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Unbounded Derivations of C*-Algebras

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Abstract. We study unbounded derivations of C^*-algebras and characterize those which generate one-parameter groups of automorphisms. We also develop a functional calculus for the domains of closed derivations and develop criteria for closeability. Some special C^*-algebras are considered $\mathfrak{B}\mathfrak{C}(\mathfrak{H})$, $\mathfrak{B}(\mathfrak{H})$, UHF algebras, and in this last context we prove the existence of non-closeable derivations.

I. Introduction

A derivation δ of a C^*-algebra $\mathfrak{A}$ is a linear mapping from a dense * subalgebra $D(\delta)\subset\mathfrak{A}$ to a subspace $R(\delta)\subset\mathfrak{A}$ satisfying the two properties

1. $\delta(AB)=\delta(A)B+A\delta(B), \quad A, B\in D(\delta)$,
2. $\delta(A^*)=-\delta(A)^*, \quad A\in D(\delta)$.

$D(\delta)$ is the domain of δ and $R(\delta)$ the range.

If $\mathfrak{A}$ contains an identity element $\mathbb{1}$ we will always assume $\mathbb{1}\in D(\delta)$ and then $\mathbb{1}^2=\mathbb{1}$ etc. immediately implies that $\delta(\mathbb{1})=0$.

It is known that if a derivation is everywhere defined, $D(\delta)=\mathfrak{A}$, then it is bounded (for this and other results on bounded derivations see, for example, [1], Chapter 4). We will be interested in unbounded derivations. Some results are already given in [2, 3].

II. General Algebras

The principal interest of unbounded derivations is that they arise as infinitesimal generators of strongly continuous one-parameter groups of *-automorphisms of $\mathfrak{A}$.

Let $A\in\mathfrak{A}\mapsto\tau_t(A)\in\mathfrak{A}$ be a one-parameter group of *-automorphisms of the C^* algebra $\mathfrak{A}$ satisfying

$$\lim_{t\to 0}\|\tau_t(A)-A\|=0, \qquad A\in\mathfrak{A}$$

and define

$$i\delta(A)=\lim_{t\to 0}(\tau_t(A)-A)/t$$

for the set $D(\delta)$ of $A\in\mathfrak{A}$ such that the limit exists. It is easily checked that δ is a derivation of $\mathfrak{A}$ and of course it corresponds to the infinitesimal generator of τ.

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The following proposition characterizes the derivations which arise in this manner.

Theorem 1. *Let δ be a derivation of a C^*-algebra $\mathfrak{A}$.*

The following conditions are equivalent

1. *δ is the infinitesimal generator of a strongly continuous one-parameter group of *-automorphisms of $\mathfrak{A}$.*

2. *δ is closed,*

$$\|\delta(A) - zA\| \geqq |\operatorname{Im} z| \, \|A\|, \qquad A \in D(\delta)$$

and $R(\delta \pm i) = \mathfrak{A}$ where

$$R(\delta \pm i) = \{B; B = \delta(A) \pm iA, \ A \in D(\delta)\}.$$

Proof. Deduction of 2 from 1 is a standard part of semi-group theory and can be found, for example, in [4].

Using condition 2 one can also use the techniques described in [4] to construct a one-parameter group of linear mappings $A \in \mathfrak{A} \mapsto \tau_t(A) \in \mathfrak{A}$, where $t \in \mathbb{R}$. In particular τ is given by a uniform limit

$$\tau_t(A) = \lim_{n \to \infty} \left(1 - i\frac{t}{n}\delta\right)^{-n} (A)$$

and the lower bound of condition 2 together with the group property ensures that

$$\|\tau_t(A)\| = \|A\|, \qquad A \in \mathfrak{A}, t \in \mathbb{R}.$$

Further note that

$$\tau_t(A)^* = \tau_t(A^*), \qquad A \in \mathfrak{A}, t \in \mathbb{R}$$

$$\tau_t(\delta(A)) = \delta(\tau_t(A)), \qquad A \in D(\delta), t \in \mathbb{R}.$$

It remains to prove that τ is in fact a group of automorphisms of $\mathfrak{A}$, e.g. to prove that

$$\tau_t(AB) = \tau_t(A)\,\tau_t(B), \qquad A, B \in \mathfrak{A}, t \in \mathbb{R}.$$

This last property may be deduced by first noting that δ has a dense set of analytic elements, i.e. a dense set of $A \in D(\delta^n)$, $n = 1, 2, \ldots$ such that

$$z \in \mathbb{C} \mapsto \sum_{n \geqq 0} \frac{z^n}{n!} \delta^n(A) \in \mathfrak{A}$$

exists and is analytic in some neighbourhood of the origin. This is established by constructing regularized elements

$$A_f = \int dt\, f(t)\, \tau_t(A),$$

where f is analytic. Now for A, B analytic one has

$$\begin{aligned}\tau_t(AB) &= \sum_{n \geqq 0} \frac{(it)^n}{n!} \delta^n(AB) \\ &= \left(\sum_{n \geqq 0} \frac{(it)^n}{n!} \delta^n(A)\right)\left(\sum_{m \geqq 0} \frac{(it)^m}{m!} \delta^m(B)\right) \\ &= \tau_t(A)\,\tau_t(B)\end{aligned}$$

for $|t| > t_0$ for some $t_0 \neq 0$. Next noting that the foregoing properties of τ ensure that $\tau_t(A)$, $\tau_t(B)$ are also analytic for $|t| > t_0$, with the same radius of convergence for the power series, one can repeat the argument to derive the product property for all t. The automorphism property then follows by density of the analytic elements.

Motivated by the above theorem we next consider closed derivations.

Theorem 2. *Let δ be a closed derivation of a C^* algebra $\mathfrak{A}$ with identity $\mathbb{1} \in D(\delta)$* [1].

If $A = A^* \in D(\delta)$ and $\lambda \notin \sigma(A)$, the spectrum of A, then

$$(\lambda\mathbb{1} - A)^{-1} \in D(\delta)$$

and

$$\delta((\lambda\mathbb{1} - A)^{-1}) = (\lambda\mathbb{1} - A)^{-1}\, \delta(A)\, (\lambda\mathbb{1} - A)^{-1}\,.$$

Proof. If $|\lambda|$ is larger than the spectral radius of A the Neumann series

$$(\lambda\mathbb{1} - A)^{-1} = \lambda^{-1} \sum_{n \geqq 0} (A/\lambda)^n$$

converges uniformly. Further $A^n \in D(\delta)$ and

$$\|\delta(A^n)\| \leqq n\, \|A\|^{n-1}\, \|\delta(A)\|\,.$$

Thus the sequence

$$\lambda^{-1} \sum_{n \geqq 0} \delta(A^n)/\lambda^n$$

converges uniformly. As δ is closed one concludes that the resolvent $R(\lambda) = (\lambda\mathbb{1} - A)^{-1} \in D(\delta)$.

Next assume λ, $\lambda_0 \notin \sigma(A)$ but $R(\lambda_0) \in D(\delta)$ and

$$|\lambda - \lambda_0| < \|R(\lambda_0)\|\,.$$

One then has

$$(\lambda\mathbb{1} - A)^{-1} = \sum_{n \geqq 0} (\lambda_0 - \lambda)^n\, ((\lambda_0\mathbb{1} - A)^{-1})^{n+1}\,.$$

By the same argument as above $R(\lambda) \in D(\delta)$.

An analytic continuation argument then allows one to conclude that $R(\lambda) \in D(\delta)$ for all $\lambda \notin \sigma(A)$.

Finally we use $\delta(\mathbb{1}) = 0$ to deduce that

$$\delta((\lambda\mathbb{1} - A)^{-1}\, (\lambda\mathbb{1} - A)) = 0$$

or, alternatively,

$$\delta((\lambda\mathbb{1} - A)^{-1})\, (\lambda\mathbb{1} - A) = (\lambda\mathbb{1} - A)^{-1}\, \delta(A)\,.$$

[1] If $\mathbb{1} \notin \mathfrak{A}$ one may extend δ to a closed derivation $\tilde{\delta}$ of $\mathfrak{A} + \mathbb{C}\mathbb{1}$ by setting $D(\tilde{\delta}) = D(\delta) + \mathbb{C}\mathbb{1}$, and $\tilde{\delta}(A + \mathbb{C}\mathbb{1}) = \delta(A)$.

Corollary 3. *Under the conditions of Theorem 2 consider $A = A^* \in D(\delta)$ and let $z \in \mathbb{C} \mapsto f(z) \in \mathbb{C}$ be a function analytic in an open simply connected set Σ_f containing $\sigma(A)$.*

It follows that $f(A) \in D(\delta)$. One has

$$f(A) = \frac{1}{2\pi i} \int_c d\lambda\, f(\lambda)\, (\lambda \mathbb{1} - A)^{-1}$$

and

$$\delta(f(A)) = \frac{1}{2\pi i} \int_c d\lambda\, f(\lambda)\, (\lambda \mathbb{1} - A)^{-1}\, \delta(A)\, (\lambda \mathbb{1} - A)^{-1},$$

where C is a simple closed rectifiable curve contained in Σ_f with $\sigma(A)$ contained in its interior.

Proof. The proof follows by approximating the integral by Riemann sums

$$\Sigma_N(f) = \frac{1}{2\pi i} \sum_{i=1}^{N} f(\lambda_i)\, (\lambda_i \mathbb{1} - A)^{-1}$$

such that $\Sigma_N(f)$ converges uniformly to $f(A)$ and noting that $\Sigma_N(f) \in D(\delta)$ with

$$\delta(\Sigma_N(f)) = \frac{1}{2\pi i} \sum_{i=1}^{N} f(\lambda_i)\, (\lambda_i \mathbb{1} - A)^{-1}\, \delta(A)\, (\lambda_i \mathbb{1} - A)^{-1}.$$

This latter sum converges uniformly to the second integral given in the corollary and as δ is closed the proof is complete.

In the sequel we use this corollary to deduce that if δ is a closed derivation and A is a positive invertible element contained in $D(\delta)$ then $A^{-\frac{1}{2}} \in D(\delta)$.

The following result is a modification of a result of Sakai's [3].

Theorem 3. *Let δ be a closed derivation of a C^*-algebra $\mathfrak{A}$ containing an identity $\mathbb{1} \in D(\delta)$.*

Given P, a projector in $\mathfrak{A}$, and $\varepsilon > 0$ there exists a projector $Q \in D(\delta)$ such that

$$\|P - Q\| < \varepsilon.$$

Proof. Following Sakai one can choose a self-adjoint $A \in D(\delta)$ such that

$$\|P - A\| < \varepsilon' = \min\{1/14, \varepsilon/8\}.$$

Since P is a projection

$$\sigma(A) \in [-6\varepsilon', 6\varepsilon'] \cup [1 - 6\varepsilon', 1 + 6\varepsilon'].$$

Let γ be the circle in C with centre at $(1, 0)$ and radius $1/2$. Define

$$Q = \frac{1}{2\pi i} \int_\gamma d\lambda (\lambda \mathbb{1} - A)^{-1}.$$

Using the spectral representation of A it is easy to see that Q is a projection and

$$\|A - Q\| < 6\varepsilon'.$$

Therefore

$$\|P - Q\| \leqq 7\varepsilon' < \varepsilon.$$

Finally we deduce that $Q \in D(\delta)$ by arguments similar to those used in Theorem 2 and Corollary 3.

The foregoing theorem will be of use in examining special algebras. This study is left for the following section.

We now give a criterion for closeability together with a criterion that ensures that δ generates an automorphism group in a suitable sense.

Theorem 4. *Let δ be a derivation of a C^* algebra $\mathfrak{A}$. Assume that $\mathfrak{A}$ possesses a state ω which generates a faithful cyclic representation $(\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega)$ and also satisfies*

$$\omega(\delta(A)) = 0, \qquad A \in D(\delta).$$

It follows that δ is closeable and there exists a symmetric operator H_δ on $\mathfrak{H}_\omega$ such that

$$D(H_\delta) = \{\psi; \psi = \pi_\omega(A)\Omega_\omega, \ A \in D(\delta)\}$$

$$\pi_\omega(\delta(A))\psi = [H_\delta, \pi_\omega(A)]\psi, \ A \in D(\delta), \psi \in D(H_\delta).$$

If further δ possesses a dense set $\mathfrak{A}^a$ of analytic elements the operator H_δ is essentially self-adjoint on $D(H_\delta)$.

Define a group of automorphisms of $\mathfrak{B}(\mathfrak{H}_\omega)$ by

$$\tau_t(A) = e^{i\tilde{H}_\delta t} A e^{-i\tilde{H}_\delta t},$$

where $\tilde{H}_\delta$ is the self-adjoint closure of H_δ. It follows that

$$\tau_t(\mathfrak{A}) = \mathfrak{A}, \qquad t \in \mathbb{R},$$

the group of automorphisms $A \in \mathfrak{A} \mapsto \tau_t(A) \in \mathfrak{A}$ is strongly continuous, and the infinitesimal generator of τ_t is the closure of the restriction of δ to $\mathfrak{A}^a$.

Proof. Define H_δ on $D(H_\delta)$ by

$$H_\delta \pi_\omega(A)\Omega_\omega = \pi_\omega(\delta(A))\Omega_\omega, \qquad A \in D(\delta)$$

then H_δ is well defined, i.e. $\pi_\omega(A)\Omega_\omega = 0$ implies $\pi_\omega(\delta(A))\Omega_\omega = 0$ by the following computation which is valid for $A, B \in D(\delta)$

$$\begin{aligned}(\pi_\omega(B)\Omega_\omega, \pi_\omega(\delta(A))\Omega_\omega) &= \omega(B^*\delta(A)) \\ &= \omega(\delta(B^*A)) + \omega(\delta(B)^*A) \\ &= (\pi_\omega(\delta(B))\Omega_\omega, \pi_\omega(A)\Omega_\omega),\end{aligned}$$

where we have used $\omega(\delta(B^*A)) = 0$ and $\delta(B^*) = -\delta(B)^*$.

The same computation shows that H_δ is symmetric, hence closeable. Next for $A, B \in D(\delta)$

$$\begin{aligned}\pi_\omega(\delta(A))\,\pi_\omega(B)\Omega_\omega &= \pi_\omega(\delta(AB))\Omega_\omega - \pi_\omega(A)\,\pi_\omega(\delta(B))\Omega_\omega \\ &= [H_\delta, \pi_\omega(A)]\,\pi_\omega(B)\Omega_\omega.\end{aligned}$$

To show that δ is closeable consider a sequence $A_n \in D(\delta)$ such that

$$\lim_{n\to\infty} \|A_n\| = 0$$

$$\lim_{n\to\infty} \|\delta(A_n) - A\| = 0 .$$

One has for each $B_1, B_2 \in D(\delta)$

$$\begin{aligned}(\pi_\omega(B_1)\Omega_\omega, \pi_\omega(A)\pi_\omega(B_2)\Omega_\omega) &= \lim_{n\to\infty} (\pi_\omega(B_1)\Omega_\omega, \pi_\omega(\delta(A_n))\pi_\omega(B_2)\Omega_\omega) \\ &= \lim_{n\to\infty} (H_\delta \pi_\omega(B_1)\Omega_\omega, \pi_\omega(A_n B_2)\Omega_\omega) \\ &\quad - \lim_{n\to\infty} (\pi_\omega(A_n^* B_1)\Omega_\omega, H_\delta \pi_\omega(B_2)\Omega_\omega) \\ &= 0 .\end{aligned}$$

Hence from the cyclicity of Ω_ω for π_ω and the density of $D(\delta)$ one has $\pi_\omega(A) = 0$. The faithfulness of π_ω then implies $A = 0$, i.e. δ is closeable.

For economy of notation we drop the indices ω and δ in the last part of the proof and identify $\mathfrak{A}$ with its representation on $\mathfrak{H}$.

If $A \in \mathfrak{A}^a$ define t_A as the radius of convergence of

$$\sum_{n\geq 0} \frac{t^n}{n!} \|\delta^n(A)\| .$$

Next note that $\mathfrak{A}^a\Omega$ is a dense set of analytic vectors for H because

$$H^n A\Omega = \delta^n(A)\Omega , \qquad A \in \mathfrak{A}^a$$

and

$$\sum_{n\geq 0} \frac{|t|^n}{n!} \|H^n A\Omega\| \leq \sum_{n\geq 0} \frac{|t|^n}{n!} \|\delta^n(A)\| < +\infty$$

for $|t| < t_A$. By a theorem of Nelson [5] H is essentially self-adjoint on $\mathfrak{A}^a\Omega$. Since a self-adjoint operator has no proper symmetric extension H is essentially self-adjoint on $D(H) = D(\delta)\Omega \supset \mathfrak{A}^a\Omega$.

Let τ_t be the group of *-automorphisms of $\mathfrak{B}(\mathfrak{H})$ corresponding to $\tilde{H}$. For $A \in \mathfrak{A}^a$ and $|t| < t_A$ define

$$\tau_t^0(A) = \sum_{n\geq 0} \frac{t^n}{n!} (i\delta)^n(A) .$$

Then one has

$$\tau_t^0(A) = \tau_t(A) , \qquad A \in \mathfrak{A}^a, |t| < t_A ,$$

as one sees by comparing powers series expansions and using

$$\delta(A) = [H, A] .$$

Since τ_t is a group of isometries it follows that

$$\|\tau_t^0(A)\| = \|A\| , \qquad A \in \mathfrak{A}^a, |t| < t_A .$$

Repeated use of the closedness of δ establishes that

$$\delta^n \tau_t^0(A) = \tau_t^0(\delta^n(A)), \qquad A \in \mathfrak{A}^a, |t| < t_A .$$

It then follows from the isometric property of τ^0 that if $A \in \mathfrak{A}^a$ and $|t| < t_A$ then $\tau_t^0 \in \mathfrak{A}^a$ and has the same radius of convergence as A. Thus from the above identity

$$\tau_t^0(\tau_t^0(A)) = \tau_t(\tau_t(A)) = \tau_{2t}(A), \qquad A \in \mathfrak{A}^a, |t| < t_A .$$

By iteration one shows that

$$\tau_t(A) = (\tau_{t/n}^0)^n (A)$$

for $A \in \mathfrak{A}^a$ and $|t/n| < t_A$. Hence τ_t maps $\mathfrak{A}^a$ into $\mathfrak{A}^a$ and by a closure argument $\tau_t(\mathfrak{A}) = \mathfrak{A}$ for all t.

Let $\tilde{\delta}$ be the infinitesimal generator of τ_t restricted to $\mathfrak{A}$. For $A \in \mathfrak{A}^a$ one has

$$\tilde{\delta}(A) = \delta(A)$$

because $\tau_t(A) = \tau_t^0(A)$ for small t. Hence

$$\tau_t(A) - A = i \int_0^t ds\, \tau_s(\delta(A)), \qquad |t| < t_A .$$

The strong continuity of τ then follows from the density of $\mathfrak{A}^a$.

The proof of the theorem will be complete if we can show that $\mathfrak{A}^a$ is a core for $\tilde{\delta}$.

To do this we will show that if $A \in \mathfrak{A}^a$ then $(\tilde{\delta} + i)^{-1}(A) \in \mathfrak{A}^a$ and then it follows that

$$(\tilde{\delta} + i)(\mathfrak{A}^a) \supseteq (\tilde{\delta} + i)(\tilde{\delta} + i)^{-1}(\mathfrak{A}^a) = \mathfrak{A}^a .$$

Thus $(\tilde{\delta} + i)(\mathfrak{A}^a)$ is dense in $\mathfrak{A}$ and $\mathfrak{A}^0$ is a core for $\tilde{\delta}$ by the inequality

$$\|(\tilde{\delta} + i)(A)\| \geqq \|A\| .$$

Therefore assume $A \in \mathfrak{A}^a$ and define

$$B = (\tilde{\delta} + i)^{-1}(A) = i \int_0^\infty dt\, e^{-t} \tau_t(A) .$$

We showed earlier in the proof that

$$\delta(\tau_t(C)) = \tau_t(\delta(C)), \qquad C \in \mathfrak{A}^a .$$

Using this, the closedness of δ, and a Riemann sum argument, n times one derives

$$\delta^n(B) = i \int_0^\infty dt\, e^{-t} \tau_t(\delta^n(A)) .$$

Hence

$$\|\delta^n(B)\| \leqq \|\delta^n(A)\|$$

and $B \in \mathfrak{A}^a$.

Problem. If δ is the infinitesimal generator of a strongly continuous one parameter group of *-automorphisms of a C^* algebra $\mathfrak{A}$ is it possible that δ has proper closed extensions, i.e. can there exist a closed derivation δ_1, such that $\delta_1 \supset \delta$? A negative reponse to this question is equivalent to a proof that

$$\|\delta_1(A) - zA\| \geqq |\mathrm{Im}\, z|\, \|A\|, \qquad A \in D(\delta_1)$$

for all closed extensions δ_1 of δ.

The difficulty in applying the foregoing criterion is finding states such that $\omega \circ \delta = 0$. This is relatively easy for the following class of derivations.

Definition 5. A derivation δ of a C^*-algebra $\mathfrak{A}$ is defined to be an inner limit derivation if there exists a directed set δ_α of bounded derivations of $\mathfrak{A}$ which converge to δ in the following sense

$$\lim_\alpha \|\delta(A) - \delta_\alpha(A)\| = 0$$

for all A in a core[2] *of $D(\delta)$.*

Inner limit derivations occur in the work of Sinai and Helemsky [2] and that of Sakai [3], on derivations of UHF algebras. We will consider their properties in various special contexts in the following section. For the moment we note the following general properties.

Corollary 6. *Let δ be an inner limit derivation of a C^*-algebra $\mathfrak{A}$ which possesses a trace state ω which generates a faithful representation.*

It follows that δ is closeable.

Proof. Note first that

$$\omega(\delta(A)) = \lim_\alpha \omega(\delta_\alpha(A))$$

for all A in a core D of $D(\delta)$. But the bounded derivations δ_α are weakly inner (see, for example [1]) i.e. there exist $H_{\delta_\alpha} = H_{\delta_\alpha}^* \in \pi_\omega^-$ such that

$$\omega(\delta_\alpha(A)) = (\Omega_\omega, [H_{\delta_\alpha}, \pi_\omega(A)]\, \Omega_\omega), \qquad A \in D.$$

Approximating H_{δ_α} weakly and using the trace property one finds $\omega(\delta_\alpha(A)) = 0$ and hence $\omega(\delta(A)) = 0$ for all $A \in D$. Consequently $\omega(\delta(A)) = 0$ for all $A \in D(\delta)$ and the conditions of the above theorem are fulfilled.

Another easy property of inner limit derivations is that their resolvents may be bounded

Theorem 7. *Let δ be an inner limit derivation of a C^*-algebra $\mathfrak{A}$.*

It follows that

$$\|\delta(A) - zA\| \geqq |\mathrm{Im}\, z|\, \|A\|, \qquad A \in D(\delta),\ z \in \mathbb{C}.$$

*Thus if $R(\bar{\delta} \pm i) = \mathfrak{A}$ then the closure $\bar{\delta}$ of δ is the infinitesimal generator of a strongly continuous one parameter group of *-automorphisms of $\mathfrak{A}$.*

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[2] The set $D \subset D(\delta)$ is defined to be a core of $D(\delta)$ if for each $A \in D(\delta)$ there is a directed set $A_\alpha \in D$ such that $\|A_\alpha - A\| \to 0$ and $\|\delta(A_\alpha - A)\| \to 0$. Normally cores are only introduced for closed operators (derivations) but the foregoing definition does not need this restriction.

Proof. Consider any representation π of $\mathfrak{A}$. In this representation the bounded derivations δ_α of A, which approximate δ are weakly inner and hence generate one-parameter groups of automorphisms τ^α of π^-. Now one has for $\operatorname{Im} z < 0$

$$(\delta_\alpha - z)^{-1}(A) = i^{-1} \int_0^\infty dt\, e^{-izt} \tau_t^\alpha(A)$$

for all $A \in \mathfrak{A}$.

Therefore

$$\|(\delta_\alpha - z)^{-1}(A)\|_\pi \leqq |\operatorname{Im} z|^{-1}\, \|A\|_\pi$$

where $\|\cdot\|_\pi$ denotes the norm associated with the representation π. Therefore

$$\|(\delta_\alpha - z)(A)\|_\pi \geqq |\operatorname{Im} z|\, \|A\|_\pi$$

and

$$\|(\delta - z)(A)\|_\pi \geqq |\operatorname{Im} z|\, \|A\|_\pi$$

for all $A \in D(\delta)$ by limiting. As this is true for all π a similar inequality is valid with the algebraic norm. $\operatorname{Im} z > 0$ is handled similarly.

III. Special Algebras

In this section we consider derivations of special classes of C^*-algebras.

A. $\mathfrak{BC}(\mathfrak{H}) \subseteqq \mathfrak{A} \subseteqq \mathfrak{B}(\mathfrak{H})$

This subsection is devoted to the study of a C^*-algebra concretely represented by bounded operators acting on a Hilbert space $\mathfrak{H}$. We assume that $\mathfrak{A}$ contains the C^*-algebra $\mathfrak{BC}(\mathfrak{H})$ of compact operators acting on $\mathfrak{H}$ as subalgebra.

Theorem 8. *Let δ be a derivation of a C^*-algebra $\mathfrak{A}$ acting on the Hilbert space $\mathfrak{H}$ and assume that*

$$\mathfrak{BC}(\mathfrak{H}) \subseteqq \mathfrak{A} \subseteqq \mathfrak{B}(\mathfrak{H}).$$

1. *If $D(\delta)$ contains a finite rank operator then δ is closeable. Conversely if δ is closeable the domain of its closure $\bar{\delta}$ contains a* rank *one projector.*

2. *If δ is closed there exists a symmetric operator H such that $AD(H) \subset D(H)$, for all $A \in D(\delta)$, and*

$$\delta(A) = [H, A], \qquad A \in D(\delta).$$

3. *If $D(\delta) \supseteqq \mathfrak{BC}(\mathfrak{H})$ then δ is bounded.*

4. *If δ is closed then $D(\delta)$ contains a *-subalgebra $\mathscr{B}$ of finite* rank *operators which is dense among the finite* rank *operators and there exists a sequence of bounded derivations δ_n such that*

$$\lim_{n \to \infty} \|\delta(A) - \delta_n(A)\| = 0, \qquad A \in \mathscr{B}.$$

Proof. First consider the special case that $D(\delta)$ contains a rank one projector P. Following [6, 7] we define a derivation δ_P by

$$D(\delta_P) = D(\delta)$$

$$\delta_P(A) = \delta(A) - [X_P, A],$$

where

$$X_P = \delta(P)P - P\delta(P).$$

One then has

$$\begin{aligned}\delta_P(P) &= \delta(P) + 2P\delta(P)P - \delta(P)P - P\delta(P)\\ &= -\delta(P^2 - P) + 2P\delta(P)P\\ &= 0,\end{aligned}$$

where we have used $P = P^2$ and $P\delta(P)P = 0$. The latter relation follows from

$$\delta(P) = P\delta(P) + \delta(P)P = \delta(P^2).$$

Next if ψ is an normalized vector in the range of P one has for $A \in D(\delta)$

$$\begin{aligned}(\psi, \delta_P(A)\psi) &= (\psi, P\delta_P(A)P\psi)\\ &= (\psi, \{\delta_P(PAP) - \delta_P(P)AP - PA\delta_P(P)\}\psi)\\ &= 0\end{aligned}$$

because PAP is a scalar multiple of P. Applying Theorem 4 it follows that δ_P, and hence δ, is closeable. Theorem 4 also establishes the existence of an H_p with the desired properties and

$$H = H_p + X_p.$$

We now reduce the general case to the special case handled above.

Let $C \in D(\delta)$ be a finite rank operator and assume, without loss of generality, that $C = C^* > 0$ and $\|C\| = 1$. There exists a rank one projector $P \in \mathfrak{A}$ such that $P \leqq C$ and

$$CPC = P.$$

Now we may choose $A \in D(\delta)$ such that $A = A^*$ and for $\varepsilon > 0$

$$\|P - A\| < \varepsilon.$$

Hence

$$\|CAC - CPC\| < \varepsilon$$

or

$$\|CAC - P\| < \varepsilon.$$

But CAC is finite rank, hermitian, and for ε small enough must possess a simple eigenvalue in the neighbourhood of 1. Let E be the corresponding rank one spectral projector. As E may be expressed as a polynomial in $CAC \in D(\delta)$ one has $E \in D(\delta)$.

The converse statement of part 1 follows from Theorem 3 above.

Part 2 was proved in the above and next we prove Part 3. If $D(\delta) \supseteqq \mathfrak{BC}(\mathfrak{H})$ it is closeable by part 1. As each $C \in \mathfrak{BC}(\mathfrak{H})$ may be decomposed as a product $C = C_1 C_2$ with $C_1, C_2 \in \mathfrak{BC}(\mathfrak{H})$

$$\delta(C) = \delta(C_1) C_2 + C_1 \delta(C_2) \in \mathfrak{BC}(\mathfrak{H}).$$

Thus the range of $\bar{\delta}$ restricted to $\mathfrak{BC}(\mathfrak{H})$ is contained in $\mathfrak{BC}(\mathfrak{H})$. Thus $\bar{\delta}$ gives an everywhere defined derivation of $\mathfrak{BC}(\mathfrak{H})$. This latter derivation is then automatically bounded (see, for example [1]). Now the desired conclusion follows from part 2 because if

$$\delta(A) = [H, A]$$

for all $A \in D(\delta)$ and δ restricted to $\mathfrak{BC}(\mathfrak{H})$ is bounded it is impossible that H is unbounded.

It remains to prove part 4. The first statement follows from Theorem 3. To prove the second we note that the symmetric operator H can always be approximated by a sequence of bounded symmetric operators H_n in the sense that

$$\lim_{n \to \infty} \|(H - H_n)\psi\| = 0, \qquad \psi \in D(H).$$

This is demonstrated by remarking that H has a self-adjoint extension K acting in a possibly larger space $\mathfrak{K} \supset \mathfrak{H}$. Let E_K denote the spectral family of K, and P the orthogonal projector on $\mathfrak{K}$ with range $\mathfrak{H}$. The family

$$H_n = P \int_{-n}^{n} dE_K(\lambda)\, \lambda\, \mathrm{P}$$

has the desired property. It also follows that if $\psi \in D(H)$, and $A \in \mathfrak{B}(\mathfrak{H})$ is such that $AD(H) \subset D(H)$ then

$$\lim_{n \to \infty} \|([H, A] - [H_n, A])\psi\| = 0.$$

Define the bounded derivations δ_n by

$$\delta_n(A) = [H_n, A], \qquad A \in \mathfrak{B}(\mathfrak{H}).$$

If P is a one dimensional projector such that $P \in \mathscr{B}$ and ψ is a normalized vector in the range of P one computes that

$$\begin{aligned} \|\delta(P) - \delta_n(P)\| &= \|(H - H_n)P - P(H - H_n)\| \\ &\leqq 2\,\|(H - H_n)\psi\| \xrightarrow[n=\infty]{} 0. \end{aligned}$$

The convergence for finite rank $A = A^* \in \mathscr{B}$ follows because A is a finite linear combination of rank one projectors each of which is contained in $\mathscr{B}$ by the same argument that was used to prove part 1.

Problem. If $\mathfrak{BC}(\mathfrak{H}) \subsetneqq \mathfrak{A} \subsetneqq \mathfrak{B}(\mathfrak{H})$ do there exist non-closeable derivations of $\mathfrak{A}$?

B. UHF Algebras

A C^*-algebra $\mathfrak{A}$ with identity is said to be uniformly hyperfinite (UHF) if there exists a set, directed by inclusion, of C^*-subalgebras $\{\mathfrak{A}_\alpha\}_{\alpha\in I}$ containing the identity such that $\mathfrak{A}$ is the uniform closure of

$$\bigcup_{\alpha\in I} \mathfrak{A}_\alpha$$

and each $\mathfrak{A}_\alpha$ is isomorphic to $\mathfrak{B}(\mathfrak{H}_\alpha)$ where each $\mathfrak{H}_\alpha$ is finite dimension.

If the UHF algebra is separable then there exists an increasing sequence $\mathfrak{A}_1 \subset \mathfrak{A}_2 \subset \cdots$ of C^*-subalgebras containing the identity such that $\mathfrak{A}$ is the uniform closure of

$$\bigcup_{n\geqq 1} \mathfrak{A}_n$$

and each $\mathfrak{A}_n$ is *-isomorphic to $\mathfrak{B}(\mathfrak{H}_n)$ where $\mathfrak{H}_n$ is finite dimension (see [1], p. 73).

As each matrix algebra $\mathfrak{A}_\alpha$ has a unique normalized trace the C^*-algebra $\mathfrak{A}$ has a unique normalized trace ω_0. Further the simplicity of the $\mathfrak{A}_\alpha$ implies that $\mathfrak{A}$ is simple ([1], p. 74) hence τ_{ω_0} is faithful. Therefore from Corollary 6 we have

Corollary 9. *Each inner limit derivation of a UHF algebra is closeable.*

A large class of the derivations of a UHF algebra are automatically inner limit derivations (but not all, see Theorem 12) the following generalization of a result of Sinai and Helemski [2] shows

Theorem 10. *Let δ be a derivation of the UHF algebra $\mathfrak{A}$ with domain $D(\delta)$ given by*

$$D(\delta) = \bigcup_{\alpha\in I} \mathfrak{A}_\alpha .$$

It follows that for each $\alpha \in I$ there exists $H_\alpha = H_\alpha^* \in \mathfrak{A}$ such that

$$\delta(A) = [H_\alpha, A]$$

for all $A \in \mathfrak{A}_\alpha$.

Therefore δ is an inner limit derivation and hence closeable.

Proof. Lemma 7 of [8] establishes the existence of H_α with the desired properties. In particular if e_{ij} is a system of matrix units for $\mathfrak{A}_\alpha$ one has

$$H_\alpha = \sum_i \delta(e_{i1}) e_{1i}$$

by explicit calculation. Thus defining δ_α by $\delta_\alpha(A) = [H_\alpha, A]$, $A \in \mathfrak{A}$. One finds that δ_α converges pointwise uniformly to δ on $D(\delta)$. Thus δ is inner limit and closeable by Corollary 9.

The foregoing theorem has a partial converse which is a generalization of a result of Sakai [3].

Theorem 11. *Let δ be a closed derivation of a separable UHF algebra $\mathfrak{A}$.*
There exists an increasing sequence of full matrix algebras B_n containing the identity such that each $B_n \subset D(\delta)$ and $\mathfrak{A}$ is the uniform closure of

$$\bigcup_n B_n .$$

Proof. The proof follows closely that of Sakai [3]. His Lemma 1 is replaced by our Theorem 3 and use is also made of Corollary 3 to deduce that if $A \in D(\delta)$, and $A > 0$, and A is invertible then $A^{-\frac{1}{2}} \in D(\delta)$. With these modifications the remainders of the proofs are identical and we refer to [3] for details.

Theorem 12. *Let $\mathfrak{A}$ be the CAR algebra and $\mathscr{B}_n$ an increasing sequence of $2^n \times 2^n$ full matrix algebras generating $\mathfrak{A}$.*
There exists a derivation δ of $\mathfrak{A}$ such that
1. $\bigcup_n \mathscr{B}_n \subset D(\delta)$.
2. *δ restricted to $\bigcup_n \mathscr{B}_n$ is zero.*
3. $\delta \neq 0$.

Hence δ is not closeable nor inner limit.

Proof. The derivation δ is constructed by representing $\mathfrak{A}$ as the discrete C^*-crossed product of the continuous functions on the Cantor set with an abelian group of automorphisms, and then differentiation, in the usual sense, on the differentiable functions is lifted to a derivation δ of $\mathfrak{A}$.

Takesaki has shown [9] that

$$\mathfrak{A} = C^*(C(G), G_0),$$

where

$$G = \bigotimes_{n=1}^{\infty} \mathbb{Z}_2$$

$$G_0 = \bigoplus_{n=1}^{\infty} \mathbb{Z}_2$$

and $\mathbb{Z}_2$ is the cyclic group with two generators.

G is equipped with the product topology and G_0 with the discrete topology. G_0 acts on G by componentwise multiplication and this defines an action of G_0 on $C(G)$, the C^*-algebra of continuous functions on G, as a group of automorphisms α. G is homeomorphic to the Cantor subset K of $\{0, 1\}$. Let η; $\{0, 1\} \mapsto \{0, 2\}$ be given by $\eta(0) = 0$, $\eta(1) = 2$. Then a homeomorphism φ; $G \mapsto K$ can be given explicitly

$$\varphi((a_1, a_2, a_3, \ldots)) = \sum_{n \geq 1} \eta(a_n)/3^n .$$

For economy of notation let us identify G and K. Let δ_0 be the usual differentiation on $C(K)$

$$\delta_0(f(t)) = \lim_{h \to 0} i \frac{f(t+h) - f(t)}{h}$$

with domain $D(\delta_0)$ given as the set of once differentiable functions with continuous derivative. It is clear that all projections $p \in C(K)$ are in $D(\delta_0)$ and that $\delta_0(p) = 0$. This is also true for finite linear combinations of projections. It is also clear that $\delta_0 \neq 0$. In fact

$$R(\delta_0) = C(K).$$

Furthermore

$$\alpha_g \circ \delta_0 = \delta_0 \circ \alpha_g$$

for all $g \in G_0$. To see this let

$$g = (a_1, a_2, \ldots a_n, 0, 0, \ldots)$$

be an arbitrary element in G_0. There exists a partition $\{K_1, \ldots, K_{2^n}\}$ of K into open subsets and the action of g on K can be described as a permutation of these subsets in such a way that if x and y belong to the same subset then

$$g(x) - g(y) = x - y.$$

There is a positive distance at least 3^{-n} between any two subsets in the partition, thus

$$\alpha_g \circ \delta_0 = \delta_0 \circ \alpha_g.$$

Let $D(\delta)$ be the elements of $\mathfrak{A}$ that are represented by functions $f; G_0 \mapsto D(\delta_0)$ such that $f(g) \neq 0$ only for finitely many $g \in G$ and define δ, on $D(\delta)$, by

$$(\delta f)(g) = \delta_0(f(g)).$$

Then

$$\begin{aligned}\delta((f h)(g)) &= \delta_0((f h)(g)) \\ &= \delta_0\Big(\sum_{g' \in G} f(g')\, \alpha_g(h(g - g'))\Big) \\ &= \sum_{g' \in G} \{(\delta_0 f)(g')\, \alpha_g(h(g - g')) + f(g')\, \alpha_g(\delta_0(h(g - g')))\} \\ &= ((\delta f) h + f(\delta h))(g).\end{aligned}$$

Also

$$\begin{aligned}(\delta f^*)(g) &= \delta_0(f^*(g)) = \delta_0(\alpha_g(f(-g))^*) \\ &= \alpha_g(\delta_0(f(-g))^*) = -\alpha_g(\delta_0(f(-g)^*)) \\ &= -(\delta f)^*(g).\end{aligned}$$

Hence δ is a derivation of $\mathfrak{A}$.

Elements in the matrix algebras $\mathscr{B}_n$ are represented by functions f from G_0 into $C(G)$ such that $f(g) \neq 0$ only for finitely many g and such that $f(g)$ is a finite linear combination of projections in $C(K)$ for such g. It follows that $\mathscr{B}_n \subset D(\delta)$ and $\delta(\mathscr{B}_n) = 0$. But $\delta \neq 0$ on $D(\delta)$. $\Big($One can even show that $\bigcup_n \mathscr{B}_n \subset R(\delta)$.$\Big)$

Corollary 13. *Let $\mathfrak{A}$ be the CAR algebra*

$$\mathfrak{A} = \overline{\bigcup_n \mathscr{B}_n}\,.$$

*There exists a linear functional φ defined on a dense *-subalgebra $D(\varphi)$ such that*

$$\bigcup_n \mathscr{B}_n \subset D(\varphi) \subset \overline{\bigcup_n \mathscr{B}_n}$$

such that

$$\begin{aligned} \varphi(AB) &= \varphi(BA), && A, B \in D(\varphi) \\ \overline{\varphi(A)} &= \varphi(A^*), && A \in D(\varphi) \\ \varphi(A) &= 0, && A \in \bigcup_n \mathscr{B}_n \end{aligned}$$

but $\varphi \neq 0$, i.e. φ is not a multiple of the unique normalized trace ω_0 on $\mathfrak{A}$.

Proof. Let δ be the derivation of Theorem 15 and define $D(\varphi) = D(\delta)$

$$\varphi(A) = i\omega_0(\delta(A)), \qquad A \in D(\delta)\,.$$

We note that there exist inner limit derivations of UHF algebras which are not infinitesimal generators of groups of automorphisms and have no extensions with this property although the approximating bounded derivations do have this property.

C. Abelian Algebras

If $\mathfrak{A}$ is abelian one can add a few extra properties not covered by the foregoing.

First note that in constructing the derivation δ of Theorem 15 we also constructed a derivation δ_0 of an abelian C^*-algebra $C(G)$ such that δ_0 was non-closeable. Hence such derivations may occur even in the commutative setting.

Theorem 17. *Let δ be a closed derivation of an abelian C^*-algebra $\mathfrak{A}$.*
Take $A = A^* \in D(\delta)$ and let $f \in C^1(\mathbb{R})$.
It follows that $f(A) \in D(\delta)$ and

$$\delta(f(A)) = \delta(A)\, f'(A)\,.$$

Proof. If P is any polynomial then by simple computation $P(A) \in D(\delta)$

$$\delta(P(A)) = \delta(A)\, P'(A)\,.$$

Next choose polynomials P_n such that $P_n \to f$ and $P_n' \to f'$ uniformly on the spectrum of A.

Then by the foregoing

$$\delta(P_n(A)) = \delta(A)\, P_n'(A)$$

converges to $\delta(A)\, f'(A)$. Hence $f(A) \in D(\delta)$ and

$$\delta(f(A)) = \delta(A)\, f'(A)\,.$$

Note that one may easily conclude that δ bounded is equivalent to $\delta = 0$.

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