_t^c\Gamma_t - \dot{C}^{-1})|_{t_i=0}\\ &= \Gamma_t S_t^c \dot{C}^{-1} S_t^c \Gamma_t - \dot{C}^{-1} = 0.\end{aligned}$$

The factor of $\frac{1}{2}$ in (2.16) is eliminated by interchanging z_1 and z_2 above. Now suppose P is even. Then

$$\begin{aligned}\ddot{S}_t = &-\tfrac{1}{2}\int S_t^c(x_1,x_2;z_1,z_2)\ddot{C}^{-1}(z_1,z_2)dz\\ &+\tfrac{1}{4}\int S_t^c(x_1,x_2;z_1,z_2;z_3,z_4)\dot{C}^{-1}(z_1,z_2)\dot{C}^{-1}(z_3,z_4)dz.\end{aligned}$$

When $t_i=0$ and $x_1^0, z_1^0 \leqq i \leqq x_2^0, z_2^0, z_3^0, z_4^0$

$$S_t^c(x_1,\dots z_4) = S_t(x_1,z_1)S_t^c(x_2,x_2;z_3,z_4).$$

There are eight terms of this form obtained by reordering the positions of the z_i. Hence

$$\begin{aligned}\left.\frac{d^2}{dt_i^2}k(t,x,y)\right|_{t_i=0} &= -\Gamma_t\ddot{S}_t\Gamma_t + 2\Gamma_t\dot{S}_t\Gamma_t\dot{S}_t\Gamma_t - \ddot{C}^{-1}(t)|_{t_i=0}\\ &= \ddot{C}^{-1} - \tfrac{8}{4}\int \dot{C}^{-1}(x,z_2)S_t^c(z_2,x_2;z_3,z_4)\Gamma_t(x_2-y)\dot{C}^{-1}(z_3,z_4)dzdx_2\\ &\quad + 2\int \dot{C}^{-1}(x,z_2)S_t^c(z_2,x_2;z_3,z_4)\Gamma_t(x_2-y)\dot{C}^{-1}(z_3,z_4)dzdx_2 - \ddot{C}^{-1}\\ &= 0.\end{aligned}$$

Lemma 2.2. *Let P be even. If l_i separates x_1, x_2 from x_3, x_4, then*

$$\frac{d^r}{dt_i^r} K(t,x)\Big|_{t_i=0} = 0 \quad \text{for} \quad 0 \leqq r \leqq 3. \tag{2.19}$$

If l_i separates x_1 from x_2, x_3, x_4, (2.19) *holds for $0 \leqq r \leqq 2$.*

Proof. First suppose $x_1^0, x_2^0 \leqq i \leqq x_3^0, x_4^0$, and let $r=2$. Then from (2.18)

$$\frac{d^2}{dt_i^2} D_{0t}^{-1}\Big|_{t_i=0} = \dot{C}^{-1}\dot{C}^{-1}|_{t_i=0}.$$

Since $K_t = D_{0t}^{-1} - D_t^{-1}$ we have

$$\frac{d^2}{dt_i^2} K\Big|_{t_i=0} = \dot{C}^{-1}\dot{C}^{-1} + D_t^{-1}\ddot{D}_t D_t^{-1} - 2D_t^{-1}\dot{D}_t D_t^{-1}\dot{D}_t D_t^{-1}|_{t_i=0}. \tag{2.20}$$

It is easy to show D_t and D_t^{-1} commute with P^+, P^- and $P^0 \equiv 1 - P^+ - P^-$ when $t_i = 0$. Note that at $t_i = 0$

$$D_t^{-1}(y_1, y_2; y_3, y_4) = \tfrac{1}{4}\Gamma_t(y_1, y_3)\Gamma_t(y_2, y_4) + \tfrac{1}{4}\Gamma_t(y_1, y_4)\Gamma_t(y_2, y_3), \tag{2.21}$$

on the range P^0. Since

$$\dot{D} = -\tfrac{1}{2}\int S_t^c(x_1, x_2; x_3, x_4; z_1, z_2)\dot{C}^{-1}(z_1, z_2)dz,$$

one can check that $P^+\dot{D}_t P^- = P^-\dot{D}_t P^+ = 0$ for $t_i = 0$. This implies that the third term on the right of (2.20) can be written

$$-2D_t^{-1}P^-\dot{D}_t P^0 D_t^{-1} P^0 \dot{D}_t P^+ D_t^{-1}|_{t_i=0}.$$

If $y_1^0 \leqq i \leqq y_2^0$ and $t_i = 0$ then

$$P^-\dot{D}_t P^0 = -\int D_t(x_1, x_2; y_1, z_1)\dot{C}^{-1}(z_1, z_2)S_t(z_2, y_2)dz.$$

Note that there are four configurations of the y_i^0 which contribute to the $\dot{D}$'s of (2.20). Moreover for each such configuration only one term on the right of (2.21) contributed to $P^0 D_t^{-1} P^0$. Hence the third term on the right of (2.20) is

$$-2\dot{C}^{-1}\dot{C}^{-1}.$$

The second term equals

$$P^-\ddot{D}_t P^+ = -\tfrac{1}{2}\int S_t^c(x_1, x_2; x_3, x_4; z_1, z_2)\ddot{C}^{-1}(z_1, z_2)dz$$
$$+\tfrac{1}{4}\int S_t^c(x_1, x_2; x_3, x_4; z_1, z_2; z_3, z_4)\dot{C}^{-1}(z_1, z_2)\dot{C}^{-1}(z_3, z_4)dz. \tag{2.22}$$

For $t_i = 0$ the first term on the right of (2.22) vanishes as above. When $z_1^0, z_3^0 \leqq i \leqq z_2^0, z_4^0$ and $t_i = 0$ the second integrand equals

$$S_t^c(x_1, x_2; z_1, z_3)S_t^c(z_2, z_4; x_3, x_4)\dot{C}^{-1}(z_1, z_2)\dot{C}^{-1}(z_3, z_4).$$

There are four such configurations of z_i which contribute to (2.22). Now (2.19) follows easily when $r = 2$. The remaining cases are similar.

Lemma 2.3. *Let Q_1 and Q_2 be as in Theorem 2. Then for $-T/2 \leqq i \leqq T/2$ we have*

$$\frac{d^r}{dt_i^r}\langle Q_1; Q_2\rangle^j(t)=0$$

at $t_i=0$ for $0 \leqq r \leqq j$.

The proof is similar to that of Lemmas 2.1 and 2.2.

3. Analyticity

Let $f(t)$ be one of the r-particle irreducible kernels of Theorems 1 or 2. By Lemmas 2.1 and 2.3 the Taylor expansion consists of only one remainder term

$$f(t=1)=\int_0^1 \prod_i \left(\frac{t_i^{r_i}}{r_i!}\partial_{t_i}^{r_i+1}\right) f(t)dt \tag{3.1}$$

when f is r_i particle irreducible across l_i. More precisely if $f=k(t,x,y)$, i ranges over $x^0 \leqq i \leqq y^0$ and we take $r_i=1$ (or 2 if P is even). In the case of the Bethe-Salpeter kernel (P even) suppose $x_1^0 \leqq x_2^0 \leqq x_3^0 \leqq x_4^0$. Then i ranges over $x_1^0 \leqq i \leqq x_4^0$. We take $r_i=3$ for $x_2^0 \leqq i \leqq x_3^0$ and $r_i=2$ otherwise. When $f(t)=\langle Q_1; Q_2\rangle^j(t)$, i ranges over the interval $(-T/2, T/2)$ and $r_i=j$ for P even, $j=1,2$.

In this section we show how to estimate the t_i derivatives by "globalizing" analyticity techniques of Fröhlich [7]. Fröhlich obtained "local" estimates on connected Schwinger functions S^c by getting bounds on the logarithm of the generating functional $J(hg(\cdot))$ for small complex h and g in local test function. By the Cauchy formula for derivatives these bounds yield estimates on S^c. In general $J(hg)$ can vanish for global g and thus destroy the analyticity of the logarithm. However for weak coupling and small $|h|$, J does not vanish even for global g because the cluster expansion converges [1]. This fact will be extremely useful as a tool in obtaining bounds for nonlocal t derivatives.

In order to motivate how we use analyticity methods to bound t derivatives suppose we want to bound $\prod_i \partial/\partial t_i \langle \Phi(x)\rangle(t)$ for $i \in I \subset Z$. From (2.16) we see that multiple derivatives yield generalized connected Schwinger functions, with numerical kernels $\dot{C}^{-1}=-C^{-1}\dot{C}C^{-1}$ localized along the lines l_i. There are two basic problems we encounter in attempting to estimate the t derivatives.

The first problem is that when we take $|I|$ derivatives we get $|I|!$ (unconnected) terms. Furthermore the kernels $\frac{\partial}{\partial t_i}C$ are not localized about a lattice line segment b as in [1] but along an entire line and it seems that one needs tree graph decay to control the dz integration in (2.16). Eckmann, Magnen and Seneor [2] have established such a decay but with an $|I|!$ coefficient.

The analyticity methods avoid both of these problems. To illustrate this consider a perturbation of V (1.1) by

$$\sum_{i\in I} h_i \Phi_i \equiv \sum h_i \int_{|x^0-i|\leqq \frac{1}{2}} \Phi(x)dx\,. \tag{3.2}$$

Let $\langle \Phi(x)\rangle(h)$ denote the expectation of $\Phi(x)$ with respect to the perturbed action (3.2). The derivatives of this expression with respect to h_i are analogous to the t_i derivatives (2.16).

The cluster expansion [1] shows that for $\lambda/m_0^2 \ll 1$ the expectation is an analytic function of h_i and that $|\langle \Phi(x) \rangle(h)| \leqq M$ for $|h_i| \leqq \varepsilon^{-1}$. We can choose ε small for large m_0 by scaling. Then

$$\prod_{i \in I} \frac{\partial}{\partial h_i} \langle \Phi(x) \rangle(h) \bigg|_{h=0} = \oint \langle \Phi(x) \rangle(h) \prod (h_i^{-2} 2\pi i) dh\,, \tag{3.3}$$

where the contour is $|h_i| = \varepsilon^{-1}$. Hence we have the bound

$$\left| \prod_{i \in I} \frac{\partial}{\partial h_i} \langle \Phi(x) \rangle(h) \right| \leqq \varepsilon^{|I|} M\,. \tag{3.4}$$

These methods may be used to give new bounds on $S^c(x_1 \dots x_n)$ of the form

$$|S^c| \leqq \mathrm{Const.}^N \prod_\Delta N(\Delta)!\,, \tag{3.5}$$

where $N(\Delta)$ is the number of x_i localized in the lattice square Δ. This bound depends on weak coupling. In statistical mechanics such bounds appear in [10].

Suppose that we wish to extend a bound of the form (3.4) to $\langle :\Phi(x_1)^6:; \dots ; :\Phi(x_n)^6: \rangle^c$. In general the perturbation $h_i \Phi_i^6$ is *not* analytic in h since $\exp(-h\Phi^6)$ becomes unbounded for negative h. So we consider the new interaction density

$$\prod_i (1 + h_{i_0} \textstyle\int_{\Delta_i} :\Phi^6(x): dx) e^{-\lambda V}\,. \tag{3.6}$$

One can show for weak coupling that the corresponding expectation is analytic in h_i for $|h_i| \leqq \varepsilon^{-1}$ and the bound (3.4) follows. Since it seems clear that $\langle \; \rangle(t)$ is not analytic in t we shall form an expression similar to (3.6) which is analytic in the h parameter and whose derivatives in h at $h=0$ yield the t derivatives.

Let us compute the t_i derivatives using integration by parts. The basic formula is [1, 8]

$$\frac{\partial}{\partial t_i} \int e^{-\lambda V} Q d\Phi(t) = \int \left(\frac{\partial}{\partial t_i} C \cdot \Delta_\Phi \right) e^{-\lambda V} Q d\Phi(t)\,,$$

where

$$\frac{\partial}{\partial t_i} C \cdot \Delta_\Phi \equiv \int dx dy \frac{\partial}{\partial t_i} C(t, x, y) \frac{\delta^2}{\delta\Phi(x)\delta\Phi(y)}\,.$$

For $I \subset Z$ let $I^{(r)}$ be the r fold disjoint union of I with itself. We order $I^{(r)}$ by defining $i^{(m)} < k^{(m')}$ if either $i < k$ or $i = k$ and $m < m'$. Here m and m' denote the copy index for i and k respectively. For $\alpha \subset I^{(r)}$ let

$$\partial_t^\alpha = \prod_i \frac{d^{r_i}}{dt_i^{r_i}}\,,$$

where r_i is the number of copies of i which are in α. Let $\beta \subset I^{(r)}$ and let $\mathscr{P}(\beta)$ be the partitions of β. Then by Leibnitz rule we have

$$\partial_t^\beta \int e^{-\lambda V} Q d\Phi(t) = \sum_{\pi \in \mathscr{P}(\beta)} \int \prod_{\alpha \in \pi} [\partial_t^\alpha C \cdot \Delta_\Phi] e^{-\lambda V} Q d\Phi(t)\,. \tag{3.7}$$

For each $\alpha \subset I^{(r)}$ we introduce a complex variable $h(\alpha)$.

Let $j=(j_1,j_2)\in Z^4$ be a localization index and define

$$C_j(x_1,x_2)=\chi_{\Delta_{j_1}}(x_1)C(x_1,x_2)\chi_{\Delta_{j_2}}(x_2),$$

where χ_Δ is the characteristic function of Δ. By (3.7) we have

$$\partial_t^\beta \int e^{-\lambda V}Q\,d\Phi(t)$$
$$=\sum_{\pi\in\mathscr{P}(\beta)}\left(\prod_{\alpha\in\pi}\frac{\partial}{\partial h(\alpha)}\right)\int\prod_{\alpha\in\pi}\prod_{j\in Z^4}(1+h(\alpha)\partial_t^\alpha C_j\cdot\Delta_\Phi)e^{-\lambda V}Q\,d\Phi(t)\Big|_{h=0}. \tag{3.8}$$

Note that if more than one copy of an integer appears in α then $\partial_t^\alpha C=0$ because $C(t)$ is only linear in the t_i. The next two lemmas enable us to make a connection with t derivatives and derivatives in the parameter $h(\alpha)$ introduced above.

Define

$$\delta_h^\beta=\sum_{\pi\in\mathscr{P}(\beta)}\prod_{\alpha\in\pi}\frac{\partial}{\partial h(\alpha)}.$$

Lemma 3.1. *Let F_1 and F_2 be smooth functions of h. Then*

$$\delta_h^\beta(F_1F_2)=\sum_{\alpha\subset\beta}(\delta_h^\alpha F_1)(\delta_h^{\beta/\alpha}F_2).$$

Proof. $\delta_h^\beta F_1F_2=\sum_{\pi\in\mathscr{P}(\beta)}\prod_{\alpha\in\pi}\frac{\partial}{\partial h(\alpha)}F_1F_2$

$$=\sum_{\pi\in\mathscr{P}(\beta)}\sum_{\pi_1+\pi_2=\pi}\left(\prod_{\alpha_1\in\pi_1}\frac{\partial}{\partial h(\alpha_1)}F_1\right)\left(\prod_{\alpha_2\in\pi_2}\frac{\partial}{\partial h(\alpha_2)}F_2\right)$$
$$=\sum_{\alpha\subset\beta}\left(\sum_{\pi_1\in\mathscr{P}(\alpha)}\prod_{\alpha_1\in\pi_1}\frac{\partial}{\partial h(\alpha_1)}F_1\right)\left(\sum_{\pi_2\in\mathscr{P}(\beta/\alpha)}\prod_{\alpha_2\in\pi_2}\frac{\partial}{\partial h(\alpha_2)}F_2\right).$$

Lemma 3.2. *Let $f_j(t,h)$ be smooth functions of t and h such that*

$$\delta_t^\beta f_j|_{h=0}=\delta_h^\beta f_j|_{h=0}$$

then

$$\partial_t^\beta\prod f_j|_{h=0}=\delta_h^\beta\prod f_j|_{h=0} \tag{3.9a}$$

and

$$\partial_t^\beta f^{-1}|_{h=0}=\delta_h^\beta f^{-1}|_{h=0}. \tag{3.9b}$$

Proof. (3.9a) follows from Lemma 3.1. We establish (3.9b) by induction on $|\beta|$, the number of elements of β. By Lemma 3.1

$$0=\delta_h^\beta f^{-1}f=\sum\delta_h^\alpha f^{-1}\delta_h^{\beta/\alpha}f.$$

Hence by induction at $h=0$ we have

$$\delta_h^\beta f^{-1}=-f^{-1}\sum_{\alpha\subsetneq\beta}\delta_h^\alpha f^{-1}\delta_h^{\beta/\alpha}f$$
$$=-f^{-1}\sum_{\alpha\subsetneq\beta}\partial_t^\alpha f^{-1}\partial_t^{\beta/\alpha}f=\partial_t^\beta f^{-1}.$$

Let Q be a polynomial in Φ. We define the expectation

$$\langle Q\rangle(t,h)=\frac{\int\prod_{\alpha\subset I^{(r)}}\prod_{j\in Z^4}(1+h(\alpha)\partial_t^\alpha C_j\cdot\Delta_\Phi)e^{-\lambda V}Qd\Phi(t)}{\int\prod_{\alpha\subset I^{(r)}}\prod_{j\in Z^4}(1+h(\alpha)\partial_t^\alpha C_j\cdot\Delta_\Phi)e^{-\lambda V}d\Phi(t)} \tag{3.10}$$

and set

$$C(t,h,x,y)=\langle\Phi(x)\Phi(y)\rangle(t,h)\quad\text{at}\quad\lambda=0\,.$$

Similarly using (2.4)–(2.6), (2.12), and (2.13) we can define

$$k(t,h),\,K(t,h),\,\langle\;\rangle^i(t,h)\,.$$

Let $f(t,h)$ denote one of the above kernels. Then (3.8) and Lemma 3.2 imply that

$$\partial_t^\alpha f(t,h)|_{h=0}=\delta_h^\alpha f(t,h)|_{h=0}\,,\qquad\alpha\subset I^{(n)}\,. \tag{3.11}$$

Notice that $f(t,h)$ is formally analytic in h.

Let $w(x_1\ldots x_n)\in L^{4/3}(\prod_{j=1}^n\Delta_{i_j})$. For $\alpha\subset I^{(r)}$ let $d(\alpha)=\infty$ if α contains more than one copy of any integer. Otherwise we define

$$\begin{aligned}&d(\alpha)=0\qquad|\alpha|=1\\&d(\alpha)=\max\{|i-j|:i,j\in\alpha\}\,.\end{aligned} \tag{3.12}$$

The following theorem estimates $S(t,h)$ as a function of h, where $h=(h(\alpha))_{\alpha\in\pi}$ and $\pi\in\mathscr{P}(I^{(r)})$.

Theorem 3. *Let $\varepsilon>0$, $r\in Z^+$ and $I\subset Z$ be given. Then for $m_0(r,\varepsilon)$ sufficiently large and $\lambda(m_0,r,\varepsilon)$ sufficiently small*

$$\int w(x)\langle\prod_i^n:\Phi(x_i)^{p_i}:\rangle(t,h)dx\equiv\langle Q_w\rangle(t,h)$$

is analytic in h, for h in the region

$$|h(\alpha)|\leqq e^{+m_0(1-\varepsilon)(d(\alpha)+1)}\,. \tag{3.13}$$

Let $w_1\cdot w_2=w$ and $T=dist.$ (supp w_1, supp w_2). Then for h in (3.13) *there is a constant a such that*

$$|\langle Q_{w_1};Q_{w_2}\rangle^c(t,h)|\leqq e^{-T}e^{m_0 a\deg Q}M(\|w\|_{L^{4/3}},\deg Q)\,. \tag{3.14}$$

The proof of this theorem is given in the following sections. The idea behind the proof is this: We shall show in Section 5 that

$$|\int\partial_t^\alpha C_t(x,y)dy|\leqq O(1)e^{-d(\alpha)m_0}\,.$$

This estimate comes from the fact that the Weiner path [see (1.4)] must hit each line l_i, $i\in\alpha$ so that it must traverse a distance of at last $d(\alpha)$. When α has more than one copy of the same integer $\partial_t^\alpha C$ vanishes so we can take $d(\alpha)=\infty$. If α contains only a single element we do not gain a convergence factor from the covariance. However for every $h(\alpha)$, $\alpha\in\pi$ there is a derivative Δ_Φ which differentiates the exponent giving us effectively a factor of $\lambda^{2/p}$ where p is the degree of the interaction. This is because

$$(\Delta_\Phi)^n\exp(-\lambda V)\approx\lambda^{2n/p}$$

as one can easily check. By choosing $\lambda \leqq e^{-m_0 p/2}$ and m_0 large we can see that

$$h(\alpha)\partial_t^\alpha C_j \Delta_\Phi$$

is small compared to 1 when h lies in the region (3.13). Hence the cluster expansion in the new parameters (s_b) can be applied as in [1] to show

$$\langle Q\rangle(h, t) = \frac{\int \prod_{\alpha \subset I^{(r)}} \prod_{j \in Z^4} (1 + h(\alpha)\partial_t^\alpha C_j \cdot \Delta_\Phi) Q e^{-\lambda V} d\Phi(t)}{\int \prod_{\alpha \subset I^{(r)}} \prod_{j \in Z^4} (1 + h(\alpha)\partial_t^\alpha C_j \cdot \Delta_\Phi) e^{-\lambda V} d\Phi(t)}$$

is analytic for h satisfying (3.13).

Next we state a lemma which will be useful for establishing Theorem 1.

Let $\zeta_j(x)$, $(j \in Z^2)$ be a continuous partition of unity with ζ_j supported in the region $|j - x| \leqq 1$.

Lemma 3.3. *Let $f(x)$ and $g(x_1, \ldots, x_4)$ be continuous functions. Under the hypothesis of Theorem 3 the expressions*

$$\int k(t, h, 0, x) f(x) \zeta_j(x) dx; \int K(t, h, x) \prod_{i=1}^4 \zeta_{j_i}(x_i) g(x) dx \tag{3.15}$$

are analytic and uniformly bounded by $|f\zeta|_\infty$, $|g\zeta|_\infty$ for h belonging to (3.13). *Here $|\,|_\infty$ is the sup norm and K may be replaced by K_1.*

We now turn to the proof of Theorem 1 using Theorem 3 and the above lemma. The proof of Theorem 2 is similar. It suffices to show that for $t = 1$ and $h = 0$, (3.15) is bounded by

$$|f|_\infty e^{-3m_0(1-\frac{\varepsilon}{2})(|j^0| - \text{Const})}; \; |g|_\infty e^{-m_0(1-\frac{\varepsilon}{2})(d_2(j_1^0, \ldots j_4^0) - \text{Const.})} \tag{3.16}$$

respectively. The analyticity of the spatial momenta $(p^1, q^1 \ldots)$ is obtained from the above bound using the Euclidean invariance of k, K, and K_1.

Let $\beta_1 = [0, j^0 - 1]_*^{(3)}$ be the three fold disjoint union of the integers in the interval $[0, j^0 - 1]$. From (3.1) and (3.11) we have

$$\begin{aligned} k(0, x)\zeta_j f(x) &= \int_0^1 (\prod t_i^2/2) \partial_t^{\beta_1} k(t, 0, x) dt \zeta_j f(x) \\ &= \int_0^1 (\prod t_i^2/2) \delta_h^{\beta_1} k(t, h, 0, x)|_{h=0} dt \zeta_j f(x) \\ &= \int (\prod t_i^2/2) \sum_{\pi \in \mathcal{P}(\beta_1)} \oint k(t, h, 0, x) \prod_{\alpha \in \pi} \frac{h(\alpha)^{-2}}{2\pi i} dh dt \zeta_j f(x) \end{aligned} \tag{3.17}$$

where the contour is given by $|h(\alpha)| = e^{m_0(1-\varepsilon)(d(\alpha)+1)}$. After integrating over x and applying Lemma 3.3 we see that each term in the sum over π is bounded by $O \exp[-m_0(1-\varepsilon)\sum_\alpha (d(\alpha)+1)]$. It is important to note that for $\pi \in \mathcal{P}(\beta_1)$,

$$\sum_{\alpha \in \pi} (d(\alpha) + 1) \geqq 3(j^0 - 1).$$

For the Bethe-Salpeter kernel we shall suppose $1 \leqq j_i^0 + 1 \leqq j_{i+1}^0 - 1$ for $i = 1, \ldots, 4$, and set

$$\beta_2 = [j_1^0 + 1, j_2^0 - 1]_*^{(3)} \cup [j_2^0 + 1, j_3^0 - 1]_*^{(4)} \cup [j_3^0 + 1, j_4^0 - 1]_*^{(3)}.$$

(Other configurations of j_i^0 can be treated similarly.) As above we have an equation entirely analogous to (3.17) and

$$\begin{aligned} &\left|\int K(x_1, \ldots, x_4) g(x) \prod_{i=1}^4 \zeta_{j_i}(x_i) dx\right| \\ &\quad \leqq \text{Const.}\, |g|_\infty \sum_{\pi \in \mathcal{P}(\beta_2)} \exp[-m_0(1-\varepsilon)\sum_{\alpha \in \pi} (d(\alpha)+1)]. \end{aligned} \tag{3.18}$$

We note that for $\pi \in \mathscr{P}(\beta_2)$

$$\begin{aligned}\textstyle\sum_{\alpha\in\pi}(d(\alpha)+1) &\geqq 3|j_2^0-j_1^0-2|+4|j_3^0-j_2^0-2|+3|j_4^0-j_3^0-2|\\ &\geqq d_2(j^0)-20\,.\end{aligned}$$

We now estimate the sum over $\pi \in \mathscr{P}=\mathscr{P}(\beta_1)$ or $\mathscr{P}(\beta_2)$. Consider all partitions π having n elements α_i, $i \leqq n$ ordered so that $\min \alpha_i \leqq \min \alpha_{i+1}$. Given a sequence of positive integers c_i we count the number of partitions for which $d(\alpha_i)+1=c_i$; $1 \leqq i \leqq n$. If we fix c and $\min \alpha$ then there are at most 2^{4c} $\alpha \subset Z^{(4)}$ such that $d(\alpha)+1=c$ because α is in fact a subset of $I^{(4)}$ where $I=Z\cap[\min \alpha,\ \min \alpha+c]$. Given the sequence c_i, we can choose $\min \alpha_i$ in at most $4c_{i-1}$ ways or else the α_i would not form a partition. Thus the sum (3.17), (3.18) is bounded by

$$\begin{aligned}&\textstyle\sum_n \sum_{c_i \geqq 1, \Sigma^n c_i \geqq c_0} \prod_i 4c_i 2^{4c_i} e^{-m_0(1-\varepsilon)c_i}\\ &\quad\leqq c_0 \textstyle\sum_{c_i \geqq 1\, \Sigma c_i \geqq c_0} \prod_i e^{-m_0(1-2\varepsilon)c_i}\\ &\quad\leqq c_0 \textstyle\sum_{c \geqq c_0} 2^c e^{-m_0(1-2\varepsilon)c} \leqq e^{-m_0(1-3\varepsilon)c_0}\,,\end{aligned} \tag{3.19}$$

where $c_0=3|j^0-1|$ or (d_2-20). We have used the fact that $\pi\in\mathscr{P}$ has at most c_0 elements and that there are less than 2^c sequences $c_i \geqq 1$ such that $\sum c_i = c$. Note that K may be replaced by K_1 in the above argument.

We now turn to the proof of Lemma 3.3 which follows the first part of Section 2.

Proof of Lemma 3.3. We express k, K using (2.5) and (2.11)

$$k(t,h)=\lambda(1+\lambda A_1(t,h))^{-1}B_1(t,h)$$

$$K(t,h)=\lambda(1+\lambda A_2(t,h))^{-1}B_2(t,h)\,.$$

By Theorem 3 $A_1(t,h,x,y)$ is in DL^4 i.e. decays exponentially in $|x-y|$ and is locally in L^4. Thus the Neumann series for $(1+\lambda A_1)^{-1}$ converges for small λ and

$$k(t,h,x,y)-\langle :P''(\Phi(x)):\rangle(t,h)\delta(x-y)$$

is in DL^4. This establishes the lemma for k. Since

$$\textstyle\int \prod_i^n \sigma_i(x_i,y_i) \langle \prod_i^n :\Phi(y_i)^{p_i}: \rangle(t,h)dy$$

is in DL^2, where $\sigma_i(x,y)=k(t,h,x,y)$ or $\delta(x-y)$, $B_2^{(k)}$, A_2', A_2'' are in DL^2. From the identity

$$A_2(t,h,x_1,\dots,x_4)=2\textstyle\int B_2(t,h,x_1,x_2,y_3,y_4)S(t,h,y_3,x_3)S(t,h,y_4,x_4)dy$$

we see that $A_2'(t,h,x_1,x_2,x_3,x_4)$ and $A_2''(t,h,x_1,x_2,x_3,x_3)$ are in DL^2. Thus the Neumann series for $(1+\lambda A_2)^{-1}$ converges as well and we see that the lemma holds for K. A similar argument works for K_1.

Remark. The continuous function $g(x_1,\dots x_4)$ is introduced to control the δ function singularities which arise in the Bethe-Salpeter kernel. By choosing λ sufficiently small we can bound (3.15) by an arbitrarily small constant times $|f\zeta|_\infty, |g\zeta|_\infty$. This is not so for K_1 since we have already factored out the coefficient λ^2.

4. The Cluster Expansion

To obtain Theorem 3 we apply the cluster expansion (see [1]) in the parameters s_b. This section reviews the cluster expansion in a slightly different form.

Fix a region Λ whose boundary $\partial\Lambda$ is a union of lattice line segments. Λ serves as a space cutoff. We define $s=(s_b)$ for $b\in\Lambda_*$ and we fix $s_b=0$ for $b\in\partial\Lambda$ which imposes Dirichlet boundary conditions on $\partial\Lambda$ in the covariance $C(s,t)$. See Section 1 for notation. If Γ is a union of line segments [i.e. $\Gamma\in U_*(\Lambda)$] let Γ^c denote the complementary set $\Lambda_*\sim\Gamma$. Since we shall expand about $s=0$ i.e. a decoupled theory, it is convenient to introduce the notation

$$s(\Gamma)_b=s_b,\, b\subset\Gamma;\; s(\Gamma)_b=0\; b\subset\Gamma^c,\partial\Lambda\,.$$

Hence $C(s(\Gamma),t)$ has Dirichlet boundary conditions on Γ^c and the corresponding Gaussian measure $d\Phi(s(\Gamma),t)$ factors over the components X_i' of $\Lambda\sim\Gamma^c$ so that

$$\Lambda\sim\Gamma^c=X_1'\cup\ldots\cup X_n'\,. \tag{4.1}$$

Let X_i be the closure of X_i'. Note that $X_i\in U(\Lambda)$.

For each $Y\in U(\Lambda)$ and each $\pi\in\mathscr{P}(I^{(r)})$ we define

$$\begin{aligned}&F_Y(t,s(Y_*),h)\\&\quad=\int\prod_{\alpha\in\pi}\prod_{j\in Z^4}\left[1+h(\alpha)\partial_t^\alpha C_j(t,s(Y_*))\cdot\Delta_\Phi\right]e^{-V(Y)}Q_Y d\Phi(t,s(Y_*))\,,\end{aligned} \tag{4.2}$$

where Q_Y is the product of those fields of the monomial Q which are localized in Y. Let $\Lambda_1\subset U(\Lambda)$ and $\Gamma\subset U_*(\Lambda_1)$. We define Y_i to be the closure of the components of $\Lambda_1-\Gamma^c$.

Lemma 4.1. *Let the family* $\{F_X(s(X_*))\}$ *be defined by* (4.2). *If* $\Gamma\in U_*(\Lambda_1)$ *then*

$$F_{\Lambda_1}(s(\Gamma))=\prod_i F_{Y_i}(s(\Gamma\cap Y_i)) \tag{4.3}$$

with Y_i *defined above.*

Proof. Note that

$$\partial_t^\alpha C_j(t,s(\Gamma))\cdot\Delta_\Phi e^{-V(Y_i)}Q_{Y_i}=0 \tag{4.4}$$

unless both j_1 and j_2 are in Y_i. Hence

$$(1+\partial_t^\alpha C_j\cdot\Delta_\Phi)e^{-V(\Lambda_1)}Q=\prod_i\left[(1+\partial_t^\alpha C_j\cdot\Delta_\Phi)e^{-V(Y_i)}Q_{Y_i}\right].$$

The lemma now follows by taking a product over α and j by the factorization property for the measure $d\Phi(s(\Gamma))$.

We say that a family of functions $\{F_X(s(X_*))\}$ *decouples at* $s=0$ if (4.3) holds for each $\Lambda_1\in U(\Lambda)$ and each $\Gamma\in U_*(\Lambda_1)$.

The cluster expansion is derived as follows. By the fundamental theorem of calçulus for any C^1 function $F(s)$ we have

$$F(s)=\sum_{\Gamma\in U_*(\Lambda)}\int_0^{s(\Gamma)}\partial_\sigma^\Gamma F(\sigma(\Gamma))d\sigma(\Gamma)\,, \tag{4.5}$$

where $\partial_\sigma^\Gamma=\prod_{b\subset\Gamma}\partial/\partial\sigma_b$. The integral does not appear when $\Gamma=\emptyset$. Let $X_0\subset\Lambda$. If F decouples at $s=0$ then each term of (4.5) is a product of the X_i defined via (4.1).

Let W be the union of all X_i which meet X_0. We hold W and $W \cap \Gamma$ fixed and resum over all $\Gamma \in U_*(\Lambda \sim W)$. This yields

$$F_\Lambda(s) = \sum_{X \supset X_0} \left(\sum_{\Gamma | X_0} \int_0^s \partial_\sigma^\Gamma F_X(\sigma(\Gamma)) d\sigma(\Gamma) \right) \cdot F_{\Lambda \sim X}(s) . \tag{4.6}$$

Here X ranges over elements of $U(\Lambda)$ and Γ ranges over elements of $U_*(X)$ (including $\Gamma = \emptyset$) such that:

$$\text{Each component of } X \sim \Gamma^c \text{ meets } X_0 . \tag{4.7}$$

If no such Γ exists for a given X the term is defined to be zero. Note that (4.7) implies each component of X must meet X_0.

Theorem 4. *Suppose F is a C^1 function of $s = (s_b)$, $(b \in \Lambda_*)$ which satisfies the following conditions*

A. $F(s)$ decouples at $s = 0$

B. $|F_\Lambda(s(\emptyset))| \geqq 1/2$.

C. $\sup_s |\partial_s^\Gamma F_X(s)| \leqq e^{0(1)|X|} e^{-K|\Gamma|}$.

Then for K sufficiently large (independent of Λ)

$$|F_{\Lambda \sim X}(s) F_\Lambda(s)^{-1}| \leqq 8^{|X|} . \tag{4.8}$$

Proof. Let $b_1 \in \Lambda_*$ and define

$$\delta F_X(s) = F_X(s(X_*)) - F_X(s(X_* \sim b_1)); \quad \text{if} \quad b_1 \subset X$$

$$= F_X(s(X_*)) \quad \text{otherwise.}$$

It is easy to check that δF decouples at $s = 0$. Note that

$$F_\Lambda(s(\Gamma \cap (\Lambda \sim X)_*) = \prod_{\Delta \subset X} F_\Delta(s(\emptyset)) F_{\Lambda \sim X}(s(\Gamma \cap (\Lambda \sim X)_*)) . \tag{4.9}$$

We reexpress (4.6) in the form

$$\delta F_\Lambda(s(\Gamma)) = \sum_{X \subset b_1} c_\Gamma(X) F_\Lambda(s(\Gamma \cap (\Lambda \sim X)_*)) , \tag{4.10}$$

where

$$c_\Gamma(X) = \prod_{\Delta \subset X} F_\Delta^{-1} \int_0^{s(\Gamma)} \sum_{\Gamma' | X} \partial_\sigma^{\Gamma'} \delta F_X(\sigma(\Gamma')) d\sigma(\Gamma') .$$

Since $X_0 = \{b_1\}$, (4.7) implies X is connected. If X is a single lattice square then $b_1 \subset \partial X$ and $\delta F_X = 0$. Thus we can assume $|X| \geqq 2$. To bound $c_\Gamma(X)$ we note that there are at most $2^{2|X|}$ choices of $\Gamma' \in U_*(X)$. Also since $X \sim \Gamma^c$ is connected

$$|X| - 1 \leqq 2|\Gamma| .$$

Combining the above observations with B and C we have

$$|c_\Gamma(X)| \leqq e^{-K(|X|-1)/3} \tag{4.11}$$

for large K. Next we state an elementary combinatorial lemma. See Proposition 5.1 of [1] for the proof.

Lemma 4.2. *Let X_0 and $|X|$ be given. The number of sets $X \in U(R^2)$ such that each component of X meets X_0 is bounded by*

$$e^{O(1)|X|}.$$

We now establish (4.8) by induction on $|\Gamma|=n$, and we set $F=F_\Lambda$. Assume that

$$\tfrac{1}{2} \leqq |F(s(\Gamma_1 \sim b_1)) \cdot F(s(\Gamma_1))^{-1}| \leqq 2$$

for $|\Gamma_1|<n$. Consequently for $\Gamma_2 \subset \Gamma_1$, $|\Gamma_1|<n$ we have

$$|F(s(\Gamma_2)) \cdot F(s(\Gamma_1))^{-1}| \leqq 2^{|\Gamma_1 \sim \Gamma_2|}. \tag{4.12}$$

By (4.10), (4.11), Lemma 4.2 and the induction hypothesis

$$\begin{aligned} |F(s(\Gamma)) \cdot F(s(\Gamma \sim b_1))^{-1} - 1| &= |\delta F(s(\Gamma)) \cdot F(s(\Gamma \sim b_1))^{-1}| \\ &\leqq \textstyle\sum_{X \supset b_1} c_\Gamma(X) |F(s(\Gamma) \cap (\Lambda \sim X)_*) F(s(\Gamma))^{-1}| \\ &\leqq \textstyle\sum_{X \supset b_1} c_\Gamma(X) 2^{2|X|} \leqq \sum_{r \geqq 2} e^{O(1)r} 4^r e^{-K(r-1)/3} \leqq \tfrac{1}{2} \end{aligned}$$

for K sufficiently large. Thus we have established (4.12). To complete the proof of the theorem we use (4.9) with $\Gamma^c = \emptyset$

$$\begin{aligned} |F_{\Lambda \sim X}(s) F_\Lambda(s)^{-1}| &= |\textstyle\prod_{\Delta \subset X} F_\Delta^{-1} F_\Lambda(s(\Lambda \sim X)_*) F_\Lambda(s)^{-1}| \\ &\leqq 2^{|X|} 2^{2|X|} = 8^{|X|}. \end{aligned}$$

In the next section we show that if F is given by (4.2) there is a constant a such that

$$|\partial_s^\Gamma F_X(t,s,h)| \leqq e^{-K|\Gamma|} e^{+O(1)|X|} \cdot e^{m_0 \alpha \deg Q} M(\|w\|_{L^{4/3}}, \deg Q) \tag{4.13}$$

for h belonging to (3.13). We now prove Theorem 3 assuming that (4.13) holds for K sufficiently large. The proof follows the lines of ([1], Section 4).

Proof of Theorem 3. Let $X_0 = X_0^1 \cup X_0^2$ where X_0^i is the localization of Q_i. We express $\langle Q_1; Q_2 \rangle^c$ by introducing an independent copy of field Φ' so that

$$2\langle Q_1; Q_2 \rangle^c(t,s,h) = \langle (Q_1 - Q_1')(Q_2 - Q_2') \rangle (t,s,h),$$

where the expectation on the right is the product expectation. Now let F be defined by

$$\begin{aligned} F_X(t,s,h) = \textstyle\int \prod_{\alpha \in \pi} \prod_{j \in Z^4} &[(1 + h(\alpha)\partial_t^\alpha C_j \cdot \Delta_\Phi) \cdot (1 + h(\alpha)\partial_t^\alpha C_j \cdot \Delta_{\Phi'})] \\ &\cdot (Q_1 - Q_1')(Q_2 - Q_2') e^{-\lambda(V(X) + V'(X))} d\Phi(s,t) d\Phi'(s,t) \end{aligned} \tag{4.14}$$

and let the normalization $Z_X(t,s,h)$ be defined as in (4.14) but with the Q's absent. F and Z are easily seen to satisfy (4.13) and decouple at $s=0$. Furthermore for small λ

$$|Z_\Delta| = |\textstyle\int e^{-V(\Delta)} d\Phi(s,t)|^2 \geqq \tfrac{1}{2}$$

hence Z satisfies the hypotheses of Theorem 4. By using the symmetry $\Phi \to \Phi'$ we observe that $F_X = 0$ if no component of X meets both X_0^1 and X_0^2. We expand F via (4.6)

$$\begin{aligned} \langle Q_1; Q_2 \rangle^c(t,s,h)|_{s=1} &= \lim_{\Lambda \uparrow R^2} |F_\Lambda Z_\Lambda^{-1}| \\ &\leqq \lim_{\Lambda \uparrow R^2} \textstyle\sum_X |Z_{\Lambda \sim X} Z_\Lambda^{-1}| |\sum_{\Gamma | X} \int \partial_s^\Gamma F_X ds(\Gamma)|. \end{aligned}$$

By Theorem 4 the first factor is bounded by $8^{|X|}$. To bound the second factor we first observe that X has at most $\deg Q$ components, hence $2|\Gamma| \geqq |X| - \deg Q$ (see 5.1 of [1]). By (4.13) we can bound the second factor by

$$Me^{\sim K|X|} e^{am_0 \deg Q}.$$

Since at least one component of X meets X_0^1 and X_0^2 we may assume that $(X) \geqq \operatorname{dist}(X_0^1, X_0^2) \geqq T$. Now we have

$$|\langle Q_1; Q_2\rangle^c(t,h)| \leqq Me^{am_0 \deg Q} \sum_{X \supset X_0} 8^{|X|} e^{-K|X|}$$
$$\leqq Me^{am_0 \deg Q} e^{-T}.$$

Here again we have used Lemma (4.2) to control the sum over X.

5. Estimate of $\partial_s^\Gamma F$

To complete the proof of Theorem 3 it suffices to establish (4.13) for K sufficiently large and h belonging to (3.13). The proof of this bound relies heavily on Sections 8–10 of [1]. The proof is identical except of course for the additional factors

$$\prod_{\beta \in B_0(X)} (1 + h(\alpha) \partial_t^\alpha C_j \cdot \Delta_\Phi),$$

where $B_0(X) = \pi_0 \times (Z^2 \cap X)^2$ and $\beta = (\alpha, j)$. [We make a change in notation: π and Y are replaced by π_0 and X respectively in (4.2).] We expand the above product in the form

$$\sum_{B \subset B_0} \prod_{\beta \in B} h(\alpha) \partial_t^\alpha C_j \cdot \Delta_\Phi.$$

To obtain (4.13) it suffices to show that for each $B \subset B_0$

$$|\partial_s^\Gamma \int [\prod_{\beta \in B} h(\alpha) \partial_t^\alpha C_j \cdot \Delta_\Phi] Q_X e^{-V(X)} d\Phi(t,s)| \leqq D \prod_{\beta \in B} e^{-d(\beta)/2}, \tag{5.1}$$

where

$$D = e^{O(1)|X|} e^{-K|\Gamma|} e^{am_0 \deg Q} \|w\|_{L^{4/3}} (2p \deg Q)^{2 \deg Q} \tag{5.2a}$$

and $d(\beta)$ are constants to be defined such that

$$\sum_{\beta \in B_0(X)} e^{-d(\beta)/2} = O(1)|X|. \tag{5.2b}$$

Here we are using the fact that by (5.2b)

$$\sum_{B_0 \supset B} \prod_{\beta \in B} e^{-d(\beta)/2} = \prod_{\beta \in B_0} (1 + e^{-d(\beta)/2}) \leqq \prod_{\beta \in B_0} \exp(e^{-d(\beta)/2}) = e^{O(1)|X|}.$$

We compute the left hand side of (5.1) as in (3.7)

$$\sum_{\Gamma_1 + \Gamma_2 = \Gamma} \int [\partial_s^{\Gamma_1} \prod_{\beta \in B} h(\alpha) \partial_t^\alpha C_j \cdot \Delta_\Phi] [\sum_{\pi \in \mathscr{P}(\Gamma_2)} \prod_{\gamma \in \pi} \partial_s^\gamma C \cdot \Delta_\Phi] e^{-\lambda V(X)} Q_X d\Phi(t,s) \tag{5.3a}$$

and again by Leibnitz rule

$$\partial_s^{\Gamma_1} (\prod_\beta \partial_t^\alpha C_j) = \sum_{\Sigma \gamma_\beta' = \Gamma_1} \prod_\beta \partial_s^{\gamma_\beta'} \partial_t^\alpha C_j. \tag{5.3b}$$

The sum $\sum_\beta \gamma'_\beta = \Gamma_1$ is over sequences of mutually disjoint subjects whose union is Γ_1.

There are at most $2^{|\Gamma|}$ terms in the sum (5.3a) over Γ_1, Γ_2. If we fix Γ_1 and Γ_2 and substitute $C = \sum_{j \in Z^4} C_j$ the sum in (5.3) ranges over

$$\pi \in \mathscr{P}(\Gamma_2), \qquad \{j_\gamma\}_{\gamma \in \pi} \qquad \{\gamma'_\beta\}_{\Sigma \gamma'_\beta = \Gamma_1}.$$

For each term in the sum (5.3) let $T(\pi, \{j\}, \{\gamma'_\beta\})$ be the number of terms coming from the differentiations Δ_Φ. Let $N(\Delta)$ be the degree of a resulting monomial of Φ in a lattice square Δ and let $|B|$ be the number of elements in B. Since there are at least $g = \max(0, 2|B|/p - \deg Q)$ differentiations of the exponent $\lambda P(\Phi)$ each term has a coefficient of $|\lambda|^g$. By Corollary 9.6 of [1] each term resulting from differentiation is bounded by

$$|\lambda|^g \|w'\|_{L^p} e^{M_0|X|} \prod_{\Delta \subset X} N(\Delta)! M_1^{N(\Delta)} \tag{5.4}$$

for all $m_0 \geqq 1$, $0 \leqq |\lambda| \leqq \lambda_0$ and $\operatorname{Re}\lambda \geqq 0$. Here w' is the w of Theorem 3 multiplied by the kernels $h(\alpha)\partial_s^\gamma \partial_t^\alpha C_j$ arising from (5.3) hence w' is of the form

$$w' = \int \prod_{(\alpha, j) \in B} h(\alpha) \partial_s^{\gamma_\beta} \partial_t^\alpha C_j \prod_{\gamma \in \pi} \partial_s^\gamma C_{j(\gamma)} \prod \delta,$$

where the integral is over the contracted variables in the δ function. See Section 9 of [1].

Next we turn to estimates on $\partial_s^\gamma \partial_t^\alpha C_j$. Let

$$d(j, \gamma) = \max_{b \subset \gamma} \{\operatorname{dist}(j_1, b) + \operatorname{dist}(j_2, b)\}, \tag{5.5a}$$

$$d(\beta) = d(j, \alpha) = \max\{d(j, \ell_{i(\alpha)}), |j_1 - j_2|\}, \tag{5.5b}$$

$$d(\gamma, \alpha) = \min_{b \subset \gamma} \operatorname{dist}(b, l_{i(\alpha)}), \tag{5.5c}$$

where $i(\alpha)$ is the least integer in α. With this definition of $i = i(\alpha)$ we define

$$|X_\alpha| = |l_{i(\alpha)} \cap X|$$

to be the length of $l_{i(\alpha)} \cap X$. Note that (5.2b) holds because

$$\sum_{\beta \in B_0(X)} e^{-d(\beta)/2} = \sum_{j, \alpha} e^{-d(j, \alpha)/2} \leqq \sum_{\alpha \in \pi} O(1)|X_\alpha| \leqq O(1) r |X|. \tag{5.6}$$

Also recall the definition of $d(\alpha)$ given by (3.12).

Lemma 5.1. *Let $1 \leqq q < \infty$, $\varepsilon > 0$, and K_2 be given. For $m_0(\varepsilon, K_2, q)$ sufficiently large there are constants $M_2(\gamma, \beta)$, $\beta \in B$ and M_3 such that $\alpha \neq \emptyset$*

$$\|\partial_s^\gamma \partial_t^\alpha C_j\|_{L_q} \leqq M_2(\gamma, \beta) e^{-d(\alpha) m_0(1-\varepsilon)} e^{-K_2|\gamma|} e^{-d(j, \alpha)}, \tag{5.7}$$

where for $\Gamma_1, \Gamma_2 \in U_*(X)$ we have

$$\sum_{\Sigma \gamma'_\beta = \Gamma_1} \prod_{\beta \in B} M_2(\gamma', \beta) \leqq e^{M_3|X|}, \qquad \alpha \neq \emptyset. \tag{5.8a}$$

For $\alpha = \emptyset$ (5.7) holds with $d(j, \gamma)$ replacing $d(j, \alpha)$ and where

$$\sum_{\pi \in \mathscr{P}(\Gamma_2)} \prod_{\gamma \in \pi} M_2(\gamma) \leqq e^{M_3|X|}. \tag{5.8b}$$

Remark. The lemma also holds where the kernel $\partial_s^\gamma \partial_t^\alpha C_j(x, y)$ above is regarded as a function of a single variable $\partial_s^\gamma \partial_t^\alpha C_j(x, x)$. The single variable kernels do not occur when the Wick order of the interaction is always defined to agree with the covariance of the Gaussian measure.

The proof of Lemma 5.1 is as in Section 8 of [1]. (See [11], Section 5 for a correction). The case $\alpha=\emptyset$ is exactly as in Section 8, hence we suppose $\alpha\neq\emptyset$. We combine two estimates. It is easy to show as in [11] and [1] that

$$\|\partial_s^\gamma\partial_t^\alpha C_j\|_{L_q}^{1-\varepsilon}\leqq\|\partial_t^\alpha C\|_{L_q}^{1-\varepsilon}\leqq e^{-d(\alpha)m_0(1-2\varepsilon)}e^{-2d(j,\alpha)} \tag{5.9}$$

By Proposition 8.1 of [1] we have

$$\|\partial_s^\gamma\partial_t^\alpha C_j\|^{\varepsilon}\leqq\|\partial_s^\gamma C_j\|_{L_q}^{\varepsilon}\leqq M_2(\gamma)e^{-K_2|\gamma|}e^{-m_0\varepsilon d(j,\gamma)/3}\,, \tag{5.10}$$

where $M_2(\gamma)=K_6^\varepsilon(\gamma)$ defined in (8.5) of [1]. Multiplying (5.9) by (5.10) we have

$$\|\partial_s^\gamma\partial_t^\alpha C_j\|_{L_q}\leqq e^{-d(\alpha)m_0(1-2\varepsilon)}e^{-K_2|\gamma|}e^{-d(j,\alpha)}M_2(\gamma)e^{-d(j,\alpha)}e^{-d(\gamma,j)}\,.$$

Let

$$M_2(\gamma,\beta)=M_2(\gamma)e^{-d(j,\alpha)}e^{-d(j,\gamma)}\,.$$

To establish (5.8a) assume

$$\sum_{\gamma\in U_*(X)}M_2(\gamma)e^{-d(j,\gamma)}\leqq O(1)\,. \tag{5.11}$$

Then

$$\begin{aligned}\sum_{\Sigma\gamma'_\beta=\Gamma_1}\prod_{\beta\in B}M_2(\gamma,\beta)&\leqq\prod_{\beta\in B}\sum_{\gamma\in U_*(X)}M_2(\gamma)e^{-d(j,\gamma)}e^{-d(j,\alpha)}\\&\leqq\prod_{\beta\in B}O(1)e^{-d(j,\alpha)}\leqq\prod_{\alpha\in\pi_0}e^{O(1)|X_\alpha|}\leqq e^{M_3|X|}\,.\end{aligned}$$

In the last line we have used (5.6) and the fact that for fixed α there are at most $O(1)|X_\alpha|$ choices of j (or of $\beta\in B$) such that $d(j,\alpha)\leqq\text{const}$.

From [1], p. 230 we can bound the left side of (5.11) by

$$\sum_{\gamma\in U_*(X)}M_2e^{-d(j,\gamma)}\leqq\sum_{\ell\in\mathscr{L}(X_*)}e^{-\varepsilon m_0|\ell|/3}e^{-d(j,\ell)}\,.$$

Here ℓ ranges over linear orderings of a subset of X_* and $\ell=\sum\text{dist}(b'_{i+1},b'_i)$. By (8.11) of [1] the number of $\ell\in\mathscr{L}(X_*)$ with $|\ell|\leqq r$ and $d(j,\ell)\leqq a$ is less than

$$O(1)a^2e^{O(1)r}$$

because there are $O(1)a^2$ choices of b'_1. Hence for large m_0

$$\sum_{\gamma\in U_*(X)}M_2(\gamma)e^{-d(j,\gamma)}\leqq\sum_{r,\sigma}O(1)a^2e^{O(1)r}e^{-\varepsilon m_0|r|/3}e^{-a}\leqq O(1)\,.$$

By Lemma 5.1 and Lemma 9.2 of [1] we have

$$\begin{aligned}|\lambda|^g\|w'\|_{L^p}\leqq\|w\|_{L^{4/3}}&e^{-K|\Gamma|}e^{rm_0\deg Q}e^{-K|B|}\\&\cdot\prod_{\beta\in B}e^{-d(j,\alpha)}M_2(\gamma'_\beta,\beta)\cdot\prod_{\gamma\in\pi}e^{-d(j_\gamma,\gamma)}M_2(\gamma)\end{aligned} \tag{5.12}$$

for $|\lambda|\leqq e^{-\frac{p}{2}(m_0+K)}$. We have used the fact that

$$\lambda^g\prod_{\beta\in B}h_\alpha e^{-m_0(1-\varepsilon)d(\alpha)}\leqq e^{rm_0\deg Q}e^{-|B|K}$$

for h in (3.13) and $|\lambda|\leqq e^{-\frac{p}{2}(m_0+K)}$.

To bound the left side of (5.1) or (5.3) we combine (5.4) and (5.12) and use (5.8) to estimate the sum over $\{\gamma'_\beta\}, \pi \in \mathscr{P}(\Gamma_2)$. This yields the bound

$$\prod_{\beta \in B} e^{-d(\beta)/2} e^{O(1)|X|} e^{-K(|\Gamma|+|B|)} e^{rm_0 \deg Q} \|w\|_{L^{4/3}}$$
$$\cdot \prod_{\beta \in B} e^{-d(\beta)/2} \sum_{\{j_\gamma\}} T \prod_{\gamma \in \pi} e^{-d(j_\gamma, \gamma)/2} \prod_{\Delta \subset X} N(\Delta)!$$

Let $n = \deg Q$, $k = |\Gamma| + |B|$ and define $M(\Delta) = M(\Delta, B, \{j_\gamma\}_{\gamma \in \pi})$ be the number of derivatives $\delta/\delta\varphi$ localized in Δ in (5.3) i.e.

$$M(\Delta) = \text{card.}\ \{\beta \in B | \Delta_{j_i} = \Delta, i = 1 \quad \text{or} \quad 2, \beta = (\alpha, j_1, j_2)\}$$
$$+ \text{card.}\ \{\gamma \in \pi | \Delta_{j_{i,\gamma}} = \Delta, i = 1 \quad \text{or} \quad 2\}\,.$$

The proof of (5.1) now follows from the following two lemmas and the inequality $e^{-|k|}|k|^n \leqq n^n$.

Lemma 5.2. *Both T and $\prod_\Delta N(\Delta)!$ are bounded by*

$$\left(\prod_\Delta M(\Delta)!^p\right)(2npk)^n p^{kp}\,.$$

Lemma 5.3. *There is a constant $M_7(q)$ independent of $m_0, \pi \in \mathscr{P}(\Gamma_2), \Gamma_2 \in U_*(X)$ and $B \subset B_0(X)$ such that*

$$\prod_{\beta \in B} e^{-d(\beta)/2} \sum_{\{j_\gamma\}_{\gamma \in \pi}} \prod_{\gamma \in \pi} e^{-d(j_\gamma, \gamma)} \prod_\Delta M(\Delta)!^q \leqq e^{M_7|X|}\,. \tag{5.13}$$

The proof of these lemmas follows Section 10 of [1].

Proof of Lemma 5.2. Let $N_0(\Delta)$ be deg. of Q in Δ. The number of terms resulting from $M(\Delta)$ differentiations in Δ is bounded by

$$(N_0(\Delta) + p)(N_0(\Delta) + 2p) \dots (N_0(\Delta) + M(\Delta)p) \leqq (N_0(\Delta) + pM(\Delta))!$$

Since

$$(a+b)! \leqq (a+b)^a b! \quad \text{and} \quad (ab)! \leqq a^{ab}(b!)^a$$

we see that

$$T \leqq \prod_\Delta (N_0(\Delta) + pM(\Delta))^{N_0(\Delta)} p^{pM(\Delta)} (M(\Delta)!)^p\,.$$

Furthermore $N(\Delta) \leqq N_0(\Delta) + pM(\Delta)$ hence the above bound holds for $\prod N(\Delta)!$. The lemma now follows from the bounds

$$\sum_\Delta N_0(\Delta) = \deg Q = n; \quad \sum_\Delta M(\Delta) \leqq 2(|\Gamma| + |B|) = 2k\,.$$

Proof of Lemma 5.3. We first estimate the sum over j_γ

$$\sum_{\{j_\gamma\}} \prod_{\gamma \in \pi} e^{-d(j_\gamma, \gamma)/2} \leqq \prod_{\gamma \in \pi} \left(\sum_{j_\gamma} e^{-d(j_\gamma, \gamma)/2}\right.$$
$$\leqq \prod_{\gamma \in \pi} O(1) \leqq e^{M'_7|X|}\,.$$

Here we have used the fact that there are at most $2|X|$ elements in $\pi \in \mathscr{P}(\Gamma_2)$. Hence it suffices to show there is a constant M''_7 such that

$$\max_{\pi, \{j_\gamma\}_{\gamma \in \pi}, B} \prod_{\beta \in B} e^{-d(\beta)/2} \prod_{\gamma \in \pi} e^{-d(j_\gamma, \gamma)/2} \prod_\Delta (M(\Delta, B, \{j_\gamma\})!^q) \leqq e^{M''_7|X|}\,.$$

Fix a lattice square Δ. There are at most $O(1)a^2$ choices of $\gamma \in \pi$ such that $d(j, \gamma) \leqq a$ and $\Delta_{j_i} = \Delta$, $i = 1$ or 2. Also there are at most $O(1)a^3$ choices of $\beta = (\alpha, j) \in B$

such that $d(\beta)\leqq a$ and $\Delta_{j_i}=\Delta$, $i=1$ or 2 because there are less than $O(1)a^2$ choices of j and ra choices of α. Thus there are less than $M(\Delta)/2$ choices of β, γ such that

$$\tfrac{1}{2}(d(\beta)+d(j_\gamma,\gamma))\leqq M(\Delta)^{1/3}\varepsilon$$

for ε sufficiently small so that there are least $M(\Delta)/2$ choices of β, γ such that

$$\tfrac{1}{2}(d(\beta)+d(j_\gamma,\gamma))\geqq \varepsilon M(\Delta)^{1/3}\,.$$

We can now bound (5.14) by

$$\prod\nolimits_{\Delta\subset X} e^{-\frac{1}{4}\varepsilon M(\Delta)^{4/3}}(M(\Delta)\,!)^q \leqq e^{O(1)|X|}$$

since $|(x\,!)^q e^{-\varepsilon|x|^{4/3}}|\leqq e^{O(1)}$.

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