Commun. math. Phys. 43, 17—32 (1975)

Holomorphic Versions of the Fabrey-Glimm Representations of the Canonical Commutation Relations*

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Received September 10, 1974; in revised form December 27, 1974

I. Introduction

Glimm and Fabrey have constructed [6; 4] a Hilbert space $\mathscr{F}_r$ for a simplified version of the $:\Phi^4:$ model in quantum field theory for 3 space-time dimensions with space cutoff by using a sequence of truncated exponentials involving $a^*(v)$ to define the dressing transformation, where

$$v(k_1, k_2, k_3, k_4) = \tilde{h}(\Sigma k_i)\, \Pi \mu(k_i)^{-1/2} (\Sigma \mu(k_i))^{-1}\ .$$

The space cutoff is h, $k_i \in \mathbb{R}^2$, $\mu(k_i) = (\mu_0^2 + |k_i|^2)^{1/2}$. For v of a more general form, lower parameter j, and upper cutoff σ, they show convergence of $(\hat{T}_{j\sigma}\phi, \hat{T}_{j\sigma}\psi)e^{-X(\sigma)}$ for ϕ, ψ in a dense subset $\mathscr{D}$ of Fock space, as $\sigma \to \infty$. $\hat{T}_{j\sigma}$ is a truncated version of $e^{a^*(v)}$ and $X(\sigma)$ is the renormalization. The closure of the inductive limit of $\hat{T}_j \mathscr{D}$ over the lower parameters defines a Hilbert space which carries a Weyl representation of the CCR (canonical commutation relations).

The Bargmann-Segal complex wave representation for the free field has as Hilbert space $H^2(K'_{cx}, d\mu)$, the completion of the tame holomorphic functionals on K'_{cx}, the complex distributions, which are square-integrable with respect to the Gaussian cylinder set measure μ on K'. The finite-dimensional case has been discussed by Bargmann [1] and the infinite dimensional case by Segal [15; 16]. Creation operators on $H^2(K', d\mu)$ are diagonalized and annihilation operators are differentiations.

We construct an analogue to the complex wave representation for the interaction case as a countable inductive limit of spaces of the following form: completion of the tame holomorphic functionals on K'_{cx} in the space of functionals which are square integrable with respect to a countably additive measure associated with T_j. This space carries a representation of the CCR for which creation is a multiplication operator and annihilation is, formally, differentiation plus multiplication by the log derivative of T_j. The representation is unitarily equivalent to the Glimm-Fabrey representation.

For a fixed lower parameter j and upper cutoff σ we construct $H^2(K', d\eta_{j\sigma})$, where $d\eta_{j\sigma} = |T_{j\sigma}|^2\, \|T_{j\sigma}\|^{-2}\, d\mu$. In order to show that the $\eta_{j\sigma}$ converge to a countably additive measure, we analyze the characteristic functions $L_{j\sigma}(h)$ of $\eta_{j\sigma}$ and,

* This work was supported in part by the National Science Foundation, GP30798X2.

using estimates derived from Fabrey's analysis and the theory of measures on the dual of a nuclear space, show that $L_j(h) = \lim_{\sigma \to \infty} L_{j\sigma}(h)$ defines a countably additive measure on K'. We define $H^2(K', d\eta_j)$ as the completion of the (tame) polynomials on K' in $L^2(K', d\eta_j)$. The creation and annihilation operators are "moving weak limits" of corresponding operators on $H^2(K', d\eta_{j\sigma})$. The creation operator $a_j^*(h)$ is multiplication by the monomial associated with $h \in K$ and the annihilation operator $a_j(h)$ is, formally,

$$a(h) + [(a(h)\, T_j)/T_j] \cdot I_0$$

where $a(h)$ is "differentiation in the h direction" and I_0 is the identity operator.

The Fabrey-Glimm construction takes an inductive limit of the spaces corresponding to different lower parameters in order to get a space on which self-adjointness of the field operator can be demonstrated. We take the inductive limit of the $H^2(K', d\eta_j)$ to get a space $\mathscr{H}$ and a representation of the CCR unitarily equivalent to the representation on $\mathscr{F}_r$.

Hepp has constructed [7, Chapter 4] a representation on a space obtained by using lower parameter 0 and the Gelfand-Naimark-Segal construction, which is unitarily equivalent to the representation on $\mathscr{F}_r$, and therefore also to the representation on H.

II. Background

A. Finite-dimensional Case (Bargmann Representation)

The Hilbert space H^2 is the entire analytic functions of n complex variables, with inner product

$$(f, g) = \pi^{-n} \int_{C^n} f(z)\, g(z)\, e^{-\Sigma |z_i|^2} d^n z\,.$$

Equivalently, this space is the completion of the polynomials on $\mathbb{C}^n$ with respect to the Gaussian measure $\pi^{-n} \exp(-|z|^2) d^n z$. We define annihilation and creation operators as follows:

$$a_i^*(f)(z) = z_i f(z) \quad \text{if} \quad z_i f(z) \in H^2$$
$$a_i(f)(z) = \partial f/\partial z_i \quad \text{if} \quad \partial f/\partial z_i \in H^2\,,$$

where $\partial f/\partial z_i = 1/2\,(\partial f/\partial x_i - i\,\partial f/\partial y_i)$. The operators $a_i^*(f)$ and $a_i(f)$ are closed, adjoints, and $[a_i, a_j^*]\, f = \delta_{ij} f$.

B. Infinite Dimensional Case (Fock Representation)

As usual, Fock space is $\mathscr{F} = \bigoplus_{n=0}^{\infty} \mathscr{F}_n$, where

$\mathscr{F}_0 = \mathbb{C}$.

$\mathscr{F}_n = SL_2(\mathbb{R}^{2n})$, the symmetric, square integrable functions of $2n$ variables, written as functions of $k_1, \ldots, k_n$, $k_i \in \mathbb{R}^2$.

$$\phi = \Sigma \phi_n \in \mathscr{F} \quad \text{if} \quad \sum_{n=0}^{\infty} \|\phi_n\|^2 = \|\phi\|^2 < \infty\,.$$

$\mathscr{D}$ will denote the subset of $\mathscr{F}$ consisting of $\phi = \Sigma \phi_n$ such that only finitely many ϕ_n are non zero and each ϕ_n has compact support.

For $f \in L_2(\mathbb{R}^2)$,

$$a^*(f)\,\phi_n = (n+1)^{1/2}\, S(f \otimes \phi_n)$$

$$(a(f)\phi_n)(k_1, \ldots, k_{n-1}) = n^{1/2} \int f(k_n)\, \phi_n(k_1, \ldots, k_n)\, dk_n$$

where $\phi_n \in \mathscr{F}_n$ and S is the symmetrization operator

$$(S\phi_n)(k_1, \ldots, k_n) = \frac{1}{n!} \sum_{\sigma} \phi_n(k_{\sigma(1)}, \ldots, k_{\sigma(n)})\,.$$

$a^*(f)$ and $a(\bar{f})$ are adjoints and $[a(f), a^*(g)] = (f, g)$.

$K(a)$ will be the space of infinitely-differentiable complex-valued functions on $\mathbb{R}^2$, with support in $\{\|x\| \leqq a\}$. A sequence $\{f_n\}$ converges to 0 in $K(a)$ if the f_n and all their derivatives converge uniformly to 0. K, the inductive limit of the $K(a)$ is usually denoted elsewhere by $\mathscr{D}(\mathbb{R}^2)$.

An element of the form $a^*(f_1) \ldots a^*(f_s)\Omega_0$, where the f_i's $\in K$ and need not be distinct, is called a *monomial* on $\mathscr{F}$. If $\phi = \Sigma \phi_i$, where each ϕ_i is a monomial and the sum is finite, then ϕ is called a *polynomial* on $\mathscr{F}$ and will be denoted by $\hat{p}\Omega_0$. In particular, $\hat{p}\Omega_0 \in \mathscr{D}$. $\hat{p}$ will be called a *polynomial operator.*

C. Infinite-dimensional Case (Bargmann-Segal Representation)

The coordinate functions z_i of the Bargmann representation will correspond to elements of K' and analytic functions of z will correspond to the completion of polynomials on K' with respect to L^2 of the infinite dimensional analogue of Gaussian measure.

A function ψ defined on K' is based on F, where F is a finite dimensional, closed (complex) subspace of K, if there is a function ψ_1 defined on K'/F° such that $\psi(q) = \psi_1(\pi q)$, where π is the natural projection: $K' \to K'/F^\circ$. A function ψ on K' is a *monomial* if there is an F, as above, $f_1, \ldots, f_n \in F$, $m_1, \ldots, m_n \in Z^+$ and a complex constant α, such that ψ can be written as

$$\psi(q) = \alpha \langle f_1, \pi q\rangle^{m_1} \ldots \langle f_n, \pi q\rangle^{m_n}\,.$$

A *polynomial* is a finite sum of monomials.

Notation and Comments. (a) $\langle f, q\rangle$ means $f \in K$, $q \in K'$ and the bracket denotes evaluation.

(b) Since F' is isomorphic to K'/F°, the second element in the bracket can equivalently be an element of K'/F°, when $f \in F$.

(c) If i denotes the natural injection of K into K', we have $\langle f, i(h)\rangle = (h, f)$, where parentheses denote the inner product on K, and the inner product is complex linear in the second variable.

(d) μ denotes the Gaussian measure on K' induced by the inner product on K.

(e) $H^2(K', d\mu) =_{\text{df.}}$ completion of the polynomials in $L^2(K', \mu)$.

(f) A polynomial on K' will be denoted by p, with or without subscripts and superscripts.

Example. Suppose

$$\psi(q) = \alpha\langle f_1, \pi q\rangle^2 \langle f_2, \pi q\rangle^3 .$$

and f_1 and f_2 are orthonormal. Then

$$\int_{K'} \psi(q)\, d\mu(q) = \pi^{-2} \int_{\mathbb{C}^2} \alpha\bar{\lambda}_1^2 \bar{\lambda}_2^3 \exp(-|\lambda_1|^2 - |\lambda_2|^2) d\lambda_1\, d\lambda_2 .$$

A polynomial on K' corresponds to an anti-polynomial on F, so the representation induced on F will be an anti-holomorphic representation of the CCR.

$H^2(K', d\mu)$ is isomorphic to Fock space, if we define the map $I: \mathscr{F} \to H^2(K', d\mu)$ as follows: $I\Omega_0 = 1$, the identity function. If $f_1, \ldots, f_s \in K$, $I(a^*(f_1) \ldots a^*(f_s)\Omega_0) = \langle f_1, \cdot\rangle \ldots \langle f_s, \cdot\rangle$. Finite sums of expressions of this form are dense in $\mathscr{F}$ so since the map thus far defined is an isometry, it can be extended to the closures. In particular, $I(\hat{p}\Omega_0) = p$.

Annihilation and creation operators on $H^2(K', d\mu)$, for $f \in K$, can be defined via the isomorphism. In particular, if $\psi \in H^2(K', d\mu)$,

$$a_H^*(f)\psi = \langle f, \cdot\rangle \psi$$

if the right-hand side is in H^2 and

$$a_H(f)\langle g, \cdot\rangle^m = m(\bar{f}, g)\langle g, \cdot\rangle^{m-1} .$$

a_H^* is a multiplication operator and $a_H(f)$ is "differentiation in the f direction".

Both are closed, $[a_H(f)]^* \supset a_H^*(\bar{f})$, and $[a_H(f), a_H^*(g)] = (\bar{f}, g) I_0$, $I_0 =$ identity operator on $H^2(K', d\mu)$, because the corresponding statements are true in Fock space.

Segal's description of the holomorphic representation of the CCR may be found in [15] and [16].

From now on, a and a^* will denote operators on $\mathscr{F}$ or $H^2(K', d\mu)$, depending on the context.

D. The Interaction Case (Renormalized Fock Space $\mathscr{F}_r$)

$\mathscr{F}_r$ is defined using expressions of the form $\lim_{\sigma\to\infty} (\hat{T}_{k\sigma}\psi, \hat{T}_{k\sigma}\psi) e^{-X(\sigma)}$ where $\hat{T}_{k\sigma}$ is a truncated version of $e^{a^{*4}(v)}$ and $X(\sigma) = 4!(v_\sigma, v_\sigma)$, v_σ satisfying certain growth conditions:

We deal with symmetric, measurable functions $v(k_1, k_2, k_3, k_4)$ which are "almost in L_2", i.e. satisfy certain growth conditions. The class of "almost in L_2" functions does not contain L_2, but does contain

$$\left\{ v \in L_2(\mathbb{R}^8) : \prod_{i=1}^{4} (\mu_0^2 + |k_i|^2)^{\varepsilon/2} v \in L_2(\mathbb{R}^8) \right\} .$$

The particular v's arising in field theory are "almost in L_2" but not in L_2.

Choose $\alpha > 1$, $n(k)$ strictly increasing but polynomial bounded, and construct $\hat{T}_{j\sigma}$ as follows [we write $\hat{T}_{j\sigma}$ for Fabrey's $T_{j\sigma}$, etc.; $k = (k_1, k_2, k_3, k_4)$]:

$$v_{\varrho\sigma}(k) = \begin{cases} v(k) & \max_{1 \leqq i \leqq 4} |k_i| \in [\varrho, \sigma) \\ 0 & \text{otherwise}. \end{cases}$$

Let $v_\sigma = v_{0\sigma}$; let $\alpha(j) = \begin{cases} \alpha^j & \text{if } j \geqq 1 \\ 0 & \text{if } j = 0. \end{cases}$

Let $v_{j\sigma}$ and v_{jk} denote $v_{\alpha(j)\sigma}, v_{\alpha(j)\alpha(k)}$, respectively. Suppose $v_\sigma \in L_2(\mathbb{R}^8)$, $j \leqq l$, and $\alpha(l) \leqq \sigma \leqq \alpha(l+1)$.

We are examining situations involving "fourth-order" creation operators. Suppose $w = w(k_1, \ldots, k_4) \in L_2(\mathbb{R}^8)$. Define $a^{*4}(w)$ as a map from $\mathscr{F}_n$ to $\mathscr{F}_{n+4}$ by

$$a^{*4}(w)\psi_n = [(n+1) \ldots (n+4)]^{1/2} S(w \otimes \psi_n)$$

where S is the symmetrization operator and $\psi_n \in \mathscr{F}_n$. Operators of this kind form the most singular part of the field operator $:\phi^4:$

In particular,

$$V_{j\sigma} = \begin{cases} a^{*4}(v_{j,j+1}) & j \leqq l-1 \\ a^{*4}(v_{l\sigma}) & j = l \end{cases}$$

$$\exp_n(x) = \sum_{l=0}^{n} x^l / l!$$

Then $\hat{T}_{j\sigma} = \prod_{j_0 \leqq j \leqq l} \exp_{n(j)} V_{j\sigma}$, and is defined on $\mathscr{D}$.

Theorem (Fabrey). *For $\phi, \psi \in \mathscr{D}$ the limit $\lim_{\sigma \to \infty} (\hat{T}_{k\sigma}\phi, \hat{T}_{l\sigma}\psi) e^{-X(\sigma)} = (\hat{T}_k \phi, \hat{T}_l \psi)_r$ exists. If $k \geqq l$, $\sigma \geqq \alpha(k)$, then $\hat{T}_{l\sigma}\psi = \hat{T}_{k\sigma}\theta$, where*

$$\theta = \prod_{j=l}^{k-1} \exp_{n(j)} \hat{V}_{j\sigma}\psi \in \mathscr{D}.$$

$(\cdot, \cdot)_r$ provides a positive definite inner product for $\bigcup_{j \geqq 0} \hat{T}_j \mathscr{D}$, whose completion is denoted $\mathscr{F}_r$. Operators $W_r(f) = e^{i\phi_r(f)}$ can be defined on $\mathscr{F}_r$ and satisfy the Weyl form of the CCR.

E. The Interaction Case (Renormalized Bargmann-Segal Space)

We would like to show that the Fabrey-Glimm expressions

$$(\hat{T}_k \hat{p}_1 \Omega_0, \hat{T}_k \hat{p}_2 \Omega_0)_r = \lim_{\sigma \to \infty} (\hat{T}_{k\sigma} \hat{p}_1 \Omega_0, \hat{T}_{k\sigma} \hat{p}_2 \Omega_0) e^{-X(\sigma)}$$

can be written as

$$\int_{K'} c_k^{-2} \bar{p}_1 p_2 \, d\eta_k(\cdot),$$

where η_k is a countably additive normalized measure on K'. To do this we use criteria for a function on K to be the characteristic function of a countably additive measure on K'.

III. Characteristic Functions and Measures on K'

Theorem III.1 (see [13] and [11]). *Let E be a complex locally convex topological vector space. Suppose L is a complex-valued function defined on E with the properties that:*

(a) $L(0)=1$.

(b) *L is positive definite, i.e., let* $g_1, \ldots, g_s \in E$, $\xi_1, \ldots, \xi_s \in \mathbb{C}$. *Then*

$$\sum_{i,j=1}^{s} L(g_i - g_j)\,\bar{\xi}_i \xi_j \geqq 0\,.$$

(c) *L continuous on finite dimensional subspaces of E, i.e. suppose* $\{g_n\} \subset F$ $=$ *space generated by* $f_1, \ldots, f_s \in E$, $g_n = \sum_{i=1}^{s} a_i^{(n)} f_i$. *Then* $\lim_{n\to\infty} a_i^{(n)} = 0$, $i=1, \ldots, s$ $\Rightarrow L(g_n) \to L(0) = 1$.

Then L is the characteristic function of some cylinder-set measure η *(possibly not countably additive) and*

$$L(g) = \int_{E'} e^{i\mathrm{Re}\langle g, \cdot \rangle} d\eta(\cdot)\,.$$

And the converse is true.

η is said to satisfy the continuity condition if, for each bounded continuous function B on $\mathbb{C}^n$, the function

$$\Phi(f_1, \ldots, f_n) = \int_{E'} B(\langle f_1, \cdot \rangle, \ldots, \langle f_n, \cdot \rangle)\, d\eta(\cdot)$$

is sequentially continuous.

If E is a countably Hilbert (complex) nuclear space, for example $K(a)$, then any positive, normalized cylinder set measure η in E', satisfying the continuity condition, is countably additive ([5], extended to the complex case).

For our purposes the following equivalent version of the continuity condition will be more useful:

For any $A > 0$, and any sequence $\{g_j\}$ converging to 0 in E we have $\lim_{j\to\infty} \eta\{q : |\langle g_j, q \rangle| \geqq A\} = 0$ [12].

If we are given a function L on all of K satisfying the hypotheses of the above theorem, and whose associated η satisfies the continuity condition, we can construct a countably additive measure on $K' = (\cup K(a))'$ and extend the measure to the Borel sets of K'.

Example 1. Suppose p is a polynomial on K'. Then $p \in L^2(K', d\mu)$ since polynomials are square integrable with respect to Gaussian measure. Define

$$L(g) = (p e^{-i/2\langle g, \cdot \rangle}, p e^{i/2\langle g, \cdot \rangle})\, \|p\|^{-2} \text{ with inner product in } K',$$

$$= \int_{K'} \bar{p} e^{i/2\overline{\langle g, \cdot \rangle}} p e^{i/2\langle g, \cdot \rangle} \|p\|^{-2} d\mu(\cdot) = \int_{K'} e^{i\mathrm{Re}\langle g, \cdot \rangle} |p|^2 \, \|p\|^{-2} d\mu\,.$$

Then L satisfies the hypotheses of Theorem III.1 and the continuity condition, and hence defines a countably additive measure η_p.

Example 2. More generally, suppose $T \in L^2(K', d\mu)$, $\|T\| \neq 0$. Then

$$\eta(Z) = \int_Z \frac{|T|^2 d\mu}{\|T\|^2}$$

defines a countably additive measure on the σ-algebra generated by the cylinder sets.

Our goal is to show initially that $I(\hat{T}_{k\sigma}\Omega_0) \in L^2(K', d\mu)$, and then that the corresponding $\eta_{k\sigma}$ converge to a measure η_k which is associated with the $(\hat{T}_k\phi, \hat{T}_k\psi)_r$.

Suppose $T \in H^2(K', d\mu)$ has the property that $I^{-1} T$ is of the form $\hat{T}\Omega_0$, where $\hat{T}\Omega_0$ is a finite sum of terms of the form $a^{*4}(f_1) \ldots a^{*4}(f_j)$, $f_i \in L^2(\mathbb{R}^8)$.

$\hat{T}$ is a bounded operator on $\bigoplus_{n \leq N} \mathscr{F}_n$ for any N, and this property forms the key to proofs of the lemmas below.

Lemma III.2. *$I(\hat{T})$, the corresponding operator on $H^2(K', d\mu)$, is multiplication by the function T, i.e. M_T. For example, $I(\hat{p}) = M_p$.*

Lemma III.3. *$T \cdot p \in H^2(K', d\mu)$ for all polynomials p. In particular, $I(\hat{T}\Omega_0) = T$.*

Lemma III.4. *We can define isometric isomorphisms E and E_T with*

$$\{\hat{T}\hat{p}\Omega_0\}^- \xrightarrow[I]{} (H^2(K', d\mu) \cap \{T \cdot p\})^- \xrightarrow[E]{} H^2(K', d\eta_T)$$

by setting

$$E(T \cdot p) = \|\hat{T}\Omega_0\| \cdot p$$
$$E_T = E \circ I$$

and extending the domains to their closures in $L^2(K', d\mu)$ and $\mathscr{F}$ respectively. In particular,

$$(\hat{T}\hat{p}_1\Omega_0, \hat{T}\hat{p}_2\Omega_0)\, \|\hat{T}\Omega_0\|^{-2} = \int_{K'} \bar{p}_1 p_2 \, d\eta_T \,.$$

Lemma III.5. *Suppose $\psi = \Sigma \psi_n \in \bigoplus_{n \leq N} \mathscr{F}_n$ for some N. Then $\hat{T}\psi \in \{\hat{T}\hat{p}\Omega_0\}^-$ and so corresponds to an element $E_T(\hat{T}\psi)$ of $H^2(K', d\eta_T)$.*

We define $a_T^*(f)$, $a_T(f)$ on $H^2(K', d\eta_T)$ using the isomorphism E.

Let $\phi \in H^2(K', d\eta_T)$.

Then

$$a_T^*(f)p = Ea^*(f)T \cdot p/\|T\| = E(T\langle f, \cdot\rangle p/\|T\|) = \langle f, \cdot\rangle p$$

and

$$a_T(f)p = Ea(f)T \cdot p/\|T\| = E[T \cdot (a(f)p) + (a(f)T) \cdot p]/\|T\|$$

which might not be in the domain of E so we do not know if $a_T(f)$ can be defined on the polynomials.

Formally, we would get

$$= a(f)p + p \cdot \text{“}a(f)T\text{”}/T = (a(f) + (\text{“}a(f)T\text{”}/T)I_0)p$$

where I_0 is the identity operator.

If $a(f)T = 0$, then $a_T(f)p$ is well defined, as the usual derivative.

We can consider the bilinear form A_T defined by

$$(p_1, A_T p_2)_T =_{\text{df.}} (E^{-1} p_1, A E^{-1} p_2)$$

for a bilinear form A.

If we let A_T correspond to $A = [a(f), a^*(g)]$, then formally,

$$\begin{aligned}(p_1, [a_T(f), a_T^*(g)])_T \text{``}=\text{''} & (p_1, A_T p_2)_T \\ &= (E^{-1} p_1, [a(f), a^*(g)]\, E^{-1} p_2) \\ &= (E^{-1} p_1, (\bar{f}, g)\, E^{-1} p_2) \\ &= (\bar{f}, g)\,(p_1, p_2)_T\end{aligned}$$

where we have assumed that the domain of $[a(f), a^*(g)]$ includes $E^{-1} p_1 \times E^{-1} p_2$ and used the commutation relations on $H^2(K', d\mu)$. This can be called a weak representation of the CCR.

The Fabrey-Glimm construction considers T's which are truncated exponentials, i.e., $T \sim e^{\langle f, \cdot \rangle^4}$. So formally,

$$(a(g)\,T)/T = e^{-\langle f, \cdot \rangle^4} \cdot 4 \langle f, \cdot \rangle^3 e^{\langle f, \cdot \rangle^4} (\bar{g}, f) = 4(\bar{f}, g) \langle f, \cdot \rangle^3 ,$$

an expression which makes sense.

Theorem III.6. *$I(\hat{T}_{k\sigma} \Omega_0) = T_{k\sigma} \in H^2(K', d\mu)$ satisfies the hypotheses of Lemma III.2 and so is an instance of Example 2 and the succeeding discussion.*

In particular,

$$(\hat{T}_{k\sigma} \hat{p}_1 \Omega_0, \hat{T}_{k\sigma} \hat{p}_2 \Omega_0)\, \|\hat{T}_{k\sigma} \Omega_0\|^{-2} = \int_{K'} \bar{p}_1 p_2 \, d\eta_{k\sigma}$$

where $\eta_{k\sigma}$ is a countably additive cylinder set measure of total mass 1.

Also

$$L_{k\sigma}(f) =_{\text{df.}} \int_{K'} e^{i \operatorname{Re} \langle f, \cdot \rangle} d\eta_{k\sigma}(\cdot)$$

is positive definite.

This expression differs from the expressions whose limits define $\mathscr{F}_r$ by a factor $(\hat{T}_{k\sigma} \Omega_0, \hat{T}_{k\sigma} \Omega_0) e^{-X(\sigma)}$. We show below that as $\sigma \to \infty$ this factor converges to a constant c_k^{-2}, where $0 < c_k^{-2} < 1$.

IV. Some Fundamental Estimates

In order to show that the Fabrey-Glimm expressions of the form $(\hat{T}_k \hat{p}_1 \Omega_0, \hat{T}_k \hat{p}_2 \Omega_0)_r$ can be written as $\int_{K'} \bar{p}_1 p_2 \, d\eta_k(\cdot)$, where η_k is a countably additive measure on K', we examine

$$L_k(f) =_{\text{df.}} \lim_{\sigma \to \infty} L_{k\sigma}(f) =_{\text{df.}} \lim_{\sigma \to \infty} (\hat{T}_{k\sigma} e^{-i/2\, a^*(f)} \Omega_0, \hat{T}_{k\sigma} e^{i/2\, a^*(f)} \Omega_0)\, \|\hat{T}_{k\sigma} \Omega_0\|^{-2}$$

where $e^{a^*(g)}$ is defined as $\sum_{k=0}^{\infty} a^*(g)^k/k!$, and show it is a characteristic function. Even $\hat{T}_{k\sigma} e^{-i/2\, a^*(f)} \Omega_0$ has an infinite number of particles, so the existence of $L_{k\sigma}$ needs to be shown.

Using the isomorphism between $\mathscr{F}$ and $H^2(K', d\mu)$ we can write

$$L_{k\sigma}(f) = \int_{K'} e^{i\mathrm{Re}\langle f, . \rangle} d\eta_{k\sigma}.$$

The operator $e^{a^*(g)}$, for $g \in K$, defines a bounded map from $\mathscr{F}_n$ to $\mathscr{F}$, for any n.

Now $(\hat{T}_{k\sigma}\phi, \hat{T}_{k\sigma}\psi) e^{-X(\sigma)} = (\phi, \hat{T}^*_{k\sigma} \hat{T}_{k\sigma}\psi) e^{-X(\sigma)}$. $\hat{T}^*_{k\sigma}\hat{T}_{k\sigma}$ is a truncated power series in V^*_σ and V_σ and can be rewritten as a sum of Wick-ordered terms with distribution kernels by using the commutation relations. Fabrey calls a term reduced if it contains no $X = V^* \underset{4}{-\!\circ\!-} V$ components, i.e. no completely contracted terms.

Fabrey shows that $(\phi, \hat{T}^*_{k\sigma} \hat{T}_{k\sigma}\psi) e^{-X(\sigma)} = \int_E h_\sigma$, where h_σ is a measurable function on the space E, and E is a direct sum of spaces associated with reduced terms in $\hat{T}^*_{k\sigma}\hat{T}_{k\sigma}$.

The following lemma is Fabrey's Lemma 3.3, with several components of the constant exhibited explicitly.

Lemma IV.1. *Suppose G is a reduced graph such that $|G| = n$, ϕ and $\psi \in \mathscr{D}_i, \mathscr{D}_j$, respectively, and ϕ, ψ vanish off a sphere of radius ϱ. Then*

$$(|\phi|, |R_0|\, |\psi|) \leqq K_1 K_2^n \alpha^{-\varepsilon c n^{1+\delta}} \|\phi\|\, \|\psi\|\, (\mu_0^2 + \varrho^2)^{a(i+j)/2} (j+1)^{2n}$$

for constants K_1, K_2.

Lemma IV.2. *(existence of $\hat{T}_k e^{a^*(f)}\Omega_0$, $\hat{T}_k e^{a^*(g)}\Omega_0)_r$). Suppose $f, g \in L^2(\mathbb{R}^2)$ and $f(k) = g(k) = 0$ for $\|k\| > \varrho > 0$. Then*

$$\lim_{\sigma \to \infty} (\hat{T}_{k\sigma} e^{a^*(f)}\Omega_0, \hat{T}_{k\sigma} e^{a^*(g)}\Omega_0) e^{-X(\sigma)} < \infty$$

and there is a uniform bound for the expressions corresponding to those f's and g's with $\|f\| < \delta$, $\|g\| < \delta$ for some $\delta > 0$.

Proof. The proof consists of several parts:

A) expressing the inner product as $\int_E h_\sigma$ where E is a measure space; showing the h_σ converge pointwise and $|h_\sigma| \leqq h$ where h is measurable on E ([3], Lemmas 3.1, 3.2, and corollary),

B) estimating $(|\phi|, |R_0|\, |\psi|)$ for a reduced term R_0,

C) writing $\int h$ as a sum, $\sum_{i,j,n=0}^{\infty}$, of expressions of the form described in B),

D) summing over i,

E) over j,

F) over n,

G) applying the dominated convergence theorem to get convergence of $\int_E h_\sigma$ as $\sigma \to \infty$.

B) Let $\phi_i = a^*(f)^i \Omega_0/i!$ Then $\|\phi_i\| \leqq (i!)^{-1/2} \|f\|^i$.

Let R_0 be a particular reduced term with graph G and $|G| = n$. Then

$$(|\phi|, |R_0| |\psi|) \leqq \sum_{i,j=0} K_1 K_2^n \alpha^{-\varepsilon c n^{1+\delta}} (\mu_0^2 + \varrho^2)^{aj/2} (j+1)^{2n} \|g\|^j (j!)^{-1/2} (\mu_0^2 + \varrho^2)^{ai/2} \|f\|^i (i!)^{-1/2}.$$

C) We want to look at

$$\int_E h = \sum_n \sum_{\substack{G \ni \\ |G| = n}} \sum_{i,j} (|\phi_i|, |R_0| |\psi_j|).$$

Lemma IV.1 estimates the inner products and Fabrey's Lemma 3.4 says there are at most $K_3^n (4n)!^2$ reduced graphs G such that $|G| = n$, where K_3 is constant. So we need to examine

$$K_1 \sum_n K_2^n K_3^n (4n)!^2 \alpha^{-\varepsilon c n^{1+\delta}} \sum_{j=0}^{\infty} (\mu_0^2 + \varrho^2)^{aj/2} (j+1)^{2n} \|g\|^j (j!)^{-1/2}$$
$$\cdot \sum_{i=0}^{\infty} (\mu_0^2 + \varrho^2)^{ai/2} \|f\|^i (i!)^{-1/2}.$$

D) The sum over i converges by the ratio test. In fact if $\|f\| < \delta$, there will be a uniform bound on the sum.

E) Fix n. The sum we want to estimate is

$$* = \sum_{n,j} K_4^n C_1^j (j+1)^{2n} (j!)^{-1/2} (4n)!^2 \alpha^{-\varepsilon c n^{1+\delta}}.$$

Using the results: $(j+1)^{2n} \leqq 4^n j^{2n}$; $(j)^{-1/2} < (e/j)^{j/2}$; $(4n)!^2 < 4^{8n} n^{8n}$, we have that

$$\begin{aligned} * &< \sum_{n,j} K_4^n C_1^j 4^n j^{2n} (e/j)^{j/2} 4^{8n} n^{8n} \alpha^{-\varepsilon c n^{1+\delta}} \\ &= \sum_{n,j} K_5^n C_2^j j^{2n - j/2} n^{8n} \alpha^{-\varepsilon c n^{1+\delta}} \\ &= \sum_n K_5^n n^{8n} \alpha^{-\varepsilon c n^{1+\delta}} \sum_j j^{j \ln C_2 / \ln j}. \end{aligned}$$

The sum over j can be shown to be bounded by $C_3^n n^{3n}$, and

F) $\sum_n C_4^n n^{11n} \alpha^{-\varepsilon c n^{1+\delta}}$ converges.

G) The dominated convergence theorem now gives convergence of $\int_E h_\sigma$ as $\sigma \to \infty$

The constants of Lemma IV.3 will be the normalizations needed to obtain measures of total mass 1.

Lemma IV.3. [3, pp. 13–14]. $\lim_{\sigma \to \infty} (\hat{T}_{k\sigma} \Omega_0, \hat{T}_{k\sigma} \Omega_0) e^{-X(\sigma)} = \lim_{\sigma \to \infty} c_{k\sigma}^{-2} = c_k^{-2}$, *where* $0 < c_k^{-2} < 1$.

Corollary IV.4.

$$\lim_{\sigma \to \infty} (\hat{T}_{k\sigma} e^{-ia^*(f)/2} \Omega_0, \hat{T}_{k\sigma} e^{ia^*(f)/2} \Omega_0) \|T_{k\sigma} \Omega_0\|^{-2} = \lim_{\sigma \to \infty} L_{k\sigma}(f) = L(f) < \infty,$$

with a uniform bound for $\{f : \|f\| < \delta\}$ *and* $\operatorname{supp} f \subset \{\|k\| \leqq \varrho\}$.

A similar sum analysis shows

Lemma IV.5. *Suppose* $\phi^1, \phi^2 \in \mathscr{D}$, $\phi^i = \sum_n \phi_n^i$, *supp* $(\phi_n^i) \subset \{\|k\| \leqq \varrho\}$ *for* $i = 1, 2$; $\varrho > 0$. *Then*

$$(\hat{T}_{k\sigma}\phi^1, \hat{T}_{k\sigma}\phi^2)e^{-X(\sigma)} \leqq C\|\phi^1\|\,\|\phi^2\| \quad \text{for all} \quad \sigma\,,$$

where C *depends on* ϱ *and on max* $\{i, j : \phi_i^1 \neq 0$ *or* $\phi_j^2 \neq 0\}$, *but is independent of* σ. *It follows that*

$$(\hat{T}_k\phi^1, \hat{T}_k\phi^2)_r \leqq C\|\phi^1\|\,\|\phi^2\|\,.$$

Because of our normalization $L_{k\sigma}(0) = 1$. We will want $L_k(f) =_{\mathrm{df.}} \lim_{\sigma\to\infty} L_{k\sigma}(f)$ to be continuous at the origin so we need:

Lemma IV.6. [equicontinuity of $L_{k\sigma}(f)$]. *Fix* $k \in Z^+$. *Suppose* $\varepsilon > 0$, $\varrho > 0$. *Then there exists a* δ *such that if* $f(k_1) = 0$ *for* $\|k_1\| > \varrho$ *and* $\|f\| < \delta$, *then*

$$|L_{k\sigma}(f) - 1| < \varepsilon \quad \text{for all} \quad \sigma\,.$$

In particular, if $\{f_i\} \subset K$ *and* $f_i \to 0$ *in* K, *then there exists an* N *such that*

$$i > N \Rightarrow |L_{k\sigma}(f_i) - 1| < \varepsilon \quad \text{for all} \quad i, \sigma\,.$$

V. Renormalized Bargmann-Segal Spaces

Theorem V.1. *Vor* $f \in K$,

$$L_k(f) = \lim_{\sigma\to\infty} (\hat{T}_{k\sigma}e^{-ia^*(f)/2}\Omega_0, \hat{T}_{k\sigma}e^{ia^*(f)/2}\Omega_0)\,\|\hat{T}_{k\sigma}\Omega_0\|^{-2}$$

defines the characteristic function of a real-valued cylinder set measure η_k *on* K', *which is countably additive. Also,*

$$\int_{K'} \bar{p}_1 p_2\, d\eta_k = c_k^2(\hat{T}_k\hat{p}_1\Omega_0, \hat{T}_k\hat{p}_2\Omega_0)_r\,.$$

Proof. The criteria of Theorem III.1 are easily verified using the results in IV, so we have a measure η_k. To show the continuity condition, and therefore countable additivity, we need to examine the moments of η_k.

The following is known about measures on $\mathbb{R}$: Suppose $\{\mu_j\}$ is a sequence of measures with moments $m_j^{(k)} = \int x^k d\mu_j(x)$ and suppose the sequences $m_j^{(k)} \to m^{(k)}$, finite. Then the limits are the moments of a measure μ such that some subsequence $\mu_{j'}$ converges weakly to μ and the $m^{(k)}$ are the moments of μ [10, p. 185]. In the case of our real measures on complex spaces, we know that the complex moments are finite; i.e. $\int \bar{p}_1 p_2\, d\eta_{k\sigma} < \infty$ and as $\sigma \to \infty$, these expressions converge to $c_k^2(\hat{T}_k\hat{p}_1\Omega_0, \hat{T}_k\hat{p}_2\Omega_0)_r < \infty$. We can verify that the real moments are also finite. Now

$$\int_{K'} \bar{p}_1 p_2\, d\eta_{k\sigma} = \int_{h\in F} \bar{p}_1(i(h))\, p_2(i(h))\, d\eta_{k\sigma,F}(h)$$

where $\langle f, i(g)\rangle = (g, f)$. But

$$\int \bar{p}_1 p_2\, d\eta_{k\sigma} = c_{k\sigma}^2(\hat{T}_{k\sigma}\hat{p}_1\Omega_0, \hat{T}_{k\sigma}\hat{p}_2\Omega_0)e^{-X(\sigma)}$$

and converges as $\sigma\to\infty$ to $c_k^2(\hat{T}_k\hat{p}_1\Omega_0, \hat{T}_k\hat{p}_2\Omega_0)_r$, where we have used Lemma IV.3. So

$$\lim_{\sigma\to\infty}\int \bar{p}_1 p_2\, d\eta_{k\sigma} = c_k^2(\hat{T}_k\hat{p}_1\Omega_0, \hat{T}_k\hat{p}_2\Omega_0)_r\,.$$

So if we let $\mu_j=\eta_{k\sigma,F}$, then μ must be $\eta_{k,F}$ and

$$\int \bar{p}_1 p_2\, d\eta_k = \lim_{\sigma\to\infty}\int \bar{p}_1 p_2\, d\eta_{k\sigma}\,,$$

where p_1 and p_2 are based on F, a finite dimensional subspace of K. The above, Chebychev's inequality, and Lemma IV.5 are used to verify the continuity condition.

We now have an isometry $E_k:\{\hat{T}_k\hat{p}\Omega_0\}\to H^2(K', d\eta_k)$ with $E_k(\hat{T}_k\hat{p}\Omega_0)=p/c_k$ and want to extend it to $\hat{T}_k\psi$ for more general ψ. Although Proposition III.5, an extension result for $\eta_{k\sigma}$, was valid for functions of arbitrary support, the extension theorem for η_k considers only functions of compact support.

Proposition V.2. *Fix $k\in Z^+$. Suppose $\phi=\Sigma\phi_n$, a finite sum, where $\phi_n\in\mathscr{D}_n$. Then there is an isometry E_k from $\{T_k\mathscr{D}\}^-$ into $H^2(K', d\eta_k)$ such that if $\hat{p}_i\Omega_0\to\phi$ in $\mathscr{F}$ and $\{\mathrm{supp}(\hat{p}_i\Omega_0)\}$ is a bounded set, then*

$$p_i/c_k = E_k(\hat{T}_k\hat{p}\Omega_0)\to E_k(\hat{T}_k\phi)$$

in $H^2(K', d\eta_k)$ and

$$\int |E_k(\hat{T}_k\phi)|^2\, d\eta_k = (\hat{T}_k\phi, \hat{T}_k\phi)_r\,.$$

The space $H^2(K', d\eta_k)$ is defined as the completion in $L^2(K', d\eta_k)$ of the polynomials on K'. We now want to define annihilation and creation operators on $H^2(K', d\eta_k)$ to get (as close as possible to) a representation of the CCR.

Let $a_k^*(f)$ be multiplication by $\langle f, \cdot\rangle$, defined on

$$\{\psi\in H^2(K', d\eta_k): \langle f, \cdot\rangle\,\psi\in H^2(K', d\eta_k)\}\,.$$

The domain contains all polynomials. $a_k^*(f)$ is closed and for $\phi\in\mathscr{D}$,

$$E_k(a^*(h)\phi) = \langle h, \cdot\rangle\, E_k(\phi)\,.$$

Let $a_k(f)$ be $(a_k^*(\bar{f}))^*$. This operator is densely defined and closed. Unfortunately we do not know if a_k and a_k^* have a common dense domain.

We now examine the relationship between $a_k^*(f)$ and $a_{k\sigma}^*(f)$, and between $a_k(f)$ and $a_{k\sigma}(f)$.

An operator B, defined on a dense subset of $H^2(K', d\eta_k)$ is called the *moving weak limit* of a sequence B_σ of operators, each defined on a dense subset of the corresponding $H^2(K', d\eta_{k\sigma})$, if the domains of all B_σ and B include all polynomials and if for all p_1, p_2

$$\int \bar{p}_1(B_\sigma p_2)\, d\eta_{k\sigma} \xrightarrow[\sigma]{} \int \bar{p}_1(Bp_2)\, d\eta_k\,.$$

Suppose $f\in$ K. Then $a_k^*(f)$ is the moving weak limit of $a_{k\sigma}^*(f)$. This result follows from V.1.

Formal Calculations

Formally we have the following:

$$a_{k\sigma}(f) = a(f) + [(a(f)\, T_{k\sigma})/T_{k\sigma}]\, I_0\,,$$

where I_0 is the identity operator (compare with the discussion after III.5).

Suppose $a_{k\sigma}$, a_k are defined on the polynomials. Then

$$\begin{aligned}\int \bar{p}_1 a_k(f) p_2\, d\eta_k &= \int (a_k^*(\bar{f}) p_1)^- p_2\, d\eta_k = \lim_{\sigma\to\infty} \int (a_{k\sigma}^*(\bar{f}) p_1)^- p_2\, d\eta_{k\sigma} \\ &= \lim_{\sigma\to\infty} \int \bar{p}_1 a_{k\sigma}(f) p_2\, d\eta_{k\sigma}\,.\end{aligned}$$

Since $a(f)p$ is a well-defined polynomial, we have that, formally,

$$(a(f)\, T_k)/T_k = \text{moving } \underset{\sigma\to\infty}{\text{weak}} \text{ limit } (a(f)\, T_{k\sigma})/T_{k\sigma}\,.$$

CCR (formally)

$$\begin{aligned}(p', [a_k(f), a_k^*(g)]\, p)_k \text{``}=\text{''} &\lim_{\sigma\to\infty} (p', [a_{k\sigma}(f), a_{k\sigma}^*(g)]\, p)_{k\sigma} \\ = &\lim_{\sigma\to\infty} (p', (\bar{f}, g)\, p)_{k\sigma} = (\bar{f}, g)\, (p', p)_k\,.\end{aligned}$$

VI. The Inductive Limit

Our goal is to produce a representation of the CCR which is unitarily equivalent to the Fabrey-Glimm representation. The latter is constructed by taking an inductive limit of the spaces $\hat{T}_i\mathscr{D}$ for $i \geqq 0$. Analogously, we take the inductive limit of the spaces $H^2(K', d\eta_i)$ for $i \geqq 0$, to form the space $\mathscr{H}$.

A collection of continuous linear maps $\beta_{ij}: H^2(K', d\eta_i) \to H^2(K', d\eta_j)$; $i \leqq j$; $i, j \in Z^+$ is called an *inductive system* if

(1) β_{ii} is the identity map on $H^2(K', d\eta_i)$,

(2) $\beta_{ik} = \beta_{jk} \circ \beta_{ij}$, $i \leqq j \leqq k$.

Proposition VI.1. *$\{H^2(K', d\eta_i), \beta_{ij}\}$ is an inductive system if we define β_{ij} as follows: Let $\beta_{ij}(p) = c_i E_j(\hat{T}_j^*(\hat{T}_{ij}\hat{p}\Omega_0))$, which is well-defined by* V.2, *and in fact equals $(c_i/c_j)\, T_{ij} \cdot p$, where $T_{ij} = I(\hat{T}_{ij}\Omega_0)$. Extend this isometry to $H^2(K', d\eta_i)$.*

We can check that $\beta_{ik} = \beta_{jk} \circ \beta_{ij}$ by using $\hat{T}_{ij}\Omega_0 = \lim_{s\to\infty} \sum_n \hat{p}_s^{(n)}\Omega_0$, where $\hat{p}_s^{(n)}\Omega_0 \in \mathscr{F}$ with appropriate supports as in V.2.

Now construct the inductive limit $\mathscr{H}$ by taking the locally convex direct sum $\bigoplus_{i\geqq 0} H^2(K', d\eta_i)$ modulo the subspace M generated by

$$\{\phi_i - \beta_{ij}(\phi_i) : \phi_i \in H^2(K', d\eta_i)\,, \quad i \leqq j\}\,.$$

Let β_j be the map taking $H^2(K', d\eta_j)$ into $\mathscr{H}$ and let $\mathscr{F}_r = \left(\bigcup_{i\geqq 0} \hat{T}_i\mathscr{D}\right)^-$.

For $p \in H^2(K', d\eta_i)$, let $\gamma_i p = c_i \hat{T}_i \hat{p}\Omega_0$. $\|c_i \hat{T}_i \hat{p}\Omega_0\|_r^2 = c_i^2 \|\hat{T}_i \hat{p}\Omega_0\|_r^2 = \int |p|^2\, d\eta_i$, by V.1. Therefore γ_i is an isometry and can be extended to all of $H^2(K', d\eta_i)$. $\gamma_i = \gamma_j \circ \beta_{ij}$.

Theorem VI.2. *There exists a unique, continuous,* 1–1, *onto, linear map* $\gamma : \mathcal{H} \to \mathcal{F}_r$ *that makes the diagram below commutative:*

$$\begin{array}{ccccc} H^2(K', d\eta_i) & \xrightarrow{\beta_{ij}} & H^2(K', d\eta_j) & \xrightarrow{\beta_j} & \mathcal{H} \\ \gamma_i \big\downarrow & & \gamma_j \big\downarrow & & \big\downarrow \gamma \\ \mathcal{F}_r & & \mathcal{F}_r & & \mathcal{F}_r \end{array}$$

Proof. The result follows because $\mathcal{F}_r$ is locally convex and the γ_i are 1–1 continuous linear maps [8].

Operators on $\mathcal{H}$

Definition. For $h \in K$ define $\tilde{a}^*(h)$ on a dense subset of $\mathcal{H}$ as follows:

$$\tilde{a}^*(h)(\beta_i p) = \beta_i(a_i^*(h) p) \qquad p \in H^2(K', d\eta_i).$$

Finite sums of expressions of the form $\beta_i p$ are dense in $\oplus H^2(K', d\eta_i)$, hence dense in $\mathcal{H}$.

Proposition VI.3. $\tilde{a}^*(h)$ *is well defined, i.e. for* $i \leqq j$ *we have*

$$a_j^*(h)(\beta_{ij} p) = \beta_{ij}(a_i^*(h) p).$$

Fabrey gives the following definition: "We say that an operator B, which maps a subspace of $\mathcal{F}_r$ into $\mathcal{F}_r$ is the weak limit of an operator A in F, written $B = \lim_\sigma A$ if the domain of B is $\bigcup_{k \geqq 0} \hat{T}_k \mathcal{D}$ and

$$(\hat{T}_k \psi_1, B \hat{T}_l \psi_2)_r = \lim_{\sigma \to \infty} (\hat{T}_{k\sigma} \psi_1, A \hat{T}_{l\sigma} \psi_2) e^{-X(\sigma)}\text{"} \qquad [3, \text{p. } 22].$$

He proves [3, p. 25] that the field operator ϕ has a weak limit $\lim_\sigma \phi \subset \phi_r$. So we have

$$a_r^*(h) = 2^{-1/2} [\phi_r(h) - i\phi_r(ih)] \supset \lim_{\sigma \to \infty} a^*(h),$$

$$a_r(\bar{h}) = 2^{-1/2} [\phi_r(h) + i\phi_r(ih)] \supset \lim_{\sigma \to \infty} a(\bar{h}).$$

Also, $\bigcup_{k \geqq 0} T_k \mathcal{D}$ is a dense set of entire vectors for $\phi_r(h)$, hence for the closed operators $a_r(\bar{h})$ and $a_r^*(h)$. The Weyl relations for a_r, a_r^* imply that $[a_r(f), a_r^*(g)]_r = (\bar{f}, g)_r$.

We can verify that $\gamma[\tilde{a}^*(h)(\beta_j p)] = a_r^*(h)(\gamma_j p)$. Since $a_r^*(h)$ is closed, we can form the closure of $\tilde{a}^*(h)$.

$\tilde{a}(\bar{h})$ is defined on $\mathcal{H}$ as $\tilde{a}^*(h)^*$, or equivalently, as the operator induced on $\mathcal{H}$ by the operator $a_r(\bar{h})$ on $\mathcal{F}_r$.

We have verified everything that is needed for the following:

Theorem VI.4. *The operators* $\tilde{a}(h)$ *and* $\tilde{a}^*(h)$ *on* $\mathcal{H}$, *for* $h \in K(\mathbb{R}^2)$, *define a representation of the* CCR *which is unitarily equivalent to the Fabrey-Glimm representation on* $\mathcal{F}_r$ *when the operators* $a_r(h)$, $a_r^*(h)$ *on* $\mathcal{F}_r$ *are considered only for* $h \in K(\mathbb{R}^2)$.

Formal Calculations

We would like to compare $\tilde{a}(h)$ with the formal annihilation operators

$$a_i(h) = a(h) + [(a(h)\,T_i)/T_i] \cdot I_0 \qquad (I_0 \text{ the identity operator})$$

on $H^2(K', d\eta_i)$, so we do the following:

Suppose the polynomials belong to the domains of all $a_i(h)$. Is it true, then, that $a_j(h) \circ \beta_{ij} = \beta_{ij} \circ a_i(h)$?

$$a_j(h) \circ \beta_{ij} p = a_j(h)\, c_i E_j(\hat{T}_j \hat{T}_{ij} \hat{p} \Omega_0) = a_j(h)\, c_i c_j^{-1}\, T_{ij} p\,.$$

As at the end of Section III, $a_j(h)$ would be defined on $H^2(K', d\eta_j)$ as the usual derivative, if $a(h)\,T_j = 0$. Then we would have

$$= c_{ij} [(a(h)\,T_{ij}) \cdot p + T_{ij} \cdot a(h) p]\,.$$

And

$$\beta_{ij} \circ a_i(h) p = \beta_{ij} \left(a(h) p + \frac{a(h)\,T_i}{T_i}\, p \right) \sim c_i E_j \left(\hat{T}_j \hat{T}_{ij} \left[a(h) \hat{p} \Omega_0 + \frac{a(h)\,\hat{T}_i}{\hat{T}_i}\, \hat{p} \Omega_0 \right] \right).$$

Now, if we write $T_i = T_{ij} T_j$ so

$$a(h)\,T_i = (a(h)\,T_{ij}) \cdot T_j + T_{ij} \cdot (a(h)\,T_j)$$
$$= (a(h)\,T_{ij}) \cdot T_j\,,$$

then

$$= c_i [T_{ij} \cdot a(h) p + (a(h)\,T_{ij}) \cdot p]\,.$$

$a(h)\,T_j = 0$ is roughly equivalent to the support of h being contained in

$$\{k \in \mathbb{R}^2 : \|k\| \leqq \alpha(j)\}\,.$$

So suppose h has compact support. Choose any i so large that $a(h)\,T_i = 0$. Then $\tilde{a}(h)\,(\beta_i p)$ can be defined as $\beta_i(a_i(h) p)$.

Acknowledgement. The author extends her thanks to Donald G. Babbitt for his help and encouragement.

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Communicated by A. S. Wightman

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