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On the Landau Diamagnetism

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Abstract. The grand-canonical partition function of an assembly of free spinless electrons in a magnetic field enclosed in a box (Dirichlet boundary conditions) is shown to be an entire function of the fugacity z and the magnetic field H, as a consequence of the trace-norm convergence of the perturbation series for the statistical semigroup. This allows to derive analyticity properties of the pressure as a function of z and H, and to express the coefficients of its power series expansion around $z=H=0$ by means of the unperturbed semigroup. Hence, the magnetic susceptibility at zero field and fixed density is expressed in terms of Green functions of the heat equation. Its asymptotic expansion for $\Lambda \to \infty$ (Fisher) along parallelepipedic domains is obtained up to $0\left(\frac{S(\Lambda)}{V(\Lambda)}\right)$. The volume term of this expansion is the Landau diamagnetism.

1. Introduction

This paper is concerned with the diamagnetic susceptibility at thermal equilibrium and zero magnetic field of an assembly of free spinless electrons in a box. The problem originates at L. D. Landau [1] in 1930, who gave a treatment in the framework of quantum statistical mechanics, in which, however, the influence of the walls of the container has been considered approximately by a semiclassical argument. His result is different from that obtained in classical statistical mechanics [2], and this gave rise to a debate on the influence of the walls (which is in fact responsible for the null magnetic moment classically obtained). This debate is still alive, because of the many contradictory results obtained. Such a situation is due either to employing approximation procedures hard to control, or to replacing the original problem with a soluble one whose connection with the former is difficult to judge. We shall therefore consider once again the original quantum statistical problem with the proper mathematical rigor.

We shall use grand-canonical quantum statistical mechanics, in which the whole information about the system is contained in the partition function:

$$\Xi_{\Lambda}(\beta, z, \omega) = \sum_{n=0}^{\infty} z^n \operatorname{tr} \exp[-\beta H_{n,\Lambda}(\omega)] \tag{1.1}$$

where $\omega = \frac{e}{c} H$ is the cyclotronic frequency in the magnetic field H, $H_{n,\Lambda}(\omega)$ is the selfadjoint n-particle Hamiltonian for electrons enclosed in a bounded domain $\Lambda \subset \mathbb{R}^3$ in the presence of the magnetic field and the trace is to be taken according with Fermi statistics. That is, for independent electrons:

$$\Xi_\Lambda(\beta, z, \omega) = \prod_i [1 + z \exp(-\beta E_i(\omega)] \tag{1.1'}$$

where $E_i(\omega)$ are the one-particle eigenenergies.

The thermodynamic equation of state can be obtained from (1.1') if an asymptotic expansion of the following kind can be proved:

$$\begin{aligned} P_\Lambda(\beta, z, \omega) &\equiv \frac{1}{\beta V(\Lambda)} \log \Xi_\Lambda(\beta, z, w) \\ &= P^{(0)}(\beta, z, \omega) + \frac{S(\Lambda)}{V(\Lambda)} P^{(1)}(\beta, z, \omega) + \mathrm{o}\left(\frac{S(\Lambda)}{V(\Lambda)}\right) \end{aligned} \tag{1.2}$$

where $V(\Lambda)$ is the volume Λ and $S(\Lambda)$ the area of $\partial\Lambda$[1]. This amounts to proving the existence of the following two limits:

$$\lim \frac{1}{V(\Lambda)} \log \Xi_\Lambda(\beta, z, \omega) = \beta P^{(0)}(\beta, z, \omega)\,, \tag{1.3}$$

$$\lim \frac{1}{S(\Lambda)} [\log \Xi_\Lambda(\beta, z, \omega) - V(\Lambda)\beta P^{(0)}(\beta, z, \omega)] = \beta P^{(1)}(\beta, z, \omega)\,, \tag{1.4}$$

when $\Lambda \to \infty$ in a suitable manner [3]. In (1.2), $P^{(0)}$ is the infinite volume pressure, while the second term gives the surfaces correction to the thermodynamic equation of state.

Although the fugacity z is the natural external parameter in the grand-canonical statistical mechanics (characterizing the "particle reservoir"), however, in most physical problems, the density is the external parameter generally used. To obtain the equation of state involving the density, one has to invert the density-fugacity relation:

$$\varrho = \beta z \frac{\partial}{\partial z} P_\Lambda(\beta, z, \omega) \tag{1.5}$$

obtaining:

$$z = g_\Lambda(\beta, \varrho, \omega) \tag{1.6}$$

and make the asymptotic expansion:

$$\begin{aligned} p_\Lambda(\beta, \varrho, \omega) &\equiv P_\Lambda(\beta, g_\Lambda(\beta, \varrho, \omega), \omega) \\ &= p^{(0)}(\beta, \varrho, \omega) + \frac{S(\Lambda)}{V(\Lambda)} p^{(1)}(\beta, \varrho, \omega) + \mathrm{o}\left(\frac{S(\Lambda)}{V(\Lambda)}\right). \end{aligned} \tag{1.7}$$

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[1] Actually, the anisotropy introduced by the external magnetic field should be reflected in the structure of the second term in (1.2) and this is in fact the case, as can be seen from the results in Sections 5 and 6. However, for shortness, we shall maintain this simplified S/V notation throughout this section.

The magnetic susceptibility at constant density and zero magnetic field is given by:

$$\chi_\Lambda(\beta, \varrho) = \left(\frac{e}{c}\right)^2 \cdot \frac{\partial^2}{\partial \omega^2}\left[p_\Lambda(\beta, \varrho, \omega) - \frac{\varrho}{\beta}\log g_\Lambda(\beta, \varrho, \omega)\right]\Bigg|_{\omega=0} \tag{1.8}$$

It will appear convenient to study first:

$$\chi_\Lambda(\beta, z) = \left(\frac{e}{c}\right)^2 \frac{\partial^2}{\partial \omega^2} P_\Lambda(\beta, z, \omega)\Bigg|_{\omega=0} \tag{1.8'}$$

which will be named magnetic susceptibility at zero field and fixed fugacity. The relation between the two turns out to be simply:

$$\chi_\Lambda(\beta, \varrho) = X_\Lambda(\beta, g_\Lambda(\beta, \varrho, 0)). \tag{1.9}$$

If $\chi_\Lambda(\beta, \varrho)$ is to be a thermodynamic quantity, then again an asymptotic expansion:

$$\chi_\Lambda(\beta, \varrho) = \chi^{(0)}(\beta, \varrho) + \frac{S(\Lambda)}{V(\Lambda)}\chi^{(1)}(\beta, \varrho) + o\left(\frac{S(\Lambda)}{V(\Lambda)}\right) \tag{1.10}$$

is to be proved. This proof (in the case when Λ are parallelepipedic domains), is the content of this paper, whith the result that $\chi^{(0)}$ is the Landau value[2]:

$$\chi^{(0)}(\beta, \varrho) = -\frac{1}{(2\pi)^{\frac{3}{2}} \cdot 12\sqrt{\beta}} \cdot f_{\frac{1}{2}}(z) \tag{1.11}$$

where z is obtained through the inversion [4] of:

$$\varrho = \frac{1}{(2\pi\beta)^{\frac{3}{2}}} \cdot f_{\frac{3}{2}}(z). \tag{1.11'}$$

Here $f_\sigma(z)$ are the well-known Fermi functions, defined for $|z| < 1$ by:

$$f_\sigma(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\sigma} \cdot z^n. \tag{1.12}$$

The question naturally arises, whether one can obtain the Expansion (1.10) by inserting the Expansion (1.7) and the corresponding expansion for g_Λ into Eq. (1.8) and taking term by term derivatives. Summarizing, for the first ("volume") terms, we have the diagram[3]:

$$\begin{array}{ccc}
\begin{array}{c} p_\Lambda(\beta, \varrho, \omega) \\ g_\Lambda(\beta, \varrho, \omega) \end{array} & \overset{(\Lambda \to \infty)}{\dashrightarrow} & \begin{array}{c} p^{(0)}(\beta, \varrho, \omega) \\ g^{(0)}(\beta, \varrho, \omega) \end{array} \\
\Big\downarrow \left.\frac{\partial^2}{\partial\omega^2}\right|_{\omega=0} & & \Big\downarrow \left.\frac{\partial^2}{\partial\omega^2}\right|_{\omega=0} \\
\chi_\Lambda(\beta, \varrho) & \xrightarrow{(\Lambda \to \infty)} & \chi^{(0)}(\beta, \varrho) \overset{?}{=} \chi_{\mathrm{Landau}}(\beta, \varrho)
\end{array} \tag{1.13}$$

[2] Actually Landau's value is twice greater due to his considering the spin degeneracy.

[3] The dotted line means "not yet firmly established". Indeed, no proof has been given thus far of the existence of the Expansion (1.7) for $\omega \neq 0$. We shall consider this problem in a subsequent paper [32].

At this point, we shall briefly survey the argumentation lines already used in connection with this subject.

The original derivation of Landau follows essentially the upper route in (1.13). As a matter of fact, Landau calculates the spectrum of the one-particle Hamiltonian for the whole space ("bulk electrons"), gives a prescription of counting states and restricts conveniently the summation over quantum numbers in the expression of the pressure to account for the finite size. His argument is that an overwhelming fraction of the electrons do not feel the surface (in a semiclassical picture, they move along helical "trajectories" which do not touch the walls). The contribution of the "surface electrons" to the pressure is therefore small.

A method which avoids considering energy levels individually was subsequently deviced by Sondheimer and Wilson [5], who used a Green function technique for calculating traces. However, their approximation of replacing the Green function appropriate to the specific boundary conditions by the Green function for the whole space can be easily seen to be identical to that of Landau.

One step farther on this way is contained in a recent paper by Ohtaka and Moriya [6]. They write down the perturbation series for the Green function in a half-space for magnetic field parallel to the surface, starting from the Green function in a half-space at zero magnetic field and then essentially follow the argument in [5]. They can write down in this way a surface correction for the susceptibility. However, neither a proof of convergence of the perturbation theory, nor a justification of the half-space approximation are given.

One thing has been realized very soon after Landau's paper [7]. Namely, when following the lower route in (1.13), that is when calculating the susceptibility from:

$$\chi_\Lambda(\beta,\varrho) = -\frac{1}{V(\Lambda)}\left(\frac{e}{c}\right)^2 \cdot \sum_i \frac{\partial^2 E_i(\omega)}{\partial\omega^2} \cdot \frac{z e^{-\beta E_i(\omega)}}{1+z e^{-\beta E_i(\omega)}} \tag{1.14}$$

where $z = g_\Lambda(\beta, \varrho, 0)$, and when making in this formula the same approximations as Landau did, one obtains a much stronger diamagnetism. This is so, because "surface electrons", though "few", behave very strongly paramagnetically. Teller and van Vleck [7] argued that this compensation finally gives the Landau value. Many attempts have thence been done to substantiate these arguments by explicitely considering the boundary conditions [8–15]. The results have been however disappointedly contradictory and the reason for such a variety remained obscure. Some of these results show pathological behaviour of the susceptibility either as a function of the magnetic field ($\chi_\Lambda \sim \omega^{-\frac{4}{3}}$ at $\omega = 0$) [10, 11] III, [12], or in the thermodynamic limit [11] IV, [13]. The origin of the first pathology seems to be the semiclassical approximation for the energy levels [15], and does not appear if these are exactly calculated in perturbation theory up to ω^2 [11] IV. The origin of the second pathology seems to be the approximate summation in (1.14), in which cancellations of large terms may appear [14]. We maintain that one can decide on this question only by starting with an actually finite system (no half-space slab approximation [6, 14]) and taking the thermodynamic limit.

A way to escape the difficulty has been to replace the hard walls by a harmonic well potential, which gives an exactly soluble problem [16–19]. Our opinion is that the results obtained in this way are of little relevance for the original problem.

The main idea of the proof we shall present here of Eq. (1.10) is to use the integral kernel of the operator $\exp[-\beta H_{1,\Lambda}(\omega)]$ (Dirichlet boundary conditions) for expressing $P_\Lambda(\beta, z, \omega)$ (at least for sufficiently small $|z|$, $|\omega|$). Two advantages result:

First, by the perturbation theory for semigroups [20–22], and the theory of infinite determinants [23], one obtains analyticity properties of P_Λ with respect to z and ω, which rule out the first pathology mentioned above and allow one to reach the physically interesting values of z by analytic continuation (Section 2).

Second, as was made clear in recent work by Nenciu [24], by this device the influence of the boundary conditions is taken into account globally (in the sense that the exact summation of the correct energy levels is automatically performed by integrating the Green function) and, moreover, in a way which allows to make estimations proving the asymptotic expansions like (1.9). (Section 5).

It turns out that the contributions of the first and second orders of the perturbation theory to X_Λ have separately bad asymptotic behaviours, which however exactly compensate each other. The identity for the Green function relevant for this cancellation is proved in Section 4.

In Section 3 and 6 the fugacity-density relation is considered and the consequences for the fixed-density susceptibility stated.

2. The Finite-Volume Magnetic Susceptibility at Zero Field and Fixed Fugacity

In this section the perturbation series for the statistical semigroup will be shown to converge in trace-norm. Analyticity properties of the pressure as a function of z and ω follow, which give sense to $X_\Lambda(\beta, z)$ defined by Eq. (1.8′) as an analytic function of z in $\mathbb{C}\setminus(-\infty, -1]$. The power series expansion of this function around $z=0$ is written down.

As already stated, we are considering free spinless electrons in a parallelepipedic box, $\Lambda = \Lambda(a_1, a_2, a_3)$, $a_1, a_2, a_3 > 0$, where:

$$\Lambda(a_1, a_2, a_3) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid |x_i| \leqq a_i, \quad i = 1, 2, 3\}. \tag{2.1}$$

The magnetic field H is taken along the x_3-axis. The Landau gauge $A = (-x_2 H, 0, 0)$ and units such that $m = \hbar = 1$ will be used throughout.

The one-particle Hamiltonian $H(\omega)$ is defined as an operator in $L_2(\Lambda)$ by:

$$H(\omega) = H_0 + \omega H_1 + \omega^2 H_2\,; \qquad \mathscr{D}(H(\omega)) = \mathscr{D}(H_0) \tag{2.2}$$

where H_0 is $-\frac{1}{2}\Delta$ supplemented with Dirichlet boundary conditions (vanishing of the wave functions on $\partial\Lambda$, i.e. infinite well potential)[4], H_1 is the differential operator $-ix_2\frac{\partial}{\partial x_1}$ with $\mathscr{D}(H_1)=\mathscr{D}(H_0)$, while H_2 is the multiplication by $\frac{1}{2}x_2{}^2$. We shall allow arbitrary complex values of ω in (2.2).

Proposition 1. (i) $-H(\omega)$ *is, for all* $\omega\in\mathbb{C}$, *the infinitesimal generator of a* (C_0)*-semigroup of trace-class oparetors,* $\{S_\omega(t)\}_{t>0}$. *For every* $t>0$, $S_\omega(t)$ *is a trace-norm entire function of* ω, *which is given by the trace-norm convergent series:*

$$S_\omega(t)=\sum_{m=0}^{\infty}(-1)^m S_\omega^{(m)}(t) \tag{2.3}$$

where:

$$S_\omega^{(0)}(t)=S_0(t);\qquad S_\omega^{(m)}(t)=\int_0^t S_0(t-\tau)\,[\omega H_1+\omega^2 H_2]\,S_\omega^{(m-1)}(\tau)d\tau \tag{2.4}$$

with trace-norm Bôchner integral in (2.4).

(ii) *For* $\omega\in\mathbb{R}$, $H(\omega)$ *is selfadjoint and positive, therefore* $0\leqq S_\omega(t)\leqq 1$.

Proof. Let us first remark that $S_0(t)$ is a selfadjoint (C_0) semigroup of finite trace operators (Gibbs semigroup). Indeed, H_0 is selfadjoint and has the following complete orthonormal set of eigenvectors:

$$\begin{aligned} e_{k_1,k_2,k_3}&(x_1,x_2,x_3)\\ &=\prod_{i=1}^{3}a_i^{-\frac{1}{2}}\sin\frac{k_i\pi}{2a_i}(x_i+a_i);\qquad k_1,k_2,k_3=1,2,\dots. \end{aligned} \tag{2.5}$$

Therefore $S_0(t)=\exp[-tH_0]$ has the eigenvalues $\exp\left[-\frac{t}{2}\sum_{i=1}^{3}\left(\frac{k_i\pi}{2a_i}\right)^2\right]$ and thus is manifestly of finite trace. In fact, $S_0(t)$, $(t>0)$ is an integral operator in $L_2(\Lambda)$ with positive C^∞ kernel [28]:

$$S_0(t)(x_1,x_2,x_3;x_1',x_2',x_3')=\prod_{i=1}^{3}G_0^{a_i}(t;x_i,x_i') \tag{2.6}$$

where

$$\begin{aligned} G_0^a&(t;x,x')\\ &=\sum_{k=1}^{\infty}\exp\left[-\frac{t}{2}\left(\frac{k\pi}{2a}\right)^2\right]\cdot a^{-\frac{1}{2}}\sin\frac{k\pi}{2a}(x+a)\cdot a^{-\frac{1}{2}}\sin\frac{k\pi}{2a}(x'+a) && \text{(2.7a)}\\ &=(2\pi t)^{-\frac{1}{2}}\sum_{m\in\mathbb{Z}}\left\{\exp\left[-\frac{(4ma+x-x')^2}{2t}\right]-\exp\left[-\frac{(4ma+2a-x-x')^2}{2t}\right]\right\} && \text{(2.7b)} \end{aligned}$$

[4] Let us remark that among the usual boundary conditions employed in statistical mechanics (i.e periodic, Dirichlet and $\frac{\partial\Psi}{\partial n}=\sigma\Psi$) [25–27], the Dirichlet conditions are the only which assure the gauge invariance of the Hamiltonian (by a straightforward adaptation of the usual proof for infinite space).

is the one-dimensional Green function of the heat equation with Dirichlet boundary conditions on $[-a, a]$.

Now, in view of the Corollary in [22], Part (i) of the proposition will be proved simply by checking that $\omega H_1 + \omega^2 H_2$ is H_0-bounded [29] and $\sup_{\omega\in K} \|(\omega H_1 + \omega^2 H_2) S_0(t)\|$ is integrable at $t=0$ for all compact sets $K \subset \mathbb{C}$. But clearly H_1 is symmetric and H_2 is bounded and selfadjoint, therefore $(\omega H_1 + \omega^2 H_2) R(\lambda; H_0)$ is bounded for all λ in the resolvent set of H_0, via the closed graph theorem. To estimate $\|H_1 S_0(t)\|$, one can make use of (2.5) and the remark that $\left\{\frac{\partial}{\partial x_1} e_{k_1,k_2,k_3}\right\}_{k_1,k_2,k_3=1,2,\dots}$ is still an orthogonal set. Then, for an arbitrary $\Psi = \sum_{k_1,k_2,k_3=1}^{\infty} \Psi_{k_1,k_2,k_3} e_{k_1,k_2,k_3} \in L_2$:

$$\begin{aligned}\|H_1 S_0(t)\Psi\|^2 &\leqq a_2^2 \sum_{k_1,k_2,k_3=1}^{\infty} |\Psi_{k_1,k_2,k_3}|^2 \left(\frac{k_1\pi}{2a_1}\right)^2 \cdot \exp\left[-t\sum_{i=1}^{3}\left(\frac{k_i\pi}{2a_i}\right)^2\right] \\ &\leqq a_2^2 \sup_{k_1\geqq 1}\left(\frac{k_1\pi}{2a_1}\right)^2 \exp\left[-t\left(\frac{k_1\pi}{2a_1}\right)^2\right] \cdot \sum_{k_1,k_2,k_3=1}^{\infty} |\Psi_{k_1,k_2,k_3}|^2 \\ &\leqq a_2^2 (et)^{-1} \|\Psi\|^2 .\end{aligned}$$

Therefore $\|H_1 S_0(t)\| \leqq a_2 (et)^{-\frac{1}{2}}$ which is integrable at $t=0$.

For $\omega \in \mathbb{R}$, $S_\omega(t)$ is selfadjoint, because every term in (2.3) is, therefore $H(\omega)$ is selfadjoint. Its positivity follows by an integration by parts from the fact that on $\mathscr{D}(H_0)$, $H(\omega)$ can be written as:

$$\frac{1}{2}\left(-i\,\mathrm{grad} - \frac{e}{c}\vec{A}\right)^2 \tag{2.8}$$

where $-i\,\mathrm{grad} - \frac{e}{c} A$ applies $\mathscr{D}(H_0)$ into $L_2(\Lambda) \oplus L_2(\Lambda) \oplus L_2(\Lambda)$. This completes the proof.

We shall next consider the grand-partition function defined by (1.1′). The infinite product converges absolutely for all $\omega \in \mathbb{R}$ and z in view of $S_\omega(\beta)$, $\beta > 0$, being trace-class operators. Because the analyticity properties of the energy levels $E_i(\omega)$ are hard to obtain in a general setting [29] and seem not to be known in the special case considered here, the infinite product representation cannot be used in obtaining analyticity properties for complex ω. However, these properties can be obtained using the theory of infinite determinants and Proposition 1.

If T is a trace-class operator and -1 is in the resolvent set of T, then $\log(1+T)$ is a trace-class operator [23]. For arbitrary trace-class T the determinant of $1+T$ is defined by:

$$\det(1+T) = \begin{cases} \exp[\mathrm{tr}\log(1+T)] & \text{for } -1 \text{ in the resolvent set of } T \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

For T selfadjoint, it can be easily seen that:

$$\det(1+T) = \prod_i (1+t_i) \tag{2.10}$$

where t_i are the eigenvalues of T. This remark and the selfadjointness of $S_\omega(\beta)$ for $\omega \in \mathbb{R}$ show that:

$$\Xi_\Lambda(\beta, z, \omega) = \det[1 + zS_\omega(\beta)], \quad \omega \in \mathbb{R}, \quad \beta > 0. \tag{2.11}$$

Our argument rests heavily on the following:

Theorem 1. [23]. a) *For every trace-class operator T:*

$$|\det(1+T)| \leqq \exp(\|T\|_1)$$

b) *If* $\|T\|_1 < 1$, *then:*

$$\det(1+T) = \exp\left[\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \operatorname{tr} T^n\right]$$

c) *If* $T(\zeta)$ *is an analytic function of* ζ *in a domain U with values in the Banach space of trace-class operators, then* $\det(1 + T(\zeta))$ *is an analytic function in U.*

One immediately concludes that $\Xi_\Lambda(\beta, z, \omega)$ is an entire function of two complex variables z, ω. Indeed, analyticity in one variable the other being held fixed follows from Proposition 1 (i) and Theorem 1 c). This implies joint analyticity via Hartog's theorem [30]. Moreover, in view of Proposition 1 (ii), -1 is in the resolvent set of $zS_\omega(\beta)$ for all $\omega \in \mathbb{R}$ and $z \in \mathbb{C} \setminus (-\infty, -1]$, therefore $\Xi_\Lambda(\beta, z, \omega)$ has no zeroes there. By a standard continuity argument, it follows that for every compact $K \subset \mathbb{C} \setminus (-\infty, -1]$ there is a neighbourhood V of the real axis, such that $\Xi_\Lambda(\beta, z, \omega) \neq 0$ for $(z, \omega) \in K \times V$. This in turn implies the analyticity of $\log \Xi_\Lambda(\beta, z, \omega)$ on $K \times V$, leading to:

Proposition 2. *Let* $\beta > 0$. *Then:*

(i) *For every compact set* $K \subset \mathbb{C} \setminus (-\infty, -1]$ *there exists a neighbourhood V of the real axis such that* $P_\Lambda(\beta, z, \omega)$ *is analytic on* $K \times V$. *In a neighbourhood of* $z = \omega = 0$:

$$P_\Lambda(\beta, z, \omega) = \frac{1}{\beta V(\Lambda)} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n} \operatorname{tr} S_\omega(n\beta). \tag{2.12}$$

(ii) *The magnetization at fixed fugacity and zero field*

$$\left.\frac{e}{c} \frac{\partial P_\Lambda}{\partial \omega}(\beta, z, \omega)\right|_{\omega=0}$$

vanishes identically for $z \in \mathbb{C} \setminus (-\infty, -1]$.

(iii) $X_\Lambda(\beta, z) = \left(\frac{e}{c}\right)^2 \left.\frac{\partial^2 P_\Lambda}{\partial \omega^2}(\beta, z, \omega)\right|_{\omega=0}$ *exists for all* $z \in \mathbb{C} \setminus (-\infty, -1]$ *and is an analytic function of z in this domain. For* $|z| < 1$:

$$X_\Lambda(\beta, z) = \left(\frac{e}{c}\right)^2 \frac{2}{\beta V(\Lambda)} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n} \mathscr{F}_\Lambda^{(2)}(n\beta) \tag{2.13}$$

where:

$$\mathscr{F}_\Lambda^{(2)}(t) = \operatorname{tr}\left[-tH_2 S_0(t) + \int_0^t d\tau (t-\tau) H_1 S_0(\tau) H_1 S_0(t-\tau)\right]. \tag{2.14}$$

Proof. Equation (2.12) follows from Theorem 1b). To Prove (ii), one has to show only that $\mathrm{tr}\, S_\omega(t) = \mathrm{tr}\, S_{-\omega}(t)$ for $\omega \in \mathbb{R}$, in view of (2.12) and the analyticity stated in (i). To this aim, denote $\mathcal{J}$ the involution of $L_2(\Lambda)$ given by complex conjugation. Then:

$$\mathrm{tr}\, S_\omega(t) = \mathrm{tr}\,[\mathcal{J}^2 S_\omega(t)] = \mathrm{tr}\,[\mathcal{J} S_\omega(t)\mathcal{J}] = \mathrm{tr}\, S_{-\omega}(t) \tag{2.15}$$

where the last equality follows from $\mathcal{J} H_1 \mathcal{J} = -H_1$ which is clear from definition. The existence and analyticity of the second derivative $\left.\frac{\partial^2 P_\Lambda}{\partial \omega^2}\right|_{\omega=0}$ follow from (i). The second derivative at $\omega = 0$ and small $|z|$ is therefore given by the coefficient of ω^2 in the series expansion of (2.12) around $\omega = 0$. Taking account of Eqs. (2.3),(2.4), this gives:

$$\mathcal{F}_\Lambda^{(2)}(t) = \mathrm{tr}\Bigg[-\int_0^t d\tau S_0(t-\tau) H_2 S_0(\tau) + \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 S_0(t-\tau_1) H_1 S_0(\tau_1-\tau_2) H_1 S_0(\tau_2)\Bigg]. \tag{2.16}$$

Then, Eq. (2.14) is obtained using the invariance of the trace under cyclic permutations, the semigroup property and an elementary change of the integration variables. The convergence properties necessary for commuting the trace with the integral are assured by Proposition 1.

3. The Finite-Volume Magnetic Susceptibility at Zero Field and Fixed Density

The aim of this section is to prove the Relation (1.9).

The results of the foregoing section allow to state the following properties of the function:

$$\varrho_\Lambda(\beta, z, \omega) \equiv \beta z \frac{\partial}{\partial z} P_\Lambda(\beta, z, \omega). \tag{3.1}$$

a) *For every fixed* $\beta > 0$ *and* $\omega \in \mathbb{R}$, ϱ_Λ *is an analytic function of* z *in* $\mathbb{C} \setminus (-\infty, -1]$.

b) *For fixed* $\beta > 0$ *and* $\omega \in \mathbb{R}$:

$$\frac{\partial \varrho_\Lambda}{\partial z}(\beta, z, \omega) > 0, \qquad z \in [0, \infty) \tag{3.2}$$

and

$$\varrho_\Lambda(\beta, [0,\infty), \omega) = [0, \infty). \tag{3.3}$$

c) *For fixed* $\beta > 0$ *and* $z \in \mathbb{C} \setminus (-\infty, -1]$, $\varrho_\Lambda(\beta, z, \omega)$ *is an even differentiable function of the real variable* ω.

Properties a) and c) are obvious from Proposition 2. Inequality (3.2) follows from:

$$\varrho_\Lambda(\beta, z, \omega) = \frac{1}{V(\Lambda)} \sum_i \frac{z e^{-\beta E_i(\omega)}}{1 + z e^{-\beta E_i(\omega)}} \tag{3.4}$$

where every term is manifestly increasing. Equation (3.3) follows remarking that every term in (3.4) has the limit $1/V(\Lambda)$ when $z\to\infty$ which implies that ϱ_Λ cannot be bounded.

Properties a) and b) of ϱ_Λ assure that the inverse function $g_\Lambda(\beta,\varrho,\omega)$ $(\beta>0, \omega\in\mathbb{R}, \varrho\geqq 0)$ is well defined, differentiable in ϱ at fixed ω, with derivative bounded on every compact in $[0,\infty)$. Then c) assures that $\frac{\partial}{\partial\omega}g_\Lambda$ exists and:

$$\frac{\partial}{\partial\omega}g_\Lambda(\beta,\varrho,\omega)=-\frac{\partial}{\partial\varrho}g_\Lambda(\beta,\varrho,\omega)\cdot\frac{\partial}{\partial\omega}\varrho_\Lambda(\beta,z,\omega)\Big|_{z=g_\Lambda(\beta,\varrho,\omega,)} \tag{3.5}$$

Moreover, because ϱ_Λ is an even function of ω:

$$\frac{\partial}{\partial\omega}g_\Lambda(\beta,\varrho,\omega)\Big|_{\omega=0}=0\,. \tag{3.6}$$

Let us now consider the constant-ϱ-susceptibility, Eq. (1.8). From the definitions of p_Λ, ϱ_Λ and g_Λ one obtains:

$$\frac{\partial}{\partial\omega}\left[p_\Lambda(\beta,\varrho,\omega)-\frac{\varrho}{\beta}\log g_\Lambda(\beta,\varrho,\omega)\right]=\frac{\partial}{\partial\omega}P_\Lambda(\beta,z,\omega)\Big|_{z=g_\Lambda(\beta,\varrho,\omega)} \tag{3.7}$$

When taking the second derivative and putting $\omega=0$, it follows from (3.6):

Proposition 3.

$$\chi_\Lambda(\beta,\varrho)=\left(\frac{e}{c}\right)^2\frac{\partial^2}{\partial\omega^2}P_\Lambda(\beta,z,\omega)\Big|_{\substack{\omega=0\\ z=g_\Lambda(\beta,\varrho,0)}}=X_\Lambda(\beta,g_\Lambda(\beta,\varrho,0))\,.$$

4. An Identity for the One-dimensional Green Function

In this section the Expressions (2.13) and (2.14) of $X_\Lambda(\beta,z)$ will be simplified and brought to a form especially convenient for taking the thermodynamic limit. Here for the first time the special shape (2.1) of Λ will play an important role.

In terms of the integral kernel (2.6) of $S_0(t)$, Eq. (2.14) reads as:

$$\begin{aligned}&\mathscr{F}_\Lambda^{(2)}(t)\\&=-\frac{t}{2}\int_{-a_1}^{a_1}dx_1\,G_0^{a_1}(t;x_1,x_1)\int_{-a_2}^{a_2}dx_2\cdot x_2^2\,G_0^{a_2}(t;x_2,x_2)\int_{-a_3}^{a_3}dx_3\,G_0^{a_3}(t;x_3,x_3)\\&\quad-\int_0^t d\tau(t-\tau\iint_{-a_1}^{a_1}dx_1\,dx_1'\,\frac{\partial}{\partial x_1}G_0^{a_1}(\tau;x_1,x_1')\cdot\frac{\partial}{\partial x_1'}G_0^{a_1}(t-\tau;x_1'\,x_1)\\&\quad\cdot\iint_{-a_2}^{a_2}dx_2\,dx_2'\cdot x_2x_2'\,G_0^{a_2}(\tau;x_2,x_2')\,G_0^{a_2}(t-\tau;x_2',x_2)\int_{-a_3}^{a_3}dx_3\,G_0^{a_3}(t;x_3,x_3)\end{aligned} \tag{4.1}$$

Writing $x_2 x_2' = \frac{1}{2}[x_2^2 + x_2'^2 - (x_2 - x_2')^2]$, one can use here the (one-dimensional) semigroup property to perform one more integration in the terms containing x_2^2, $x_2'^2$, thus obtaining:

$$\begin{aligned}
&\mathscr{F}_{\Lambda}^{(2)}(t)\\
&= -\int_{-a_3}^{a_3} dx_3 G_0^{a_3}(t; x_3, x_3)\left\{\int_{-a_2}^{a_2} dx_2 \cdot x_2^2 G_0^{a_2}(t; x_2, x_2)\left[\frac{t}{2}\int_{-a_1}^{a_1} dx_1 G_0^{a_1}(t; x_1, x_1)\right.\right.\\
&\left.+\int_0^t d\tau(t-\tau)\iint_{-a_1}^{a_1} dx_1 dx_1' \frac{\partial}{\partial x_1} G_0^{a_1}(\tau; x_1, x_1')\frac{\partial}{\partial x_1'} G_0^{a_1}(t-\tau; x_1', x_1)\right]\\
&-\frac{1}{2}\int_0^t d\tau(t-\tau)\iint_{-a_1}^{a_1} dx_1 dx_1' \frac{\partial}{\partial x_1} G_0^{a_1}(\tau; x_1, x_1')\cdot\frac{\partial}{\partial x_1'} G_0^{a_1}(t-\tau; x_1', x_1)\\
&\left.\cdot\iint_{-a_2}^{a_2} dx_2 dx_2'(x_2 - x_2')^2 G_0^{a_2}(\tau; x_2, x_2') G_0^{a_2}(t-\tau; x_2', x_2)\right\}
\end{aligned} \tag{4.2}$$

Due to the fact that $G_0^a(t; x, x)$ becomes practically constant as a function of x, while $G_0^a(t; x, x')$ is bounded by a rapidly decreasing function of $|x - x'|$, when $a \to \infty$, the second term in curly brakets in Eq. (4.2) will give a good behaviour $[0(V(\Lambda))]$ when $\Lambda \to \infty$, while the first must behave like a_2^3, unless the expression in square brakets vanishes identically. This pathological behaviour is in fact removed by proving the following identity for the Green function:

Proposition 4.

$$\begin{aligned}
&\int_0^t d\tau(t-\tau)\iint_{-a}^{a} dx dx' \frac{\partial}{\partial x} G_0^a(\tau; x, x')\frac{\partial}{\partial x'} G_0^a(t-\tau; x', x)\\
&= -\frac{t}{2}\int_{-a}^{a} dx G_0^a(t; x, x)\,.
\end{aligned} \tag{4.3}$$

Proof. Though a more far-reaching proof using gauge invariance can be given, we shall show here (4.3) by explicitely evaluating the l.h.s. This is done by using the formula (2.7a) for the Green function. When inserting this in (4.3) and performing the integrals over x, x', one obtains for the l.h.s.:

$$\begin{aligned}
&-\left(\frac{4}{\pi}\right)^2\left(\frac{\pi}{2a}\right)^2 \cdot \sum_{\substack{k_1, k_2, = 1\\ (-1)^{k_1} \neq (-1)^{k_2}}}^{\infty} \frac{k_1^2 k_2^2}{(k_2^2 - k_1^2)^2}\\
&\int_0^t d\tau(t-\tau)\exp\left[-\frac{\tau}{2}\left(\frac{k_1\pi}{2a}\right)^2 - \frac{t-\tau}{2}\left(\frac{k_2\pi}{2a}\right)^2\right]\\
&= \left(\frac{4}{\pi}\right)^2 \cdot 2t \cdot \sum_{\substack{k_1, k_2 = 1\\ (-1)^{k_1} \neq (-1)^{k_2}}}^{\infty} \frac{k_1^2 k_2^2}{(k_2^2 - k_1^2)^3} e^{-\frac{t}{2}\left(\frac{k_2\pi}{2a}\right)^2}\,.
\end{aligned} \tag{4.4}$$

The sum over k_1 can be performed using the identities [31] (1.421,1,3)

$$\frac{\pi}{4x}\tan\frac{\pi x}{2}=\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2-x^2} \tag{4.5}$$

$$\frac{1}{2x^2}-\frac{\pi}{4x}\operatorname{cotan}\frac{\pi x}{2}=\sum_{k=1}^{\infty}\frac{1}{(2k)^2-x^2} \tag{4.6}$$

from which, by derivations with respect to x, one obtains the sums appearing in (4.4) for k_2 even and odd, respectively. Namely:

$$\sum_{\substack{k_1=1\\(-1)^{k_1}\neq(-1)^{k_2}}}^{\infty}\frac{k_1^2k_2^2}{(k_1^2-k_2^2)^3}=\frac{1}{4}\left(\frac{\pi}{4}\right)^2\qquad \forall\, k_2\in\mathbb{Z}\,. \tag{4.7}$$

With (4.7) one obtains for the l.h.s. of (4.3): $-\frac{t}{2}\sum_{k=1}^{\infty}e^{-\frac{t}{2}\left(\frac{k\pi}{2a}\right)^2}$ which clearly equals the r.h.s.

We close this section by writing the simplified expression for $\mathscr{F}_A^{(2)}(t)$:

$$\begin{aligned}\mathscr{F}_A^{(2)}(t)=&\frac{1}{2}\int_{-a_3}^{a_3}dx_3\,G_0^{a_3}(t;x_3,x_3)\int_0^t d\tau(t-\tau)\\ &\cdot\iint_{-a_1}^{a_1}dx_1\,dx_1'\frac{\partial}{\partial x_1}G_0^{a_1}(\tau;x_1,x_1')\frac{\partial}{\partial x_1'}G_0^{a_1}(t-\tau;x_1',x_1)\\ &\cdot\iint_{-a_2}^{a_2}dx_2\,dx_2'(x_2-x_2')^2\,G_0^{a_2}(\tau;x_2,x_2')\,G_0^{a_2}(t-\tau;x_2',x_2)\,.\end{aligned} \tag{4.8}$$

5. The Thermodynamic Limit and Surface Correction for the Susceptibility at Fixed Fugacity

In this section, we shall give a representation of $X_A(\beta,z)$ manifestly analytic in the cut plane and study on this representation the limit $a_1,a_2,a_3\to\infty$.

The idea is to use the expression for the Green functions G_0^a obtained by the method of images, Eq. (2.7b). When inserting this expression into (4.8) and (2.13), one obtains X_A as a multiple infinite sum over images of expressions like (2.13), where however every Green function in (4.8) is replaced by exponentials like those appearing in (2.7b). We shall show that every term in this sum is analytic in the cut plane. Moreover, we shall obtain bounds on these terms which assure the uniform convergence of the series.

The typical term has the following form[5]:

$$I(z)=\beta^{-\frac{1}{2}}(2\pi)^{-\frac{5}{2}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}z^n}{n^{\frac{7}{2}}}\int_0^\beta d\tau(\beta-\tau)\iint_{-a_1}^{a_1}dx_1dx_1'\iint_{-a_2}^{a_2}dx_2dx_2'\int_{-a_3}^{a_3}dx_3$$
$$\cdot\frac{\alpha_1(x_1,x_1')\alpha_2(x_1',x_1)}{[\tau(\beta-\tau)]^2}(x_2-x_2')^2$$
$$\cdot\exp\left[-\frac{1}{2n}\left(\frac{\alpha_1(x_1,x_1')^2}{\tau}+\frac{\alpha_2(x_1',x_1)^2}{\beta-\tau}\right.\right.$$
$$\left.\left.+\frac{\alpha_3(x_2,x_2')^2}{\tau}+\frac{\alpha_4(x_2',x_2)^2}{\beta-\tau}+\frac{\alpha_5(x_3,x_3)^2}{\beta}\right)\right] \tag{5.1}$$

where $\alpha_i' s$ are expressions appearing in the brakets at the exponent in (2.7b), i.e. $(4ma_i+x_i-x_i')$ or $(4ma_i+2a_i-x_i-x_i')$, $i=1,2,3$.

We shall first consider the terms for which the exponent in (5.1) never vanishes on the integration domain, that is:

$$\Gamma(r,r',\beta,\tau)^2\equiv\frac{\alpha_1(x_1,x_1')^2}{\tau}+\frac{\alpha_2(x_1',x_1)^2}{\beta-\tau}+\frac{\alpha_3(x_2,x_2')^2}{\tau}$$
$$+\frac{\alpha_4(x_2',x_2)^2}{\beta-\tau}+\frac{\alpha_5(x_3,x_3)^2}{\beta}$$
$$\geqq A^2>0$$
$$\text{for all}\quad r=(x_1,x_2,x_3),\ r'=(x_1',x_2',x_3')\in\Lambda\quad\text{and}\quad 0<\tau<\beta\,. \tag{5.2}$$

We shall use:

$$e^{-\frac{\Gamma^2}{2n}}=\sqrt{\frac{n}{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{n}{2}}e^{iu\Gamma}du$$
$$=\sqrt{\frac{n}{2\pi}}\,\Gamma^{-m}\int_{-\infty}^{\infty}P_m(u,n)\,e^{-\frac{n}{2}u^2}e^{iu\Gamma}du \tag{5.3}$$

where the second equality follows by m integrations by parts and $P_m(u,n)$ is some polynomial of degree m in variables u and n. We shall write this polynomial separating different powers of n, as:

$$P_m(u,n)=\sum_i p_{i,m}(u)n^{s_i}\,;\qquad 0\leqq S_i\leqq m\,. \tag{5.4}$$

Inserting (5.3) into (5.1) and interchanging the sum over n with all integrals (which is allowed at small $|z|$), we obtain:

$$I(z)=\beta^{-\frac{1}{2}}(2\pi)^{-3}\int_0^\beta d\tau(\beta-\tau)\iint_{-a_1}^{a_1}dx_1dx_1'\iint_{-a_2}^{a_2}dx_2dx_2'$$
$$\cdot\int_{-a_3}^{a_3}dx_3\frac{\alpha_1(x_1,x_1')\alpha_2(x_1',x_1)}{[\tau(\beta-\tau)]^2}(x_2-x_2')^2 \tag{5.5}$$
$$\cdot[\Gamma(r,r',\beta,\tau)]^{-m}\int_{-\infty}^{\infty}du\sum_i p_{i,m}(u)f_{1-s_i}\left(ze^{-\frac{u^2}{2}}\right)e^{iu\Gamma(r,r',\beta,\tau)}$$

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[5] We have made the change of variable $\tau\to n\tau$ in (4.8) written for $t=n\beta$.

where the Fermi functions $f_\sigma(z)$ are known to be analytical in the complex plane cut along $(-\infty, -1]$. Moreover, for every compact set K of $\mathbb{C}\setminus(-\infty, -1]$:

$$|f_\sigma(z)| \leqq M(K, \sigma)|z|, \qquad z \in K \tag{5.6}$$

for a suitable constant $M(K, \sigma) > 0$. This inequality is used to show that the integrals in (5.5) are absolutely convergent, which implies the analyticity of $I(z)$ in the cut plane. Indeed, for $z \in K$:

$$\begin{aligned}
|I(z)| \leqq |z|\beta^{-\frac{1}{2}}(2\pi)^{-3} \int_0^\beta d\tau(\beta-\tau)[\tau(\beta-\tau)]^{-\frac{3}{4}} \iint_{-a_1}^{a_1} dx_1 dx_1' \iint_{-a_2}^{a_2} dx_2 dx_2' \int_{-a_3}^{a_3} dx_3 \\
\cdot[|\alpha_1(x_1, x_1')\alpha_2(x_1', x_1)\alpha_3(x_2, x_2')\alpha_4(x_2', x_2)|^{-\frac{1}{4}} \cdot \Gamma(r, r', \beta, \tau)^{-4}] \\
\cdot\left[\frac{(x_2 - x_2')^2}{|\alpha_3(x_2, x_2')\alpha_4(x_2', x_2)|}\right] \\
\cdot\left[\frac{\alpha_1(x_1, x_1')^2}{\tau} \cdot \frac{\alpha_2(x_1', x_1)^2}{\beta-\tau} \cdot \frac{\alpha_3(x_2, x_2')^2}{\tau} \cdot \frac{\alpha_4(x_2', x_2)^2}{\beta-\tau}\right]^{\frac{5}{8}} \\
\cdot\, \Gamma(r, r', \beta, \tau)^{4-m}\left[\sum_i M(K, s_i) \int_{-\infty}^{\infty} |p_{i,m}(u)| \cdot e^{-\frac{u^2}{2}} du\right],
\end{aligned} \tag{5.7}$$

on which it is appearant that all integrals are convergent, in view of

$$\frac{(x_2 - x_2')^2}{|\alpha_3(x_2, x_2')\alpha_4(x_2', x_2)|} \leqq 1, \qquad -a_2 \leqq x_2, x_2' \leqq a_2 \tag{5.8}$$

and of the elementary inequality, following from (5.2):

$$\begin{aligned}
\left[\frac{\alpha_1(x_1, x_1')^2}{\tau} \cdot \frac{\alpha_2(x_1', x_1)^2}{\beta-\tau} \cdot \frac{\alpha_3(x_2, x_2')^2}{\tau} \cdot \frac{\alpha_4(x_2', x_2)^2}{\beta-\tau}\right]^{\frac{5}{8}} \cdot \Gamma(r, r', \beta, \tau)^{4-m} \\
\leqq A^{9-m}, \qquad \forall r, r' \in \Lambda .
\end{aligned} \tag{5.9}$$

As we are interested in the behaviour of $I(z)$ for $a_i \to \infty$, we shall also derive a bound for the integrals which are left, e.g.:

$$\begin{aligned}
&\iint_{a_1}^{a_1} dx_1 dx_1' |\alpha_1(x_1, x_1')\alpha_2(x_1', x_1)|^{-\frac{1}{4}} \Gamma(r, r', \beta, \tau)^{-2} \\
&\quad \leqq 4 \iint_{-a_1}^{a_1} dx_1 dx_1' \left[|\alpha_1(x_1, x_1')|^{-\frac{1}{4}}\left(A + \frac{|\alpha_1(x_1, x_1')|}{\beta}\right)^{-1}\right] \\
&\quad\quad \cdot\left[|\alpha_2(x_1', x_1)|^{-\frac{1}{4}}\left(A + \frac{\alpha_2(x_1', x_1)|}{\sqrt{\beta}}\right)^{-1}\right] \\
&\quad \leqq 8a_1\beta \int_{-\infty}^{\infty} d\alpha |\alpha|^{-\frac{1}{2}}(\sqrt{\beta}A + |\alpha|)^{-2}
\end{aligned} \tag{5.10}$$

where we used the inequality (5.2), the Schwarz inequality, changed to the integration variables z and α $(= \alpha_1(x_1 . x_1')$ or $\alpha_2(x_1', x_1))$ and extended the integral over α. Collecting the estimates, we obtain finally that, for $\Lambda \to \infty$:

$$|I(z)| \leqq C_p(K, \beta) \cdot a_1 a_2 a_3 |z| \cdot \Lambda^{-p}, \qquad z \in K \tag{5.11}$$

with $C_p(K, \beta) > 0$ independent of $a_i (i = 1, 2, 3)$ and Λ, where p can be still chosen at our will.

One can easily see that all but a finite number of terms in the sums over images in (4.8) obey inequality (5.2). Indeed, if any one of the five α_i's appearing in (5.1) is of the form $(4ma + x - x')$ with $m \neq 0$, or $(4ma + 2a - x - x')$ with $m \neq 0, -1$, then $\Lambda^2 \geqq \frac{4m^2 a^2}{\beta}$. Moreover, if e.g. $\alpha_1(x_1, x_1') = 2a_1 - x_1 - x_1'$ and $\alpha_2(x_1', x_1) = -2a_1 - x_1' - x_1$, then

$$\Lambda^2 > \beta^{-1} \cdot \min_{-a_1 \leqq x_1, x_1' \leqq a_1} [\alpha_1(x_1, x_1')^2 + \alpha_2(x_1', x_1)^2] = 8\beta^{-1} a_1^2 .$$

All these terms sum up to an analytic function of z in the cut plane because the sum converges absolutely and uniformly on compacts, by virtue of (5.11) with p chosen sufficiently large.

Because Λ^{-1} behaves as $0(a_i^{-1})$ with $i = 1, 2$ or 3 when $a_1, a_2, a_3 \to \infty$, one concludes from (5.11) that for every term $I(z)$ satisfying (5.2), $V(\Lambda)^{-1} I(z)$ and $S(\Lambda)^{-1} I(z)$ tend to zero uniformly on compacts if $a_1, a_2, a_3 \to \infty$ in such a manner that, for at least one p, all ratios $a_i^{-p} a_j (i, j = 1, 2, 3)$ tend to zero. This condition is in particular satisfied if $\Lambda \to \infty$ in the sense of Fisher [3].

In this way, we are left to consider only those terms (5.1) in which the α_i's are chosen at will among the following posibilities:

a) For α_1 and α_2:
1°. $\alpha_1(x_1, x_1') = \alpha_2(x_1, x_1') = x_1 - x_1'$.
2°. $\alpha_1(x_1, x_1') = x_1 - x_1'$; $\alpha_2(x_1, x_1') = \pm 2a_1 - x_1 - x_1'$
(or the same with α_1, α_2 interchanged).
3°. $\alpha_1(x_1, x_1') = \alpha_2(x_1, x_1') = \pm 2a_1 - x_1 - x_1'$.

b) For α_3 and α_4:
1°. $\alpha_3(x_2, x_2') = \alpha_4(x_2, x_2') = x_2 - x_2'$.
2°. $\alpha_3(x_2, x_2') = x_2 - x_2'$; $\alpha_4(x_2, x_2') = \pm 2a_2 - x_2 - x_2'$
(or the same with α_3, α_4 interchanged).
3°. $\alpha_3(x_2, x_2') = \alpha_4(x_2, x_2') = \pm 2a_2 - x_2 - x_2'$.

c) For α_5:
1°. $\alpha_5(x_3, x_3') = x_3 - x_3'$.
2°. $\alpha_5(x_3, x_3') = \pm 2a_3 - x_3 - x_3'$.

In calculating the contributions of these terms, we shall freely extend the integration domains in the space variables, taking care to add only quantities, R, which obey inequality (5.2). By the same reasoning as above, all the contributions added in this way to $X_\Lambda(\beta, z)$ sum up to an analytic function of z which affects only the $o\left(\frac{S(\Lambda)}{V(\Lambda)}\right)$ term in the asymptotic expansion of $X_\Lambda(\beta, z)$.

The following types of integrals appear:
In Cases a) 1° and b) 1°:

$$\iint_{-a}^{a} dx\,dx'(x-x')^2 \exp\left[-\frac{(x-x')^2\beta}{2n\tau(\beta-\tau)}\right]$$

$$=\left[\iint_{-2a\leqq x+x'\leqq 2a} -4\iint_{\substack{x\geqq 0\\ x+x'\leqq 2a}}\right](x-x')^2\exp\left[-\frac{(x-x')^2\beta}{2n\tau(\beta-\tau)}\right]+R$$

$$=2a\sqrt{2\pi}\left[\frac{n\tau(\beta-\tau)}{\beta}\right]^{\frac{3}{2}}-4\left[\frac{n\tau(\beta-\tau)}{\beta}\right]^2+R\,.$$

In case a) 2°, the integral vanishes by a parity argument (interchange x_1, x_1').
In Case b) 2°:

$$\iint_{-a}^{a} dx\,dx'(x-x')^2\exp\left[-\frac{(x-x')^2}{2n\tau}-\frac{(2a-x-x')^2}{2n(\beta-\tau)}\right]$$

$$=\iint_{\substack{a-x\geqq 0\\ a-x'\geqq 0}} dx\,dx'(x-x')^2\exp\left[-\frac{(x-x')^2}{2n\tau}-\frac{(2a-x-x')^2}{2n(\beta-\tau)}\right]+R$$

$$=\tfrac{1}{2}n^2\tau^{\frac{3}{2}}(\beta-\tau)^{\frac{1}{2}}\int_{-\arctan\sqrt{\frac{\tau}{\beta-\tau}}}^{\arctan\sqrt{\frac{\tau}{\beta-\tau}}} d\varphi\cos^2\varphi\int_0^\infty d\varrho\,\varrho^3 e^{-\frac{\varrho^2}{2}}+R$$

$$=n^2\tau^{\frac{3}{2}}(\beta-\tau)^{\frac{1}{2}}\left[\arctan\sqrt{\frac{\tau}{\beta-\tau}}+\frac{2}{\beta}\sqrt{\tau(\beta-\tau)}\right]+R$$

where we changed to polar coordinates

$$\frac{x-x'}{\sqrt{n\tau}}=\varrho\cos\varphi\,;\qquad \frac{2a-x-x'}{\sqrt{n(\beta-\tau)}}=\varrho\sin\varphi\,.$$

In Case a) 3°:

$$\iint_{-a}^{a} dx\,dx'(2a-x-x')^2\exp\left[-\frac{(2a-x-x')^2\beta}{2n\tau(\beta-\tau)}\right]$$

$$=\iint_{\substack{a-x\geqq 0\\ a-x'\geqq 0}} dx\,dx'(2a-x-x')^2\exp\left[-\frac{(2a-x-x')^2\beta}{2n\tau(\beta-\tau)}\right]+R$$

$$=2\left[\frac{n\tau(\beta-\tau)}{\beta}\right]^2+R\,.$$

In Case b) 3°, with the same extension of the integration domain:

$$\iint_{-a}^{a} dx\,dx'(x-x')^2\exp\left[-\frac{(2a-x-x')^2\beta}{2n\tau(\beta-\tau)}\right]=\frac{2}{3}\left[\frac{n\tau(\beta-\tau)}{\beta}\right]^2+R\,.$$

In Case c) 1°, the integrand is constant and in Case c) 2°:

$$\int_{-a}^{a} dx \exp\left[-\frac{(2a-2x)^2}{2n\beta}\right] = \frac{1}{2}\int_{0}^{\infty} d\xi\, e^{-\frac{\xi^2}{2n\beta}} + R = \frac{1}{4}(2\pi n\beta)^{\frac{1}{2}} + R\,.$$

When introducing these formulae in (5.1), one can perform easily the elementary integrations over τ and, then, the summation over n gives Fermi functions of different indices. Collecting all terms, one finally obtains the desired asymptotic expansion:

Proposition 5. *When $\Lambda = \Lambda(a_1, a_2, a_3) \to \infty$ in the sense of Fisher:*

$$\left(\frac{c}{e}\right)^2 X_\Lambda(\beta, z) = -\frac{1}{12}(2\pi)^{-\frac{3}{2}}\beta^{-\frac{1}{2}} f_{\frac{1}{2}}(z) + \frac{S_{||}(\Lambda)}{V(\Lambda)}\cdot\frac{3}{2^8\pi} f_0(z) + \frac{S_\perp(\Lambda)}{V(\Lambda)}\cdot\frac{1}{3\cdot 2^5\pi} f_0(z) + \mathrm{o}\left(\frac{S(\Lambda)}{V(\Lambda)}\right),$$

where $S_{||}(\Lambda) = 8(a_1 + a_2)a_3$ and $S_\perp(\Lambda) = 8a_1 a_2$.

6. The Thermodynamic Limit and Surface Correction for the Susceptibility at Fixed Density

We shall show in this section that the result of the section before implies the existence of the asymptotic expansion of $\chi_\Lambda(\beta, \varrho)$.

To this aim we need the asymptotic expansions of the functions $\varrho_\Lambda(\beta, z, 0)$ and $g_\Lambda(\beta, \varrho, 0)$ defined in Section 3.

Proposition 6. *When $\Lambda = \Lambda(a_1, a_2, a_3) \to \infty$ in the sense of Fisher:*

$$\varrho_\Lambda(\beta, z, 0) = (2\pi\beta)^{-\frac{3}{2}} f_{\frac{3}{2}}(z) - \frac{S(\Lambda)}{V(\Lambda)}(8\pi\beta)^{-1} f_1(z) + \mathrm{o}\left(\frac{S(\Lambda)}{V(\Lambda)}\right) \quad (6.1)$$

uniformly for z in compacts of the cut plane, and:

$$g_\Lambda(\beta, \varrho, 0) = g^{(0)}(\beta, \varrho) + \frac{S(\Lambda)}{V(\Lambda)}\cdot\tfrac{1}{4}(2\pi\beta)^{\frac{1}{2}} z \left.\frac{f_1(z)}{f_{\frac{1}{2}}(z)}\right|_{z=g^{(0)}(\beta,\varrho)} + \mathrm{o}\left(\frac{S(\Lambda)}{V(\Lambda)}\right) \quad (6.2)$$

where $g^{(0)}(\beta, \varrho) = f_{\frac{3}{2}}^{-1}((2\pi\beta)^{\frac{3}{2}}\varrho)$.

Proof. For $|z| < 1$:

$$\varrho_\Lambda(\beta, z, 0) = \frac{1}{V(\Lambda)}\sum_{n=1}^{\infty}(-1)^{n-1} z^n \int_{-a_1}^{a_1} dx_1\, G_0^{a_1}(n\beta; x_1, x_1) \cdot \int_{-a_2}^{a_2} dx_2\, G_0^{a_2}(n\beta; x_2, x_2) \int_{-a_3}^{a_3} dx_3\, G_0^{a_3}(n\beta; x_3, x_3)$$

and arguments similar to, but much simpler than those used in Section 5 prove (6.1).

Let now $\varrho_0 > 0$ be given, and denote $z_0 = g^{(0)}(\beta, \varrho_0)$ and $z_1 = g_\Lambda(\beta, \varrho_0, 0)$.

Because $\frac{d}{dz}f_{\frac{3}{2}}(z_0) > 0$ and in view of the uniform convergence of $\varrho_\Lambda(\beta, z, 0)$ to $(2\pi\beta)^{-\frac{3}{2}}f_{\frac{3}{2}}(z)$, one concludes that $z_1 \to z_0$ for $\Lambda \to \infty$ (Fisher). One has:

$$z_1 = z_0 - [\varrho_\Lambda(\beta, z_0, 0) - (2\pi\beta)^{-\frac{3}{2}} f_{\frac{3}{2}}(z_0)] \cdot \frac{1}{\frac{\partial}{\partial z}\varrho_\Lambda(\beta, z', 0)}$$

for a suitable z' between z_0 and z_1. Now, for $\Lambda \to \infty$ (Fisher):

$$\frac{\partial}{\partial z}\varrho_\Lambda(\beta, z', 0) = (2\pi\beta)^{-\frac{3}{2}} \frac{d}{dz} f_{\frac{3}{2}}(z_0) + o(1)$$

which together with (6.1) proves (6.2).

Proposition 7. *For $\Lambda = \Lambda(a_1, a_2, a_3) \to \infty$ in the sense of Fisher, and every $\beta > 0$, $\varrho > 0$:*

$$\chi_\Lambda(\beta, \varrho) = \chi^{(0)}(\beta, \varrho) + \frac{S_{||}(\Lambda)}{V(\Lambda)} \chi_{||}^{(1)}(\beta, \varrho) + \frac{S_\perp(\Lambda)}{V(\Lambda)} \chi_\perp^{(1)}(\beta, \varrho) + o\left(\frac{S(\Lambda)}{V(\Lambda)}\right) \quad (6.3)$$

where:

$$\left(\frac{c}{e}\right)^2 \chi^{(0)}(\beta, \varrho) = -\frac{1}{12\sqrt{\beta}} (2\pi)^{-\frac{3}{2}} f_{\frac{1}{2}}(g^{(0)}(\beta, \varrho)), \quad (6.4)$$

$$\left(\frac{c}{e}\right)^2 \chi_{||}^{(1)}(\beta, \varrho) = \frac{3}{2^8\pi} f_0(g^{(0)}(\beta, \varrho)) - \frac{1}{3 \cdot 2^5\pi} \cdot \frac{f_1(g^{(0)}(\beta, \varrho)) \cdot f_{-\frac{1}{2}}(g^{(0)}(\beta, \varrho))}{f_{\frac{1}{2}}(g^{(0)}(\beta, \varrho))},$$

$$(6.5)$$

$$\left(\frac{c}{e}\right)^2 \chi_\perp^{(1)}(\beta, \varrho) = \frac{1}{3 \cdot 2^5\pi} \left[f_0(g^{(0)}(\beta, \varrho)) - \frac{f_1(g^{(0)}(\beta, \varrho)) \cdot f_{-\frac{1}{2}}(g^{(0)}(\beta, \varrho))}{f_{\frac{1}{2}}(g^{(0)}(\beta, \varrho))} \right].$$

Proof. This is immediate from Propositions 3, 2 (iii), 5, and 6.

For the sake of comparison with previous results, we shall write down the leading terms of the asymptotic expansions of $\chi^{(0)}$, $\chi_{||}^{(1)}$ and $\chi_\perp^{(1)}$, for $\beta \to 0$ and $\beta \to \infty$:

(i) Low temperature limit ($\beta \to \infty$):

$$\chi^{(0)}(\beta, \varrho) = -\frac{1}{12\pi^{\frac{3}{2}}} \left[\frac{3\sqrt{\pi}}{4} \varrho\right]^{\frac{1}{3}} \left(\frac{e}{c}\right)^2 + 0(\beta^{-2})$$

$$\chi_{||}^{(1)}(\beta, \varrho) = \frac{5}{3 \cdot 2^8\pi} \left(\frac{e}{c}\right)^2 + 0(\beta^{-2})$$

$$\chi_\perp^{(1)}(\beta, \varrho) = \frac{1}{3 \cdot 2^6\pi} \left(\frac{e}{c}\right)^2 + 0(\beta^{-2}).$$

(ii) High temperature limit ($\beta \to 0$):

$$\chi^{(0)}(\beta, \varrho) = -\frac{\beta \varrho}{12}\left(\frac{e}{c}\right)^2 + 0(\beta^{\frac{5}{2}})$$

$$\chi_{\|}^{(1)}(\beta, \varrho) = \frac{1}{3 \cdot 2^8 \pi}(2\pi\beta)^{\frac{3}{2}} \varrho \left(\frac{e}{c}\right)^2 + 0(\beta^3)$$

$$\chi_{\perp}^{(1)}(\beta, \varrho) = 0(\beta^3).$$

References

1. Landau, L. D.: Z. Physik **64**, 629 (1930)
2. Bohr, N.: Dissertation, Copenhagen (1911)
   Lorentz, H. A.: Göttinger Vorträge über die kinetische Theorie der Materie, p. 188 Leipzig 1914
   van Leeuwen, J. H.: Dissertation, Leiden (1919); J. phys. et radium **2**, 361 (1921)
3. Ruelle, D.: Statistical mechanics, p. 14. New York: Benjamin 1969
4. Leonard, A.: Phys. Rev. **175**, 221 (1968)
5. Sondheimer, E. H., Wilson, A. H.: Proc. Roy. Soc. (London) A **210**, 173 (1951)
   Wilson, A. H.: The theory of metals, p. 160ff. Cambridge: Cambridge Univ. Press 1965
6. Ohtaka, K., Moriya, T.: J. Phys. Soc. Japan **34**, 1203 (1973)
7. Teller, E.: Z. Physik **67**, 311 (1931)
   Van Vleck, J. H.: Theory of electric and magnetic susceptibilities, p. 353ff. London: Oxford Univ. Press 1932
8. Papapetrou, A.: Z. Physik **106**, 9 (1937); **107**, 387 (1937); **112**, 587 (1939)
9. Welker, H.: Sitzber. bayer. Akad. Wiss. **14**, 115 (1938)
10. Osborne, M. F. M.: Phys. Rev. **88**, 438 (1952)
    Steele, M. C.: Phys. Rev. **88**, 451 (1952)
11. Dingle, R. B.: Phys. Rev. **82**, 966 (1951)
    Dingle, R. B.: I—V: Proc. Roy. Soc. (London) A **211**, 500 (1952); A **211**, 517 (1952); A **212**, 38 (1952); A **212**, 47 (1952); A **216**, 118 (1953)
12. Ham, F. S.: Phys. Rev. **92**, 1113 (1953)
13. Band, W.: Phys. Rev. **91**, 249 (1953)
    Dingle, R. B.: Phys. Rev. **92**, 1320 (1953)
14. Friedman, L.: Phys. Rev. **134** A, 336 (1964)
15. Nedorezov, S. S.: JETP **60**, 1938 (1971) (Russian); JETP Pis'ma **14**, 597 (1971) (Russian)
16. Darwin, C. G.: Proc. Cambridge Phil. Soc. **27**, 86 (1930)
17. Childers, D., Pincus, P.: Phys. Rev. **177**, 1036 (1969)
18. Denton, R. V.: Z. Physik **265**, 119 (1973)
19. Thomas, R. B. Jr.: Phys. Rev. B **7**, 4399 (1973)
20. Hille, E., Phillips, R. S.: Functional analysis and semigroups
    Providence, R. I.: Amer. Math. Soc. 1957
21. Uhlenbrock, D. A.: J. Math. Phys. **12**, 2503 (1971)
22. Angelescu, N., Nenciu, G., Bundaru, M.: On the perturbation of Gibbs semigroups, Commun. math. Phys. **42**, 29—30 (1975)
23. Kuroda, S. S.: Sci. Papers Coll. Gen. Educ. Univ. Tokyo **11**, 1 (1961)
24. Nenciu, G.: J. Stat. Phys. **7**, 119 (1973)
25. Robinson, D. W.: The thermodynamic pressure in quantum statistical mechanics. Lecture Notes in Physics, Vol. 9. Berlin-Heidelberg-New York: Springer 1971
26. Novikov, I. D.: Funct. Anal. Appl. **3**, 71 (1969) (Russian)
27. Angelescu, N., Nenciu, G.: Commun. math. Phys. **29**, 15 (1973)

28. Friedman, A.: Partial differential equations of parabolic type. Englewood Cliffs, N. J.: Prentice Hall Inc. 1964
29. Kato, T.: Perturbation theory for linear operators. New York: Springer 1966
30. Bochner, S., Martin, T.: Several complex variables. Princeton, N. J.: Princeton Univ. Press 1948
31. Ryjik, I. M., Gradstein, I. S.: Tables of integrals, sums, series and products (3rd ed.). Moscow: Gostehteorizdat 1951 (Russian)
32. Angelescu, N., Corciovei, A.: On free quantum gases in a homogeneous magnetic field, to appear in Rev. Roum. Phys.

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