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Three Applications to $SO(4)$ Invariant Systems of a Theorem of L. Michel Relating Extremal Points to Invariance Properties

Francesco Pegoraro

Scuola Normale Superiore, Pisa, Italy

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Abstract. We consider a theorem due to Michel [1] which relates the invariance properties in peculiar directions in a linear space on which we represent a Lie group G to the extremal points of an arbitrary smooth G-invariant function.

The group we are interested in is $SO(4)$ and we apply the mathematical results to the following problems:

i) mixed linear Stark Zeeman effect in a hydrogen atom,

ii) perturbation of a finite Robertson-Walker metric,

iii) gas evolutions preserving angular momentum and vorticity.

Introduction

Among the solutions of a theory covariant under the action of a group G there may be peculiar ones which are invariant under a subgroup of G.

For example among the orbits of a mass point in a central, stationary and attractive field the circular ones are invariant under rotations around the axis perpendicular to the plane of the orbit.

As it is well known the orbits of a mass point in a central stationary field are specified, modulo a transformation of the covariance group, by the energy E and by the square of the angular momentum J^2.

Now, if we consider the orbits with a fixed value of J^2

$$E = \frac{m\dot{r}^2}{2} + \frac{J^2}{2mr^2} + U(r)$$

we see that the allowed values of E[1] go from $E_{\min}$ to infinity. Moreover $E_{\min}$ corresponds to a circular orbit and vice versa.

This example illustrates a property frequently satisfied by highly symmetric solutions, if suitably normalized. Such solutions are extremal i.e. each invariant smooth function defined on the space of "normalized" solutions has a vanishing differential on them.

In the case of Lie groups Michel [1] has proved a theorem which relates invariance properties to extremal points of the invariant functions.

In this paper we want to apply this theorem to the study of three problems which are invariant under the group $SO(4)$.

[1] If $\lim_{r\to 0} r^2 |U(r)| < 1$.

In section one we quote Michel's theorem, we list some general results concerning abstract $SO(4)$ representations and we explicitly analyze some consequences of Michel's theorem.

A simple application concerning extremal splittings of the energy levels of a hydrogen atom in a magnetic and electric field is given in section two. In section three after examining the properties of a perturbation of a metric tensor in General Relativity we consider a finite Robertson Walker metric. Then we decompose a generic perturbation of a Robertson Walker metric into irreducible components and we look for extremal perturbations.

In section four we consider a class of uniform or quasiuniform evolutions of a gas cloud which are $SO(4)$ invariant.

1. Critical Orbits and Extremal Points

Let us recall the following definitions.

Definition 1. Let G be a transformation group acting on a set S. We call G-orbit of s, $s \in S$ the set O_s defined by

$$O_s = \{\gamma \circ s; s \in S, \forall \gamma \in G\} . \tag{1.1}$$

Definition 2. We define the isotropy subgroup of $s \in S$ the subgroup G_s of G:

$$\gamma \circ s = s \quad \forall \gamma \in G_s . \tag{1.2}$$

The isotropy groups of points belonging to the same orbit are conjugated i.e.

$$\exists \gamma \in G : \gamma \circ G_s \circ \gamma^{-1} = G_{s'} \ \ s' = \gamma \circ s . \tag{1.3}$$

Definition 3. The orbits whose elements have conjugated isotropy groups belong by definition to the same stratum.

Let us now assume S to be a manifold.

Definition 4. We call an orbit O_s isolated in its stratum if $\forall s' \in O_s$ a neighbourhood $I_{s'}$ of s' exists such that $\forall s'' \in I_{s'} : s'' \notin O_s$ the isotropy group of s'' is not conjugated to that of s.

Definition 5. We call an orbit O_s critical if for each G-invariant real smooth function f defined on S the differential vanishes on the orbit i.e.

$$\forall f, \forall s' \in O_s : df_{s'} = 0 .$$

The following two theorems hold [1].

Theorem 1. *Let G be a compact Lie group acting smoothly on a smooth paracompact manifold S; then a G-orbit on S is critical if and only if it is isolated in its stratum.*

Theorem 2. *If the isotropy group of a stratum (modulo a conjugation) is maximal then the stratum is closed and if S is compact either it contains only a finite number of orbits, which then obviously are isolated, or for each G-invariant smooth function f from S into $\mathbb{R}$ $df = 0$ for at least two orbits in the stratum (the actual position of these orbits depends in general on the function f).*

Definition 6. Given two spaces A and B and a group G acting on them (i.e. given two maps $G \times A \to A$ and $G \times B \to B$ with the due composition laws) we call a map from A into B G-equivariant if it commutes with the group action. The use of equivariant maps allows us to study the representations of the group G on abstract linear spaces.

In the remaining part of this section we will study orbits and strata of some $SO(4)$ representations and we will give a general result about $SO(3)^d$ invariant vectors. The application of these results to physical problems will be given in the following sections.

Since each stratum is characterized by an isotropy subgroup we give now a list of those we will meet in the following analysis.

Let L_α and K_β $\alpha, \beta = 1, 2, 3$, be the infinitesimal generators of $SO(4)$: their commutation relations are

$$\begin{aligned} [L_\alpha, L_\beta] &= \varepsilon_{\alpha\beta\gamma} L_\gamma \\ [L_\alpha, K_\beta] &= \varepsilon_{\alpha\beta\gamma} K_\gamma \\ [K_\alpha, K_\beta] &= \varepsilon_{\alpha\beta\gamma} L_\gamma \end{aligned} \tag{1.4}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the usual three index antisymmetrical symbol. Let $J = \frac{1}{2}(L + K)$ and $H = \frac{1}{2}(L - K)$: then we have

$$\begin{aligned} [J_\alpha, J_\beta] &= \varepsilon_{\alpha\beta\gamma} J_\gamma \\ [H_\alpha, H_\beta] &= \varepsilon_{\alpha\beta\gamma} H_\gamma \\ [J_\alpha, H_\beta] &= 0 \end{aligned} \tag{1.5}$$

i.e. $SO(4)$ is isomorphic to $SO(3)_1 \times SO(3)_2$ (where 1 labels the group generated by the J's and 2 the other one). The universal covering group of $SO(4)$ is $SU(2)_1 \times SU(2)_2$.

The four dimensional subgroups are $SO(3)_1 \times O(2)_2$ and $O(2)_1 \times SO(3)_2$ which are both maximal. The three dimensional subgroups we will be interested in are:

$$\begin{aligned} &SO(3)_1, \\ &SO(3)_2, \\ \text{and } &SO(3)^d \end{aligned}$$

(and all the conjugated ones) where d means diagonal, which is generated by the L's and is maximal.

All two dimensional subgroups are Abelian and are conjugated to $O(2)_1 \times O(2)_2$.

All the irreducible representations of $SO(4)$ are finite dimensional and can be labelled by a pair of integer or half integer positive numbers (p, q) such that $p + q$ is an integer.

a) The (1/2, 1/2) *Representation*

It is the lowest dimensional faithful representation. Let x be a vector of $\mathbb{R}^4$: the action of G on $\mathbb{R}^4$ is given by the law

$$x'_i = O_{ij} x_j \tag{1.6}$$

where O is an orthogonal 4×4 matrix.

From the decomposition

$$(1/2, 1/2)\otimes(1/2, 1/2) = (0, 0)\oplus(1, 1) \tag{1.7}$$

it follows that there is only one independent bilinear scalar form (which is positive definite since the group is compact) namely

$$\langle x, y\rangle = \sum_i x_i y_i \qquad x, y \in \mathbb{R}^4 . \tag{1.8}$$

Each vector belonging to $\mathbb{R}^4$ is conjugated through $SO(4)$ to a vector of the form $(\alpha, 0, 0, 0)$ where $0 \leqq \alpha = \langle x, x\rangle^{\frac{1}{2}}$. Thus the invariant $\langle x, x\rangle = \alpha^2$ is sufficient to label the orbits.

Moreover only two strata exist, namely the origin [$\alpha = 0$, isotropy group $SO(4)$] and $\mathbb{R}^4$ minus the origin [$\alpha > 0$ isotropy group $SO(3)^d$].

In our physical applications we will be interested in normalized vectors.

If we normalize to $\alpha = 1$ there is only one stratum with only one orbit (obviously on the sphere $\alpha = 1$ all invariant functions are constant) and thus the criticity of the orbit becomes trivial.

b) The Reducible Representation $(1/2, 1/2)\oplus(1/2, 1/2)$

The group action law follows from Eq. (1.6) i.e.

$$\begin{gathered} \mathbb{R}^8 = \mathbb{R}^4_1 \oplus \mathbb{R}^4_2 \qquad x_1 \in \mathbb{R}^4_1 \qquad x_2 \in \mathbb{R}^4_2 \\ x = x_1 \oplus x_2 \qquad x' = x'_1 \oplus x'_2 = O x_1 \oplus O x_2 \end{gathered} \tag{1.9}$$

(the same orthogonal matrix O for both x_1 and x_2). From

$$\otimes^2[(1/2, 1/2)\oplus(1/2, 1/2)] = \oplus \cdots \oplus 4\cdot(0, 0)\oplus \cdots \tag{1.10}$$

it follows that there exist four independent scalar bilinear forms, i.e.

$$\langle x_1, y_1\rangle, \quad \langle x_2, y_2\rangle, \quad \langle x_1, y_2\rangle, \quad \langle x_2, y_1\rangle . \tag{1.11}$$

Each vector belonging to $\mathbb{R}^8$ is conjugated through the action of the group to a vector of the form

$$\begin{gathered} \underline{x} = \underline{x}_1 \oplus \underline{x}_2 \\ \underline{x}_1 = (\alpha, 0, 0, 0) \qquad \alpha = \langle x_1, x_1\rangle^{\frac{1}{2}} \\ \underline{x}_2 = (\beta, \gamma, 0 0) \qquad \alpha\beta = \langle x_1, x_2\rangle \qquad \beta^2 + \gamma^2 = \langle x_2, x_2\rangle \\ \alpha, \gamma \geqq 0 . \end{gathered}$$

Thus the orbits are labelled by the three invariants

$$\sigma_1 = \langle x_1, x_1\rangle \qquad \sigma_2 = \langle x_2, x_2\rangle \qquad \tau = \langle x_1, x_2\rangle .$$

The strata into which $\mathbb{R}^8$ is partitioned are:

i) $\alpha = \beta = \gamma = 0 \Rightarrow \sigma_1 = \sigma_2 = 0$. The isotropy group is $SO(4)$,

ii) $\gamma = 0$ and either α or β or both are different from zero $\Rightarrow$ either $\sigma_2 = \lambda\tau = \lambda^2\sigma_1$ or $\sigma_1 = \mu\tau = \mu^2\sigma_2$ $\lambda, \mu \in \mathbb{R}$. The isotropy group is $SO(3)^d$,

iii) $\alpha \neq 0 \neq \beta \neq 0 \neq \gamma$ isotropy group $O(2)^d$.

The following two normalizations are useful:

a) $\langle x_1, x_1\rangle = \langle x_2, x_2\rangle = 1$.

In $S^3 \times S^3$ (where S^3 is the unit sphere in $\mathbb{R}^4$) there are two strata: the collinear (ii) and the general (iii) ones.

Only two orbits belong to the collinear stratum and are isolated; thus from Theorem 1 they are critical.

b) $\langle x_1, x_1\rangle + \langle x_2, x_2\rangle = 1$.

In S^7 we have two strata as before but there are no longer critical orbits.

Moreover since the collinear stratum is closed Theorem 2 holds.

We will now verify explicitly in these cases Theorems 1, 2. Let f be a real smooth $SO(4)$-invariant function defined on $\mathbb{R}^8$, then

$$f = f(\sigma_1, \sigma_2, \tau).$$

Compiuting the differential with the constrains $\sigma_1 = \sigma_2 = 1$ we have

$$\begin{aligned} 2\frac{\partial f}{\partial \sigma_1} x_1 + \frac{\partial f}{\partial \tau} x_2 + 2\lambda x_1 &= \frac{\partial f}{\partial x_1} \\ 2\frac{\partial f}{\partial \sigma_2} x_2 + \frac{\partial f}{\partial \tau} x_1 + 2\mu x_2 &= \frac{\partial f}{\partial x_2} \end{aligned} \tag{1.12}$$

where λ and μ are the Lagrange multipliers.

The condition $df = 0\ \forall f$ leads to:

$$\begin{aligned} 2\frac{\partial f}{\partial \sigma_1} x_1 + \frac{\partial f}{\partial \tau} x_2 - \left(\tau \frac{\partial f}{\partial \tau} + 2\frac{\partial f}{\partial \sigma_1}\right) x_1 = 0 \\ 2\frac{\partial f}{\partial \sigma_2} x_2 + \frac{\partial f}{\partial \tau} x_1 - \left(\tau \frac{\partial f}{\partial \tau} + 2\frac{\partial f}{\partial \sigma_2}\right) x_2 = 0 \end{aligned} \tag{1.13}$$

i.e. x_1 parallel (and thus equal or opposite) to x_2.

Vice versa if $x_1 // x_2$

$$\frac{\partial f}{\partial x_1} = \delta x_1 \quad \text{and} \quad \frac{\partial f}{\partial x_2} = \zeta x_2 \tag{1.14}$$

$\delta, \zeta \in \mathbb{R}$ and thus

$$df_{|\sigma_2 = \sigma_2 = 1} = 0$$

since the gradient is orthogonal to the hypersurface

$$\sigma_1 = \sigma_2 = \text{const}.$$

In case b) from $df = 0\ \forall f$ we get:

$$\begin{aligned} 2\frac{\partial f}{\partial \sigma_1} x_1 + \frac{\partial f}{\partial \tau} x_2 + 2\lambda(x_1 + x_2) = 0 \\ 2\frac{\partial f}{\partial \sigma_2} x_2 + \frac{\partial f}{\partial \tau} x_1 + 2\lambda(x_1 + x_2) = 0 \end{aligned} \tag{1.15}$$

i.e.

$$2\frac{\partial f}{\partial \sigma_1}x_1 - 2\frac{\partial f}{\partial \sigma_2}x_2 + \frac{\partial f}{\partial \tau}(x_2 - x_1) = 0 \tag{1.16}$$

from which it follows $x_1 = kx_2$ and $k = k(f)$ and since the conditions $\alpha^2 + \beta^2 = 1$ and $\gamma = 0$ define a circle in the orbit space, $f|_{x_1//x_2}$ must have at least a maximum and a minimum. Vice versa if $x_1//x_2$ from Eq. (1.14) it follows that df is orthogonal to the hypersurface $\sigma_1 + \sigma_2 = 1$ at the point $\delta x_1 = \zeta x_2$ where δ and ζ depend on f.

c) The (1, 1) *Representation*

It is a nine dimensional representation. If we realize $\mathbb{R}^9$ as the linear space of real symmetric traceless 4×4 matrices, $\boldsymbol{m} \in \mathbb{R}^9$, the action law is

$$\boldsymbol{m}' = O\boldsymbol{m}\,{}^t O \tag{1.17}$$

where $\boldsymbol{m} \in \mathbb{R}^9$ and $O \in SO(4)$.

Another useful realization of $\mathbb{R}^9$ is as the linear space of the 3×3 matrices with the action law

$$\boldsymbol{n} \in \mathbb{R}^9 \qquad \boldsymbol{n}' = O_1 \boldsymbol{n}\,{}^t O_2 \tag{1.18}$$

where $O_1 \in SO(3)_1$ and $O_2 \in SO(3)_2$. This realization is interesting in relation to the similar $(3, \bar{3}) \oplus (\bar{3}, 3)$ representation of $SU(3) \times SU(3)$, see [2], and will be used in Section 4.

If n_{ij} are the elements of the 3×3 matrix the elements of the corresponding 4×4 are

$$m_{\alpha\beta} = \frac{1}{2}\left|\begin{matrix} n_{11}+n_{22}+n_{33} & n_{23}-n_{32} & n_{31}-n_{13} & n_{12}-n_{21} \\ n_{23}-n_{32} & n_{11}-n_{22}-n_{33} & n_{12}+n_{21} & n_{31}+n_{13} \\ n_{31}-n_{13} & n_{12}+n_{21} & n_{22}-n_{33}-n_{11} & n_{23}+n_{32} \\ n_{12}-n_{21} & n_{31}+n_{13} & n_{23}+n_{32} & n_{33}-n_{11}-n_{22} \end{matrix}\right| \tag{1.19}$$

$\alpha, \beta = 1, \ldots, 4$.

From the tensor decomposition (symmetrical part)

$$\otimes^2[(1, 1)] = \cdots \oplus (0, 0) \oplus (1, 1) \tag{1.20}$$

it follows that there exists only one bilinear form and one G equivariant symmetrical algebra: i.e.

$$\langle \boldsymbol{m}, \boldsymbol{m}' \rangle = \tfrac{1}{2}\,\mathrm{tr}\,\boldsymbol{m}\boldsymbol{m}', \tag{1.21}$$

$$\boldsymbol{m} \vee \boldsymbol{m}' = \tfrac{1}{2}(\boldsymbol{m}\boldsymbol{m}' + \boldsymbol{m}'\boldsymbol{m}) - \tfrac{1}{4}\,\mathrm{tr}\,\boldsymbol{m}\boldsymbol{m}'\,{}^2. \tag{1.22}$$

This algebra contains: nihil potent vectors

$$\boldsymbol{m} \vee \boldsymbol{m} = 0 \tag{1.23}$$

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[2] The three dimensional version of this algebra is

$$\boldsymbol{n} \vee \boldsymbol{n}' = \tfrac{1}{2}\,1(\mathrm{tr}\,\boldsymbol{n}\,\mathrm{tr}\,\boldsymbol{n}' - \mathrm{tr}\,\boldsymbol{n}\boldsymbol{n}') - \tfrac{1}{2}\,{}^t\boldsymbol{n}\,\mathrm{tr}\,\boldsymbol{n}' - \tfrac{1}{2}\,{}^t\boldsymbol{n}'\,\mathrm{tr}\,\boldsymbol{n} + \tfrac{1}{2}(\boldsymbol{n}\boldsymbol{n}' + \boldsymbol{n}'\boldsymbol{n}).$$

all conjugated to $\boldsymbol{m}_0$

$$\boldsymbol{m}_0 = \begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & -\alpha & \\ & & & -\alpha \end{pmatrix} \quad \alpha \in \mathbb{R} \tag{1.24}$$

idempotent vectors

$$\boldsymbol{m} \vee \boldsymbol{m} = \lambda \boldsymbol{m} \tag{1.25}$$

all conjugated to $\boldsymbol{m}_1$

$$\boldsymbol{m}_1 = \alpha \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -3 \end{pmatrix} \tag{1.26}$$

$$\lambda = -2\alpha^2 .$$

Each $\boldsymbol{m} \in \mathbb{R}^9$ is conjugated to a diagonal matrix

$$\boldsymbol{m} = \begin{pmatrix} \alpha & & & \\ & \beta & & \\ & & \gamma & \\ & & & \delta \end{pmatrix} \tag{1.27}$$

$$\alpha \geqq \beta \geqq \gamma \geqq \delta = -(\alpha + \beta + \gamma) .$$

The orbits can thus be labelled by the numbers α, β, and γ or equivalently by the three invariants

$$\begin{aligned} \langle \boldsymbol{m}, \boldsymbol{m} \rangle &= \sigma_1 \\ \langle \boldsymbol{m} \vee \boldsymbol{m}, \boldsymbol{m} \rangle &= \sigma_2 \\ \langle \boldsymbol{m} \vee \boldsymbol{m}, \boldsymbol{m} \vee \boldsymbol{m} \rangle &= \sigma_3 . \end{aligned} \tag{1.28}$$

The strata are:

		isotropy group
$\alpha = \beta = \gamma = \delta = 0$	trivial	$SO(4)$
$\begin{cases} \alpha = \beta = \gamma \\ \beta = \gamma = \delta \end{cases}$	idempotent	$SO(3)^d$
$\alpha = \beta = -\gamma$	nihil potent	$O(2)_1 \times O(2)_2$
$\begin{cases} \alpha = \beta \neq \lvert\gamma\rvert \\ \text{or} \\ \alpha \neq \beta = \gamma \\ \text{or} \\ \gamma = \delta \neq \alpha \end{cases}$		$O(2)^d$
$\alpha \neq \beta \neq \gamma \neq \delta$	general stratum .	

The natural normalization is $\langle \boldsymbol{m}, \boldsymbol{m} \rangle = 1$. In S^8 there are three critical orbits i.e. two idempotent and the nihil potent vectors of the symmetric algebra as it follows easily from the following equations: $f = f(\sigma_1, \sigma_2, \sigma_3)$,

$$\frac{\partial \sigma_1}{\partial \boldsymbol{m}} = \boldsymbol{m} \qquad \frac{\partial \sigma_2}{\partial \boldsymbol{m}} = \tfrac{3}{2}\, \boldsymbol{m} \vee \boldsymbol{m} \qquad \frac{\partial \sigma_3}{\partial \boldsymbol{m}} = 2\boldsymbol{m} \vee \boldsymbol{m} \vee \boldsymbol{m}\,. \tag{1.29}$$

d) The Reducible Representation (1, 1)⊕(1, 1)

The group action law follows from Eq. (1.17) or from Eq. (1.18). Each pair of matrices $\boldsymbol{m} = \boldsymbol{m}_1 \oplus \boldsymbol{m}_2 \in \mathbb{R}^9 \oplus \mathbb{R}^9$ is conjugated to a pair whose first element is diagonal according to Eq. (1.27). The orbits are labelled by the twelve invariants

$$\begin{array}{lll}
\langle \boldsymbol{m}_1, \boldsymbol{m}_1 \rangle & \langle \boldsymbol{m}_1 \vee \boldsymbol{m}_1, \boldsymbol{m}_1 \rangle & \langle \boldsymbol{m}_1 \vee \boldsymbol{m}_1, \boldsymbol{m}_1 \vee \boldsymbol{m}_1 \rangle \\
\langle \boldsymbol{m}_2, \boldsymbol{m}_2 \rangle & \langle \boldsymbol{m}_2 \vee \boldsymbol{m}_2, \boldsymbol{m}_2 \rangle & \langle \boldsymbol{m}_2 \vee \boldsymbol{m}_2, \boldsymbol{m}_2 \vee \boldsymbol{m}_2 \rangle \\
\langle \boldsymbol{m}_1, \boldsymbol{m}_2 \rangle & \langle \boldsymbol{m}_1 \vee \boldsymbol{m}_1, \boldsymbol{m}_2 \rangle & \langle \boldsymbol{m}_1 \vee \boldsymbol{m}_1, \boldsymbol{m}_1 \vee \boldsymbol{m}_2 \rangle \\
 & \langle \boldsymbol{m}_2 \vee \boldsymbol{m}_2, \boldsymbol{m}_1 \rangle & \langle \boldsymbol{m}_1 \vee \boldsymbol{m}_1, \boldsymbol{m}_2 \vee \boldsymbol{m}_2 \rangle \\
 & & \langle \boldsymbol{m}_1 \vee \boldsymbol{m}_2, \boldsymbol{m}_2 \vee \boldsymbol{m}_2 \rangle\,.
\end{array}$$

Besides the general stratum the others are

		isotropy group
$\boldsymbol{m}_1 = \boldsymbol{m}_2 = 0$	trivial	$SO(4)$
$\boldsymbol{m}_1$ and $\boldsymbol{m}_2$	idempotent and parallel	$SO(3)^d$
$\boldsymbol{m}_1$ and $\boldsymbol{m}_2$	nihil potent and parallel	$O(2)_1 \times O(2)_2$
$\boldsymbol{m}_1$ and $\boldsymbol{m}_2$	invariant under the same $O(2)^d$	$O(2)^d$.

With the normalization $\langle \boldsymbol{m}_1, \boldsymbol{m}_1 \rangle = \langle \boldsymbol{m}_2, \boldsymbol{m}_2 \rangle = 1$ the second and the third stratum are critical.

e) The Adjoint Representation

The adjoint representation (1, 0)⊕(0, 1)[3] is six dimensional and can be realized on the linear space of the real antisymmetrical 4×4 matrices with the transformation law.

$$\boldsymbol{m} \in \mathbb{R}^6 \qquad O \in SO(4)$$

$$\boldsymbol{m}' = O\boldsymbol{m}\,{}^tO\,. \tag{1.31}$$

We define $\boldsymbol{m}^* \in \mathbb{R}^6$

$$\boldsymbol{m}^*_{kl} = \varepsilon_{klij}\, \boldsymbol{m}_{ij} \tag{1.32}$$

where ε is the completely antisymmetrical four index symbol. Thus the decomposition of $\boldsymbol{m}$ into irreducible components becomes:

$$\begin{aligned} \boldsymbol{m}_+ &= \tfrac{1}{2}(\boldsymbol{m} + \boldsymbol{m}^*) \in (1, 0) \\ \boldsymbol{m}_- &= \tfrac{1}{2}(\boldsymbol{m} - \boldsymbol{m}^*) \in (0, 1)\,. \end{aligned} \tag{1.33}$$

[3] Which is reducible since the group is not semi-simple.

From

$$\otimes^2[(1,0)\oplus(0,1)] = \cdots \oplus 2\cdot(0,0)\oplus\cdots \tag{1.34}$$

it follows that there exist two independent bilinear scalar forms.

The orbits can be labelled by the two invariants

$$\begin{aligned} \sigma_1 &= -\operatorname{tr} \boldsymbol{m}_+^2 \\ \sigma_2 &= -\operatorname{tr} \boldsymbol{m}_-^2 \,. \end{aligned} \tag{1.35}$$

The strata are

		isotropy group
$\sigma_1=\sigma_2=0$	trivial	$SO(4)$
$\sigma_1=0$		$SO(3)_1\times O(2)_2$
$\sigma_2=0$		$O(2)_1\times SO(3)_2$
$\sigma_1 \neq 0 \neq \sigma_2$		$O(2)_1\times O(2)_2$.

If we normalize to $\sigma_1+\sigma_2=1$ the second and the third stratum contain only one orbit which is critical.

f) Criticity of a $SO(3)^d$ Invariant Orbit on a Reducible Representation

Untill now we have examined only single irreducible representations or direct sum of at most two of them. Now we want to give a general result concerning a $SO(3)^d$ invariant vector X belonging to an arbitrary finite dimensional, reducible, representation.

In the decomposition of the vector X the multiplicity of a given irreducible representation can be greater than one.

From the assumption of $SO(3)^d$ invariance it follows that in the decomposition of X only the representations of the form (j,j) can appear, where j is an integer or a half integer.

Let Γ_s be the set of these representations.

Each (j,j) representation contains once and only once the scalar representation of $SO(3)^d$.

Thus for each representation belonging to Γ_s there is a one dimensional linear space to which all $SO(3)^d$ invariant vectors belong.

Let us consider a $SO(4)$-equivariant operator from $(j,j)\otimes(k,k)$ into (l,m) [obviously (l,m) must appear in the decomposition of $(j,j)\otimes(k,k)$]. If it is applied to $SO(3)^d$ invariant vectors [belonging to the (j,j) and (k,k) representations respectively] the image vector must be $SO(3)^d$ invariant. Thus if $l\neq m$ the image vector must be the null vector whereas if $l=m$ it must belong to the one-dimensional, $SO(3)^d$-invariant sub space.

Thus the following theorem holds

Theorem 3. *Given a set $\tilde{\Gamma}_s$ of representations belonging to Γ_s, the direct sum of the $SO(3)^d$ invariant linear spaces belonging to these representations is closed for all the G-equivariant operators from $\gamma_1\otimes\gamma_2\to\gamma_3$ $\gamma_1,\gamma_2,\gamma_3\in\tilde{\Gamma}_s$.*

This is a generalization of the concept of idempotent vector that we have met in the study of the $(1,1)$ representation.

Let now f be a real $SO(4)$ invariant smooth function defined on a linear space acted on by a representation which is the direct sum of a finite number of irreducible representations belonging to Γ_s. Then $f(X) = f(\alpha_1(X), \ldots, \alpha_p(X))$ where $\alpha_i \; i = 1 \ldots p$ are all the invariants which are necessary and sufficient to label the orbits on $\tilde{\Gamma}_s$ and X is a generic vector. Then

$$\frac{\partial f}{\partial X} = \sum_{1}^{p}{}_i \frac{\partial f}{\partial \alpha_i} \frac{\partial \alpha_i}{\partial X}. \tag{1.36}$$

Since all the invariants α_i can be obtained using scalar products and $SO(4)$-equivariant operators it follows from the previous theorem that if X is $SO(3)^d$ invariant then $\forall i \frac{\partial \alpha_i}{\partial X}$ belongs to the space of $SO(3)^d$ invariant vectors.

Thus:

i) if we normalize the vector X to

$$\langle X_\gamma, X_\gamma \rangle = 1 \quad \forall_\gamma \in \Gamma_s$$

where X_γ is the γ-th irreducible component of X, then the $SO(3)^d$ stratum consists of a finite number of isolated orbits which are critical

ii) if we normalize the vector X to

$$\langle X, X \rangle \stackrel{\text{def}}{=} \sum_\gamma \langle X_\gamma, X_\gamma \rangle = 1$$

the stratum is closed and since $SO(3)^d$ is maximal there must be at least two stationary points[4].

We will now apply the foregoing mathematical concepts to the cases of physical interest.

2. The Mixed Stark Zeeman Effect on the Hydrogen Atom

The results of the previous section can be applied to all problems in which we are looking for maxima and minima (or extremal points in general) of an $SO(4)$ invariant function f. This method does not give the actual positions of all the extremal points for each given f but it exhibits a certain number of them that must necessarily be found in well defined positions. This can be an important a priori information. The simplest example is given by an hydrogen atom in a magnetic and electric field.

It is well known [3] that it is possible to classify the bound eigen states of the spinless non relativistic hydrogen atom using the (j, j) representations of $SO(4)$. (In Section 1 we called the set of these representations Γ_s.)

This classification follows from the conservation in a Coulomb field of the angular momentum and of the Lentz vector which are the group generators L and K.

[4] This theorem can be easily generalized to $SU(3) \times SU(3)$ and applied to the $SU(3)^d$ invariant orbits belonging to an arbitrary representation. The vectors of these "critical" orbits satisfy a quadratic equation since they are idempotent (see [2]).

The eigenvalues of the Hamiltonian are

$$E_n = -\frac{1}{2n^2} \quad \text{(in atomic units)} \tag{2.1}$$

where $n = 2j + 1$.

Let us now apply the linear perturbation theory to the determination of the energy corrections to the states of a hydrogen atom in a uniform electro-magnetic field (mixed Stark and Zeeman effects). Let $|\psi\rangle$ be a normalized energy eigen vector describing a bound state. It belongs to the complex Hilbert space of the hydrogen atom states and thus to apply the results of the previous section we must write it as the direct sum of its real and imaginary parts which transform like two equivalent real $SO(4)$ representations.

The correction to the mean energy is[5]

$$\Delta E = \mu_0 \mathscr{H}_i \langle\psi | L_i | \psi\rangle - \tfrac{3}{2}(-2E)^{-\frac{1}{2}} \langle\psi | K_i | \psi\rangle \mathscr{E}_i \tag{2.2}$$

where μ_0 is the Bohr magneton (in atomic units) $\mathscr{H}$ and $\mathscr{E}$ are the magnetic and electric fields and E is the unperturbed energy eigen value.

From Eq. (2.2) it follows that the perturbation term of the Hamiltonian operator on the bound hydrogen energy eigen state transforms like a vector depending on $\mathscr{E}$ and $\mathscr{H}$ which belongs to the $(1,0)\oplus(0,1)$ $SO(4)$ representation. Thus $\Delta E = \Delta E(\psi, \mathscr{E}, \mathscr{H})$ is a $SO(4)$ invariant function and we can look for extremal points of ΔE i.e. for those fields configurations and atomic states for which ΔE is critical.

The energy difference ΔE is invariant under the phase transformation group $U(1)$ whose action on $|\psi\rangle$ is

$$|\psi\rangle \to e^{i\phi} |\psi\rangle \tag{2.3}$$

while it acts trivially on the perturbation hamiltonian.

The real and the imaginary parts of $|\psi\rangle$ belong to a $SO(4)\times U(1)$ representation which is irreducible on the real numbers.

From Section 1 it follows that (apart from the trivial case $n = 1$) $O(2)_1 \times O(2)_2$ is the largest isotropy subgroup of a vector belonging to an arbitrary $(1,0)\oplus(0,1)$ $\oplus(j,j)$ representation of $SO(4)\times U(1)$ with the normalization conditions

$$\langle\psi|\psi\rangle = 1 \qquad \mu_0^2 \mathscr{H}^2 + 9\frac{\mathscr{E}^2}{4} n^2 = \alpha^2 \qquad \mathscr{E}\cdot\mathscr{H} = \beta\,. \tag{2.4}$$

The corresponding $SO(4)\times U(1)$ strata contain $(2j+1)^2$ critical orbits: the orientation of the electric and magnetic fields is arbitrary and $|\psi\rangle$ is an eigenvector of the perturbed Hamiltonian (cf. the results of the secular equation).

A representative element of these orbits can be chosen in the following way: $\mathscr{E}$ and $\mathscr{H}$ lie along the z-axis and $|\psi\rangle$ is an eigen vector of J_z and H_z (see Section 1).

The results in terms of $SO(4)$ strata would have been different: if j is an integer there is only one critical orbit $O(2)_1 \times O(2)_2$ invariant corresponding to $\Delta E = 0$ $\forall \alpha, \beta$ in Eq. (2.4) if j is a half integer there is one closed $O(2)^d$ invariant stratum.

[5] See for example Landau Lifshitz Vol. 3 Quantum Mechanics p. 269.

3. Perturbations of a Robertson Walker Metric

a) Perturbations of the Metric Tensor, Linearized Einstein Equations

Let (W, g) be a Riemannian space. W is an open subset of a four dimensional manifold and $g(x)$, $x \in W$, is the metric tensor.

Let us now perturb slightly the metric tensor and let $(W, g+\delta g)$ be the perturbed Riemannian space.

If $\begin{Bmatrix} l \\ ij \end{Bmatrix}$ and $\begin{Bmatrix} l \\ ij \end{Bmatrix}'$ are the Christoffel symbols of (W, g) and $(W, g+\delta g)$ respectively, it follows

$$\begin{Bmatrix} l \\ ij \end{Bmatrix}' - \begin{Bmatrix} l \\ ij \end{Bmatrix} = Q^l_{ij} \tag{3.1}$$

where Q is a tensor field.

Equation (3.1) is equivalent to

$$\nabla_k(g_{ij}+\delta g_{ij}) \stackrel{\text{def}}{=} \nabla_k(\delta g_{ij}) = (g_{il}+\delta g_{il})\, Q^l_{jk} \tag{3.2}$$

where ∇ is the covariant derivative of the Riemannian space (W, g).

The Riemann and the Ricci tensors of the two spaces are related by

$$\begin{aligned} R'^i_{jkl} - R^i_{jkl} &= \nabla_k Q^i_{jl} - \nabla_l Q^i_{jk} \\ &\quad + Q^h_{lj} Q^i_{hk} - Q^h_{jk} Q^i_{hl} \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} R'_{jk} - R_{jk} &= \nabla_k Q^i_{ji} - \nabla_i Q^i_{jk} \\ &\quad + Q^h_{ji} Q^i_{hk} - Q^h_{jk} Q^i_{hi}\,. \end{aligned} \tag{3.4}$$

Similar formulae relate the tensors we obtain by covariant differentiation of the Riemann tensor.

All these equations involve linear terms in δg and its derivatives as well as higher order terms.

Definition 7. $(W, g+\delta g)$ is a perturbation of (W, g) if in Eqs. (3.2), (3.3), (3.4) and in all the equations we obtain by differentiating Eq. (3.3) for $\forall x \in W$ the higher order terms are much smaller than the linear ones.

According to this definition δg and all its derivatives must be small compared to the corresponding unperturbed quantities.

Thus restricting ourselves to first order terms it follows

$$\nabla_k\, \delta g_{ij} = Q_{kij} \stackrel{\text{def}}{=} g_{kl} Q^l_{ij}\,, \tag{3.5}$$

$$\delta R_{ik} \stackrel{\text{def}}{=} R'_{ik} - R_{ik} = \nabla_k Q^j_{ij} - \nabla_j Q^j_{ik}\,. \tag{3.6}$$

From (3.6) the linearized Einstein equations follow

$$\nabla_k Q^i_{ij} - \nabla_j Q^j_{ik} = -8\pi G(\delta T_{ik} - \tfrac{1}{2} g_{ik}\, \delta T^j_j - \tfrac{1}{2} T^j_j\, \delta g_{ik}) \tag{3.7}$$

where T_{ik} is the momentum energy tensor and δT_{ik} its first variation.

Let us now suppose that the isometry group of (W, g) is a Lie group G.

Then the covariant derivative is invariant in form under the transformations of G.

If we write Eq. (3.6) in the form

$$\mathscr{D}(\delta g) = \delta R \tag{3.8}$$

from (3.5) and (3.6) it follows that $\mathscr{D}$ is a G-scalar operator i.e. it commutes with the isometry transformations of (W, g). Thus if $\gamma \in G$

$$\begin{aligned} \mathscr{D}(\delta g) &= \delta R \\ \mathscr{D}(\delta g') &= \delta R' \\ \delta g' &= \gamma \circ \delta g \end{aligned} \tag{3.9}$$

then

$$\delta R' = \gamma \circ \delta R\,.$$

Thus in discussing Eq. (3.8) it is meaningful to subdivide the infinite dimensional linear space of symmetric tensor fields (to which the tensor fields δg belong) into G-invariant subspaces (possibly irreducible).

The definition or perturbed metric given previously is to a certain extent ambiguous.

Indeed the perturbation δg is determined up to terms

$$\nabla_j \xi_i + \nabla_i \xi_j \tag{3.10}$$

where $x'^i = x^i + \xi^i$ is an infinitesimal coordinate transformation acting on the unperturbed metric which leaves the unperturbed momentum energy tensor invariant [6].

We can use this ambiguity by choosing ξ to simplify the form of $\mathscr{D}$.

For example if the Ricci tensor of (W, g) vanishes (empty space) it is always possible to fix the gauge conditions so that

$$g^{ij} Q^l_{ij} = 0 \tag{3.11}$$

and if in particular (W, g) is flat $\mathscr{D}$ becomes the usual d'Alambertian operator.

In the general case we must use G-covariant gauge conditions such as Eq. (3.11).

b) Finite Robertson Walker Metrics
Decomposition into Irreducible Representations of a Perturbation of a R.W. Metric

A well known class of cosmological models is based on a Robertson Walker metric i.e. [4]

$$ds^2 = -R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right\} + dt^2 \tag{3.12}$$

where $k = \pm 1$, $kr^2 < 1$ and $R(t)$ is an arbitrary function of time which can be interpreted as the radius of the universe. These models describe in a comoving frame a homogeneous and isotropic world with a hydrodynamic energy momentum tensor; the mass density ϱ and the pressure p depend only on time.

[6] For example a linearized plane gravitational wave has only two independent helicity states while δg has ten independent components.

The isometry group of a Robertson Walker metric is the product of the group of space rotations and space "quasi-translations". The latter are defined by

$$\vec{x}' = \vec{x} + \vec{a}\left\{(1 - kx^2)^{\frac{1}{2}} - |1 - (1 - ka^2)^{\frac{1}{2}}|\left(\frac{\vec{x}\cdot\vec{a}}{a^2}\right)\right\} \tag{3.13}$$

where $\vec{x}$ is the set of the three cartesian space-coordinates and $\vec{a}$ is an arbitrary three-vector satisfying

$$ka^2 < 1\,. \tag{3.14}$$

The Lie algebra of the isometry group is equivalent to that of $SO(4)$ for $k=1$ and to that of $SO(3, 1)$ for $k=-1$.

In both cases the minimal invariant submanifolds are three dimensional and coincide with the physical space at a given time.

If $k=1$, from the conditions $kr^2 < 1$ and $ka^2 < 1$ it follows that both the group and the minimal invariant submanifolds are compact i.e. the universe at a given time is finite.

If $k=-1$ the isometry group is not compact and the universe is infinite.

The results of the previous sections can be applied to the compact case and only partially to the non compact one.

The energy momentum tensor of a R.W. metric is

$$T^{ij} = p(t)g^{ij} - (p(t) + \varrho(t))\delta^i{}_0\,\delta^j{}_0\,g^{00} \tag{3.15}$$

where the pressure p and the density ϱ are both positive definite.

If $p \neq 0$ the coordinate transformations commuting with T^{ij} are those and only those which commute with g^{ij}; the gauge is therefore fixed. If $p=0$ (pure dust) they are all the invertible transformations

$$\begin{aligned} \vec{x}' &= \vec{x}'(\vec{x}, t) \\ t' &= t\,. \end{aligned} \tag{3.16}$$

In this case we can impose the gauge condition

$$\begin{gathered} g^{\alpha\beta}\,Q^l_{\alpha\beta} = 0 \\ \alpha, \beta = 1, 2, 3 \end{gathered} \tag{3.17}$$

which is evidently covariant under the isometry group.

Let us consider a perturbation of a given finite R.W. metric. We want to decompose it into components belonging to irreducible representations of the isometry group.

The following result is well known: since $SO(4)$ is compact each function from S^3 into $\mathbb{R}$ where S^3 is the unit sphere in $\mathbb{R}^4$, i.e. each real function of $\vec{x}$ in our case, can be written as [5]

$$f(\vec{x}) = \sum_{\gamma, m} f^m_\gamma\, Y^m_\gamma\,. \tag{3.18}$$

where: $\gamma \in \Gamma$ and Γ is the set of the irreducible inequivalent representations of $SO(4)$; m is an index varying from 1 to n_γ where n_γ is the dimension of the γth representation; f^m_γ are real numbers and Y^m_γ form a basis in the linear space on which the γth representation acts.

Equation (3.18) says that it is possible to write f as a sum of finite dimensional irreducible components.

We want now to decompose the tensor field δg into irreducible representations.

The transformation law of δg under the action of the isometry group is

$$\delta g'_{ij}(\vec{x}', t) = \frac{\partial x^l}{\partial x'^i} \frac{\partial x^m}{\partial x'^i} \delta g_{lm}(\vec{x}, t) \tag{3.19}$$

where $x = (\vec{x}, t)$, $x' = (\vec{x}', t)$ and $\vec{x}' = \vec{x}'(x)$ is a transformation belonging to the isometry group.

Thus it follows that under the action of the isometry group δg_{00}, $\delta g_{0\alpha}$, and $\delta g_{\alpha\beta}$ transform like tensor fields of ranks 0, 1, 2 respectively, $\alpha, \beta = 1, 2, 3$.

In the decomposition of δg_{00} we can use Eq. (3.18).

For $\delta g_{0\alpha}$[7] we must construct a linear space of vector fields such that:

i) it is invariant under $SO(4)$,

ii) the vectors, tangent to the unit sphere in $\mathbb{R}^4$, belong to it,

iii) it is possible to write in a unique way

$$\delta g_{0\alpha} = \sum_p h_{\alpha p} v^p \tag{3.20}$$

where v^p are vector fields which form a basis in the linear space and $h_{\alpha p}$ are functions.

Let x, y, z, w be a set of cartesian coordinates in $\mathbb{R}^4$ and let $\mathbb{i}_x$, $\mathbb{i}_y$, $\mathbb{i}_z$, $\mathbb{i}_w$ be four vector-fields with constant components $(1, 0, 0, 0)$, $(0, 1, 0, 0)$ etc. respectively: $\mathbb{R}^4$ and the linear space of the vector fields $\mathbb{i}$ transform under $SO(4)$ like two irreducible $(\frac{1}{2}, \frac{1}{2})$ representations. Among the components of their tensor product the six-dimensional linear space which transforms like the $(1, 0) \oplus (0, 1)$ representation satisfies the previous requirements (see Appendix).

Now applying Eq. (3.18) to the functions $h_{\alpha p}$ we obtain the decomposition of the vector fields $\delta g_{0\alpha}$ into irreducible components i.e. symbolically[8]

$$\delta g_{0\alpha} = \left[\sum_{ij} a_{ij}(i, j)\right] \otimes [(1, 0) \oplus (0, 1)] \tag{3.21}$$

where a_{ij} are real numbers and (i, j) label the representations into which the functions h have been decomposed.

In an analogous way we have

$$\delta g_{\alpha\beta} = \left[\sum_{ij} b_{ij}(i, j)\right] \otimes (1, 1) \tag{3.22}$$

where $(1, 1)$ denotes the nine-dimensional representation on the linear space of the tensor fields we obtain by tensor product of the $(1, 0)$ and $(0, 1)$ representations given by Eq. (A.3).

Now if we try to apply these results to a $SO(3, 1)$ invariant (i.e. infinite) R.W. metric we find that:

i) finite dimensional representations are no longer unitary and do not exhaust the set of irreducible representations,

[7] See for example (with some changes) [5].

[8] In the decomposition of the tensor product the scalar representation $(0, 0)$ is authomatically excluded.

ii) Eq. (3.18) is no longer true,

iii) the relation between isolated and critical orbits is no longer true since $SO(3, 1)$ is not compact (see Section 1). We can find exceptional isolated orbits which are not critical (light like orbits).

c) Scalar Functionals of the Perturbations

In order to apply the mathematical results of Section 1 to a perturbation of a Robertson Walker metric we will construct two classes of smooth $SO(4)$-invariant functions describing the perturbation and its evolution. It is clear that, since the unperturbed metric is $SO(4)$ invariant, any physical information on the linearized perturbation can be written as a $SO(4)$-invariant function.

Let $\delta g(t_0)$ and $\left.\frac{d\,\delta g}{dt}\right|_{t=t_0}$ be the perturbation and its time derivative at a given initial time $t=t_0$. Then from the linearized Einstein equations (3.7) and the conservation of the momentum energy tensor we know $\delta g(t)\ \forall t>t_0$.

Since the unperturbed metric is $SO(4)$ invariant from

$$\delta g'(t_0)=\gamma\circ\delta g(t_0) \qquad \gamma\in SO(4) \tag{3.23}$$

it follows

$$\delta g'(t)=\gamma\circ\delta g(t) \qquad \forall t>t_0\,.$$

Let $d_1, \ldots, d_{10}$ be the ten algebraicly independent functions we obtain contracting δg and g. We shall call the quantity

$$A_{\bar t}(\delta g)=\sum_{i=1}^{10}\left[\int \sqrt{-g_3}\,(d_i)^2\, d^3x\right]_{t=\bar t} \tag{3.24}$$

the amplitude of the perturbation δg at the time $t=\bar t$ where g_3 is the spatial determinant of the unperturbed metric. $A_t(\delta_g)$ is a $SO(4)$ invariant functional of δg.

If we decompose $\delta g(t)$ into irreducible representations $A_t(\delta g)$ becomes a smooth function of the irreducible components of $\delta g(t)$. Since $\delta g(t)$ is determined by $\delta g(t_0)$ and $\frac{d}{dt}\,\delta g|_{t=t_0}$ i.e. by a vector (which we call X_0 and gives the initial conditions) belonging to an $SO(4)$ reducible representation, we can define a $SO(4)$ invariant function f_t

$$f_t(X_0)=A_t(\delta g)\,. \tag{3.25}$$

Now if we normalize the vector X_0 to one, we can study the stability ($f_t(X_0)$ $\leqq\lambda\ \forall f\ \forall\delta g\ \lambda\in\mathbb{R}$) or the asymptotic stability $\left(\lim_{t\to\bar t} f_t(X_0)=0\ \forall\,\delta g\right)$ of a Robertson Walker metric looking at the behaviour of the functions f_t.

The study of the maxima and the minima of the amplitude of the perturbation obviously determines the stability or the instability of the unperturbed metric.

From the results of the previous sections we know

i) where stationary points must be necessarily found if δg belongs to a given representation (possibly reducible),

ii) that there must be stationary points of the amplitude among the spherically symmetric perturbations.

Another class of physically interesting $SO(4)$ invariant functions is given by the integrals

$$\int \sqrt{-g_3}(\delta\phi_\alpha)^2\, d^3x \tag{3.26}$$

where

ϕ_α is one of the functions obtained by contracting the Riemann tensor, the tensors covariantly derived from it and the products of these tensors with the metric tensor

and $\delta\phi_\alpha$ is its first variation when we perturb (W, g) to $(W, g+\delta g)$. The search of the extremal points of the functionals (3.26) amounts to look for extremal variations of the space mean values of the square of the pressure, the density, the tidal forces etc. according to the physical meaning of the function ϕ.

4. Spinning Gas Cloud

There is a class of hydrodynamical systems [6] whose evolution can be described in term of a time dependent 3×3 matrix F which relates the Euler coordinates $\underline{x}=(x_1, x_2, x_3)$ to the Lagrange coordinates $\underline{a}=(a_1, a_2, a_3)$ i.e.

$$x_i(t) = F_{ij}(t)\, a_j\,. \tag{4.1}$$

These uniform motions of the fluid are completely determined by the knowledge of F as a function of time: the hydrodynamical problem is thus reduced to a mechanical one with, in general nine degrees of freedom.

Under suitable assumptions [6] such systems are invariant under the transformations of a $SO_1(3)\times SO_2(3)$ group with the following infinitesimal generators

$$J = F^t\dot{F} - \dot{F}^t F \tag{4.2}$$

i.e. the components of the angular momentum and

$$K = {}^tF\dot{F} - {}^t\dot{F}F = -{}^tF\zeta F \tag{4.3}$$

where ζ is the vorticity.

The action of $SO_1(3)\times SO_2(3)$ on F is given by (see Section 1c)

$$F' = O_1 F^t O_2 \tag{4.4}$$

$O_1 \in SO_1(3)$, $O_2 \in SO_2(3)$.

The matrix F can be canonically decomposed as

$$F = T D^t S \tag{4.5}$$

where T and S are orthogonal matrices and D is a diagonal one whose elements are the semi-axes of the elipsoidal distribution of matter. (Thus D must be a positive definite matrix.)

In the case of an isothermal spinning gas cloud, Dyson [6] finds the following equation of motion for F

$$\ddot{F} + \partial U/\partial F = 0 \tag{4.6}$$

where U, the internal energy density, is a function of $\det F$. Equation (4.6) can be derived from the following Lagrangian

$$L(F,\dot{F}) = \tfrac{1}{2}\,\mathrm{tr}\,\dot{F}^t\dot{F} - U(F) \tag{4.7}$$

which is explicitly $SO(3)\times SO(3)$ invariant.

The corresponding Hamiltonian is

$$H(F,\dot{F}) = \tfrac{1}{2}\,\mathrm{tr}\,\dot{F}^t\dot{F} + U(F)\,. \tag{4.8}$$

If we use the four dimensional notation (see Section 1 c) and we call Φ the 4×4 traceless symmetric matrix which corresponds to F we can write [6]

$$\begin{aligned} H &= \tfrac{1}{2}\,\mathrm{tr}\,\dot{\Phi}^t\dot{\Phi} + U(\Phi)\\ &= \tfrac{1}{2}\,\mathrm{tr}\,\dot{\Delta}^t\dot{\Delta} + \tfrac{1}{2}\,\mathrm{tr}\,\Gamma^{2\,t}\Lambda^2 + U(\Delta) \end{aligned} \tag{4.9}$$

where Δ is the four dimensional counter party of D.

$\Gamma_{\alpha\beta} = \Gamma_{\alpha\beta}(D)$ is defined by

$$\Gamma_{\alpha\beta} = \begin{vmatrix} 0 & D_2+D_3 & D_3+D_1 & D_1+D_2\\ -D_2-D_3 & 0 & D_1-D_2 & D_1-D_3\\ -D_3-D_1 & D_2-D_1 & 0 & D_2-D_3\\ -D_1-D_2 & D_3-D_1 & D_3-D_2 & 0 \end{vmatrix} \tag{4.10}$$

and

$\Lambda_{\alpha\beta}$ is the six dimensional angular velocity i.e., in the notation of Section 1 e), the matrix m with m_+ and m_- equal to the derivatives of T and S respectively.

The term $\frac{1}{2}\,\mathrm{tr}\,\dot{\Delta}^t\dot{\Delta}$ can be interpreted as the expansion kynetic energy, $\frac{1}{2}\,\mathrm{tr}\,\Gamma^{2\,t}\Lambda^2$ is the kynetic energy due to rotation and vorticity and Λ is the six dimensional angular velocity. This decomposition is $SO(3)\times SO(3)$ invariant.

Since the equation of motion is a second order differential equation, $\Phi(t)$ is uniquely determined by the matrices $\Phi(0)$ and $\dot{\Phi}(0)$. Thus any invariant function of Φ and its derivatives can be written as an invariant function of $\Phi(0)$ and $\dot{\Phi}(0)$ i.e. of a vector belonging to a $(1,1)\oplus(1,1)$ representation of $SO(3)\times SO(3)$.

Equation (4.9) gives the most natural normalization for this problem i.e.

$$H = \tfrac{1}{2}\,\mathrm{tr}\,\dot{\Phi}^t\dot{\Phi} + U(\Phi) = \tfrac{1}{2}\,\langle\dot{\Phi},\dot{\Phi}\rangle + U(\langle\Phi\vee\Phi,\Phi\rangle) = E \tag{4.11}$$

where E is a fixed value of the energy of the system.

From Theorem 3 it follows that each smooth real invariant function defined on the submanifold of $\mathbb{R}^{18}$ $H(\dot{\Phi}_0,\Phi_0)=E$ must have at least two extremal points belonging to the $SO(3)^d$ stratum i.e. to the set of the evolutions of the gas cloud [9] whose initial matter and velocity distributions are spherical symmetric.

If in particular we consider the function $\mathrm{tr}\,\dot{\Delta}^t\dot{\Delta}$ which gives the square of the expansion velocity at time t, from Eq. (4.9) it follows that one of the two points

[9] Since for these evolutions Φ is idempotent the equations of motion become very simple i.e.

$$\ddot{d} = f(d)\,d$$

where d is the radius of the sphere.

must be a maximum since from the assumption of $SO(3)^d$ symmetry we have $\Lambda \equiv 0$ and U in general is a decreasing function of $\det \Phi$.

Let us now consider a class of evolutions more general than those described by Eq. (4.1).

We assume the following relation between Euler and Lagrange coordinates [10]

$$x_i(t) = X_i^0(t) + F_{ij}(t)\,a_j + A_{ijk}(t)\,a_j a_k \tag{4.12}$$

where A is a tensor symmetric in the last two indices, $X_i^0(0)$ and $\dot{X}_i^0(0)$ can be set equal to zero without loss of generality and the third term is much smaller than the second one. This means that Eq. (4.12) is assumed to be valid for all t and for all a such that $|F(t)a| \ll |A(t)aa|$. Up to higher order terms Eq. (4.12) can be inverted in the form

$$a_i = F_{ij}^{-1}(x_j - X_j^0) - F_{ij}^{-1} A_{jkl} F_{kt}^{-1} F_{ls}^{-1}(x_t - X_t^0)(x_s - X_s^0)\,. \tag{4.13}$$

Thus we have

$$\text{i)}\quad U_i = \dot{X}_i^0 + \dot{F}_{ij} a_j + \dot{A}_{ijk} a_j a_k \tag{4.14}$$

where U is the fluid velocity

$$\text{ii)}\quad \frac{\partial}{\partial x_i} = \frac{\partial a_j}{\partial x_i}\,\frac{\partial}{\partial a_j} = (F_{ij}^{-1} - 2F_{il}^{-1} A_{lkm} a_k F_{mj}^{-1})\,\frac{\partial}{\partial a_j} \tag{4.15}$$

$$\begin{aligned}\text{iii)}\quad \operatorname{div} U &= (F_{ij}^{-1} - 2F_{il}^{-1} A_{lkm} a_k F_{mj}^{-1})(\dot{F}_{ji} + 2\dot{A}_{jki} a_k) \\ &\simeq \frac{d}{dt}(\log \det F + 2\,\mathrm{tr}\, F^{-1} A a)\,.\end{aligned} \tag{4.16}$$

From the equation of continuity

$$\dot{\varrho} = -\varrho \operatorname{div} U \tag{4.17}$$

where ϱ is the density we get

$$\begin{aligned}\varrho &= f(a)\det(F^{-1})\,e^{2\,\mathrm{tr}\,F^{-1}Aa} \\ &\simeq f(a)\,\phi^{-1}(1 - 2\,\mathrm{tr}\,F^{-1} A a)\end{aligned} \tag{4.18}$$

where f is an arbitrary positive function and $\phi = \det F$.

From

i) the equation of state

$$p = R\varrho T \tag{4.19}$$

where p is the pressure and T the temperature,

ii) the dependence of the internal energy density upon temperature

$$U = U(T)\,, \tag{4.20}$$

iii) the energy equation

$$\varrho \dot{U} = (P/\varrho)\dot{\varrho}\,, \tag{4.21}$$

[10] This is a special case of an expansion of the Euler coordinates in powers of the Lagrange coordinates in which the first order term is much greater than the following ones.

iv) the assumption that the only dependence of T on a is through $\dot{\varrho}/\varrho$ we deduce:

$$RT \simeq -\phi(1+2\,\mathrm{tr}\,F^{-1}Aa)\frac{dU_0}{d\phi} - 2\phi^2\,\mathrm{tr}\,F^{-1}Aa\frac{d^2U_0}{d\phi^2} \tag{4.22}$$

and

$$U = U[\phi(1+2\,\mathrm{tr}\,F^{-1}Aa)] = U_0(\phi) + U_1\,. \tag{4.23}$$

The Euler equations of motion are

$$\begin{aligned}\ddot{X}_i^0 + \ddot{F}_{ij}a_j + \ddot{A}_{ijk}a_ja_k = {}& -RT(F_{ji}^{-1} - 2F_{jl}^{-1}A_{lkt}a_kF_{ti}^{-1})\cdot \\ &\cdot\left(\frac{1}{f}\frac{\partial f}{\partial a_j} - 2F_{mn}^{-1}A_{mnj}\right).\end{aligned} \tag{4.24}$$

The most general form of f we can assume in Eq. (4.24), possibly redefining the a's, is

$$f \propto e^{-\frac{1}{2}a_ja_j + \frac{1}{2}W_{ijk}a_ia_ja_k} \tag{4.25}$$

where W_{ijk} is a constant small tensor.

Collecting terms according to the powers of a and neglecting terms of second order in A in the case $W \equiv 0$ the equations of motion become

$$\begin{aligned}\ddot{X}_i^0 &= -\phi\frac{dU_0}{d\phi}F_{ji}^{-1}2F_{mn}^{-1}A_{mnj} \\ &= -2\frac{\partial U_0}{\partial F_{ij}}F_{mn}^{-1}A_{mnj}\,,\end{aligned} \tag{4.26}$$

$$\ddot{F}_{ij} = \frac{-\partial U_0}{\partial F_{ij}}\,, \tag{4.27}$$

$$\begin{aligned}\ddot{A}_{ijk} = {}& \frac{\partial U}{\partial F_{it}}(F_{jl}^{-1}A_{lkt} + F_{kl}^{-1}A_{ljt}) \\ &-\left(\frac{\partial U}{\partial F_{ij}} + \phi\frac{\partial^2 U}{\partial F_{ij}\,\partial\phi}\right)(F_{ls}^{-1}A_{lsk}) \\ &-\left(\frac{\partial U}{\partial F_{ik}} + \phi\frac{\partial^2 U}{\partial F_{ik}\,\partial\phi}\right)(F_{ls}^{-1}A_{lsj})\,.\end{aligned} \tag{4.28}$$

From these equations it follows that

i) The equations of motion (4.26, 27, 28) are $SO(3)\times SO(3)$ covariant where we transform X^0, F, and A as vectors of the (1,0), (1,1), and $(1,2)\oplus(1,0)$ representations respectively.

ii) X^0 is a small quantity.

iii) The equation for F is unchanged up to second order terms [see Eq. (4.6)].

iv) The expression of the angular momentum and of the mean vorticity are not changed up to second order terms.

Thus equations (4, 2.3) are still valid and J and K are conserved as it follows from Eq. (4.27).

These statements are valid only if W vanishes since it explicitly breaks the invariance.

This problem represents almost uniform evolutions of (the central part of) a spinning gas cloud. If an inhomogeneous and a (small) quadratic term [see Eq. (4.12)] are added to Eq. (4.1) the divergence of the velocity and thus $\dot{\varrho}/\varrho$ and, through assumption (iv), T and U differ from the corresponding uniform quantities by a small linear term in x (up to second order corrections). If the function $f(a)$ of Eq. (4.18) is assumed to be isotropic (i.e. $W=0$) the gas is still $SO(3)\times SO(3)$ invariant (if second order terms are neglected) since all the linear terms in x average to zero. It is thus possible to analyze small departures from a uniform evolution in the frame work of the $SO(3)\times SO(3)$ invariance.

Any invariant function of $\dot{X}^0(t)$, $F(t)$, and $A(t)$ and their derivatives can be written as an invariant function of the initial data $F(0)$, $\dot{F}(0)$, $A(0)\, e\, \dot{A}(0)$ i.e. as a function of a vector belonging to a $(1,1)\oplus(1,1)\oplus(1,2)\oplus(1,2)\oplus(1,0)\oplus(1,0)$ representation of $SO(3)\times SO(3)$ [satisfying the restriction D positive definite see Eq. (4.5)]. Since no nontrivial vector belonging to a $(1,2)\oplus(1,0)$ representation can be $SO(3)^d$ invariant, it follows that if we normalize to an $n-1$ dimensional submanifold, n being the dimension of the representation, (e.g. $\langle F(0), F(0)\rangle + \langle \dot{F}(0), \dot{F}(0)\rangle + \langle A(0), A(0)\rangle + \langle \dot{A}(0), \dot{A}(0)\rangle = 1$) there is only one closed stratum. It is the $SO(3)^d$ invariant stratum with $A=\dot{A}=0$ ($\Rightarrow X^0(t)\equiv 0$ and $A(t)\equiv 0\,\forall t$). Thus among the linear spherical-symmetric evolutions there are at least two extremal points of any normalized invariant function. One of these functions is the mean energy of the gas (second order terms included).

On the other hand if we require that the non linear perturbation does not vanish i.e. if we normalize to a $n-2$ submanifold [such as for example $\langle F(0), F(0)\rangle + \langle \dot{F}(0), \dot{F}(0)\rangle = \alpha$ $\langle A(0), A(0)\rangle + \langle \dot{A}(0), \dot{A}(0)\rangle' = \beta$] the stratum with the largest invariance is the $O(2)^d$ one. Since it is closed there must be among the axial evolutions at least two extremal points for each invariant function. One of these functions, for example, can be the mean value of the square at time t of the quadratic correction to $x(t)$, proportional to $\langle A(t), A(t)\rangle$, or the square of the inhomogeneous correction $\langle X^0(t), \dot{X}^0(t)\rangle$. We can use these functions to judge the stability of the linear evolutions (see Section 3).

If instead the function f in Eq. (4.18) is not assumed isotropic the system is no longer $SO(3)\times SO(3)$ symmetric since the distribution of matter is asymmetric also in the limit $A(0)=\dot{A}(0)=0$.

Nevertheless we can look for extremal breaking configurations by constructing the following class of invariant functions:

a) write the complete equations of motion with W different from zero [they are not $SO(3)\times SO(3)$ invariant],

b) define a mean solution by the average of the initial conditions over a given compact $SO(3)\times SO(3)$ invariant manifold [e.g. $\langle F(0), F(0)\rangle = \alpha$ $\langle \dot{F}(0), \dot{F}(0)\rangle = \beta$ $\langle A(0), A(0)\rangle = \gamma$ $\langle \dot{A}(0), \dot{A}(0)\rangle = \delta$],

c) write a scalar function of this mean solution $\bar{X}$, $\bar{F}$, and $\bar{A}$ at a given time t and integrate it over a $SO(3)\times SO(3)$ invariant domain in the a-space (e.g. for all $a\leqq\bar{a}: |F\bar{a}| \gg |A\bar{a}\bar{a}|$).

The function thus constructed is a $SO(3)\times SO(3)$ invariant function which can be used to weight the effect of the breaking term W [11].

W transforms as a $(0,3)\oplus(0,1)$ representation. If we normalize to $\langle W, W\rangle = 1$ the only closed stratum is the $SO(3)_1\times O(2)_2$ invariant one.

Thus extremal breaking effects are found when $f(a)$ is axisymmetrical.

Appendix

Let us introduce in $\mathbb{R}^4$ the polar coordinates $\varrho\,\theta_1\,\theta_2\,\theta_3$ [in the notation of Eq. (2.1) $\phi=\theta_3, \theta=\theta_2, r=\varrho\cos\theta_1$].

We have

$$\begin{aligned} x &= \varrho\cos\theta_1\cos\theta_2\cos\theta_3\\ y &= \varrho\cos\theta_1\cos\theta_2\sin\theta_3\\ z &= \varrho\cos\theta_1\sin\theta_2\\ w &= \varrho\sin\theta_1\,. \end{aligned} \tag{A.1}$$

The tangent vectors to the unit sphere of $\mathbb{R}^4$ are [12]

$$\begin{aligned} e_1 &= (-\sin\theta_1\cos\theta_2\cos\theta_3,\ -\sin\theta_1\cos\theta_2\sin\theta_3,\ -\sin\theta_1\sin\theta_2, \cos\theta_1)\\ e_2 &= (-\sin\theta_2\cos\theta_3,\ -\sin\theta_2\sin\theta_3, \cos\theta_2, 0)\\ e_3 &= (-\sin\theta_3, \cos\theta_3, 0, 0)\,. \end{aligned} \tag{A.2}$$

A basis for the $(1,0)\oplus(0,1)$ representation linear space is

$$\begin{aligned} v_i^{10} &= (\varepsilon_{ijk}-\delta_{ik}\delta_{j0}\pm\delta_{i0}\delta_{jk})x_j\,\mathbb{1}_k\\ v_i^{01} &= (\varepsilon_{ijk}+\delta_{ik}\delta_{j0}\mp\delta_{i0}\delta_{jk})x_j\,\mathbb{1}_k\\ x_j &= x, y, z \qquad x_0 = w\,. \end{aligned} \tag{A.3}$$

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[11] To illustrate the meaning of these functions let us consider the following simple mechanical system

$$m\ddot{x} = a - k\dot{x} \quad \text{where} \quad -k\dot{x} \quad \text{is the breaking term}\,.$$

The unperturbed solutions are

$$x(t) = \frac{1}{2}\,\frac{a}{m}\,t^2 + v_0 t + x_0$$

the mean unperturbed solution $v_0\in[-1,1]$, $x_0\in[-1,1]$ is

$$\bar{x}(t) = \frac{1}{2}\,\frac{a}{m}\,t^2\,.$$

The perturbed solutions are

$$x_p(t) = \frac{a}{k}\,t + \left(\frac{ma}{k^2} - \frac{m}{k}\,v_0\right)(e^{-k/mt}-1) + x_0$$

the mean solution is

$$\bar{x}_p(t) = \frac{a}{k}\,t + \frac{ma}{k^2}(e^{-k/mt}-1)$$

if we choose τ so that $k\tau/m \gg 1$

$$\bar{x}_p(\tau)/\bar{x}(\tau) = m/k\tau \quad \text{very far from one}\,.$$

Thus the "breaking" effect becomes larger and larger.

[12] The linear space generated by e_1, e_2, and e_3 is *not* invariant under $SO(4)$.

Then we can write

$$\begin{aligned}
v_1^{10} + v_1^{01} &= \cos\theta_1 \sin\theta_3\, e_2 - \cos\theta_1 \sin\theta_2 \cos\theta_3\, e_3 \\
v_1^{10} - v_1^{01} &= \cos\theta_2 \cos\theta_3\, e_1 + \sin\theta_1 \sin\theta_2 \cos\theta_3\, e_2 + \sin\theta_1 \sin\theta_3\, e_3 \\
v_2^{10} + v_2^{01} &= -\cos\theta_1 \cos\theta_3\, e_2 - \cos\theta_1 \sin\theta_2 \sin\theta_3\, e_3 \\
v_2^{10} - v_2^{01} &= \cos\theta_2 \sin\theta_3\, e_1 + \sin\theta_1 \sin\theta_2 \sin\theta_3\, e_2 - \sin\theta_1 \cos\theta_3\, e_3 \\
v_3^{10} + v_3^{01} &= \cos\theta_1 \cos\theta_2\, e_3 \\
v_3^{10} - v_3^{01} &= \sin\theta_2\, e_1 - \sin\theta_1 \cos\theta_2\, e_2 \,.
\end{aligned} \tag{A.4}$$

Since at every point the linearly independent vectors are only three, the decomposition given by Eq. (3.20) is unique.

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Francesco Pegoraro
Massachusetts Institute
of Technology
Room 26-213
Research Laboratory of Electronics
Cambridge, Mass. 02139, USA