Commun. math. Phys. 40, 7—13 (1975)

Ideal, First-kind Measurements in a Proposition-State Structure

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Received May 9, 1974

Abstract. Let $\mathscr{L}$ be an orthomodular lattice and $\mathscr{S}$ a strongly ordering set of probability measures on $\mathscr{L}$ such that supports of measures exist in $\mathscr{L}$. Then we show the existence of a set of mappings of $\mathscr{S}$ into $\mathscr{S}$ that are physically interpretable as ideal, first-kind measurements.

1.

In the conventional formulation [1–3] of the so called logic of quantum mechanics the basic mathematical structure associated to a physical system is the pair $(\mathscr{L}, \mathscr{S})$ where $\mathscr{L}$ is the set of propositions (yes-no experiments), $\mathscr{S}$ the set of states; each $\alpha \in \mathscr{S}$ defines on $\mathscr{L}$ a probability measure $\alpha : \mathscr{L} \to [0, 1]$, and $\alpha(a)$, $a \in \mathscr{L}$, is interpreted as the probability of the yes answer of a, when the initial state of the system is α. $\mathscr{L}$ is given the appropriate structure of orthomodular lattice by means of suitable axioms which elude any requirement on the transformations of the state of the system caused by the measurement of a: the usual postulates of quantum theory of measurement are considered as independent from the structure of $\mathscr{L}$ (for details and further bibliography we refer to [4]).

Pool [5, 6] has suggested an alternative approach which uses as basic mathematical structure the proposition-state-operation triple $(\mathscr{L}, \mathscr{S}, \Omega)$[1], where the operation $\Omega_a \in \Omega$ associated to $a \in \mathscr{L}$ is understood as the transformation of the state of the system induced by an ideal, first-kind measurement (with yes answer) of the proposition a. By use of the postulates of quantum theory of measurement and of the remarkable connections between orthomodular lattices and Baer*-semigroups, he deduces for $\mathscr{L}$ the structure of orthomodular lattice (see also [7]).

In this paper, we shall examine the possibility of reversing the Pool approach: we are going to study whether the assumption of a $(\mathscr{L}, \mathscr{S})$ structure, with $\mathscr{L}$ orthomodular lattice, is sufficient to deduce the

[1] $(\mathscr{L}, \mathscr{S})$ is denoted by Pool as $(\mathscr{E}, \mathscr{S}, P)$ and called an event-state structure; $(\mathscr{L}, \mathscr{S}, \Omega)$ is denoted as $(\mathscr{E}, \mathscr{S}, P, \Omega)$ and called an event-state-operation structure.

existence of a set of transformations of $\mathscr{S}$ into $\mathscr{S}$ which admit the physical interpretation of ideal, first-kind measurements, hence the existence of operations in the sense of Pool.

2.

We sum up several usual assumptions about $\mathscr{L}$ and $\mathscr{S}$ by the following

Axiom 1. *$\mathscr{L}$ is a complete orthomodular lattice, $\mathscr{S}$ is a strongly ordering, σ-convex set of probability measures on $\mathscr{L}$.*

Here, and in the sequel, the definitions and the notations about lattices are from [8], the definitions about probability measures on orthomodular posets are from [5, 6].

We say that $a \in \mathscr{L}$ is support of $\alpha \in \mathscr{S}$ when

$$\alpha(b) = 0, \quad b \in \mathscr{L} \Leftrightarrow \quad a \perp b .$$

For every $\alpha \in \mathscr{S}$ there exists at most one $a \in \mathscr{L}$ such that a is support of α: we shall denote such element, when it exists, by $\sigma(\alpha)$. We refer to [6] for properties of supports: the following will be used

$$\{a \in \mathscr{L} : a \perp \sigma(\alpha)\} = \{a \in \mathscr{L} : \alpha(a) = 0\},$$
$$\{a \in \mathscr{L} : \sigma(\alpha) \leqq a\} = \{a \in \mathscr{L} : \alpha(a) = 1\} .$$

Then we adopt, as in [6], the following

Axiom 2. *If $\alpha \in \mathscr{S}$ then the support of α exists in $\mathscr{L}$. If $a \in \mathscr{L} \setminus \{\mathbf{0}, \mathbf{1}\}$ then there exists $\alpha \in \mathscr{S}$ such that $a = \sigma(\alpha)$.*

Hence it is defined a surjective mapping

$$\alpha \mapsto \sigma(\alpha) : \mathscr{S} \to \mathscr{L} \setminus \{\mathbf{0}, \mathbf{1}\} ,$$

which is always canonically decomposable in the composition of a surjective mapping ω of $\mathscr{S}$ onto $\mathscr{S}/\sigma$ (the set of the equivalence classes of the states with the same support) with a bijection f of $\mathscr{S}/\sigma$ onto $\mathscr{L} \setminus \{\mathbf{0}, \mathbf{1}\}$. Thus f defines an one-to-one correspondence between $\mathscr{S}/\sigma$ and $\mathscr{L} \setminus \{\mathbf{0}, \mathbf{1}\}$ such that every element of $\mathscr{S}/\sigma$ corresponds to the common support of the states belonging to that element.

Let $S(\mathscr{L})$ be the Baer*-semigroup of residuated mappings (or emimorphisms) of $\mathscr{L}$[2], let $P'(S(\mathscr{L}))$ be the lattice of the closed projections of $S(\mathscr{L})$. For every $a \in \mathscr{L}$, let

$$\varphi_a(b) = (b \vee a^{\perp}) \wedge a, \quad b \in \mathscr{L} .$$

[2] We refer to [5, 6, 8] for definitions, properties and bibliography on Baer *-semigroups, residuated mappings of orthomodular lattices, and their connections.

φ_a is an element of $P'(S(\mathscr{L}))$; conversely every element of $P'(S(\mathscr{L}))$ is of the form φ_a for some $a \in \mathscr{L}$.

The pair $(\mathscr{L}, \mathscr{S})$ will be definitely assumed to satisfy Axioms 1 and 2: we shall call it the $(\mathscr{L}, \mathscr{S})$ structure.

Lemma 1. *For any given $a \in \mathscr{L}$ there exists a unique mapping $\omega_a : \mathscr{S} \to \mathscr{S}/\sigma$ such that*

(i) *the following diagram is commutative*

$$\begin{tikzcd} \mathscr{S} \arrow[r, "\omega_a"] \arrow[d, "\sigma"'] & \mathscr{S}/\sigma \arrow[d, "f"] \\ \mathscr{L}\setminus\{0,1\} \arrow[r, "\varphi_a"'] & \mathscr{L}\setminus\{0,1\} \end{tikzcd}$$

(ii) *the domain $\mathscr{D}[\omega_a]$ of ω_a is given by*

$$\mathscr{D}[\omega_a] = \{\alpha \in \mathscr{S} : \sigma(\alpha) \nleq a\}.$$

Proof. Given $a \in \mathscr{L}$, $\alpha \in \mathscr{S}$, a known property of closed projections of $S(\mathscr{L})$ ensures that $\varphi_a(\sigma(\alpha)) \neq \mathbf{0}$ if and only if $\sigma(\alpha) \nleq a$. Thus, if $\alpha \in \{\beta \in \mathscr{S} : \sigma(\beta) \nleq a\}$, $\varphi_a(\sigma(\alpha)) \in \mathscr{L}\setminus\{\mathbf{0}, \mathbf{1}\}$. Since f is a bijection it is well defined $f^{-1}(\varphi_a(\sigma(\alpha))) \in \mathscr{S}/\sigma$. Then we put

$$\omega_a(\alpha) = f^{-1}(\varphi_a(\sigma(\alpha))) \quad \text{if} \quad \sigma(\alpha) \nleq a\,.$$

In this way we have defined a mapping of $\mathscr{S}$ into $\mathscr{S}/\sigma$ whose domain is $\mathscr{D}[\omega_a] = \{\alpha \in \mathscr{S} : \sigma(\alpha) \nleq a\}$. Clearly ω_a makes the diagram commutative and is uniquely determined by the Properties (i) and (ii). Q.E.D.

Lemma 2. *Given $a \in \mathscr{L}$, $\alpha \in \mathscr{D}[\omega_a]$,*

(i) *if $\alpha(a) = 1$ then $\omega_a(\alpha) = \{\alpha\}$, where $\{\alpha\}$ is the equivalence class of α in $\mathscr{S}/\sigma$;*

(ii) *if $\beta \in \omega_a(\alpha)$ then $\beta(a) = 1$.*

Proof. (i) $\alpha(a) = 1$ implies $\sigma(\alpha) \leqq a$, hence $\sigma(\alpha) C a$, hence $\sigma(\alpha) C a^\perp$ [in an orthomodular lattice we say that a commutes with b, and we write $a\,C\,b$ when $a = (a \wedge b) \vee (a \wedge b^\perp)$]. Therefore a, $a^\perp$, $\sigma(\alpha)$ form a distributive triple so that

$$\varphi_a(\sigma(\alpha)) = (\sigma(\alpha) \vee a^\perp) \wedge a = (\sigma(\alpha) \wedge a) \vee (a^\perp \wedge a) = \sigma(\alpha)\,,$$

$$\text{whence } f^{-1}(\varphi_a(\sigma(\alpha))) = \omega_a(\alpha) = \{\alpha\}.$$

(ii) Owing to the commutativity of the diagram of Lemma 1 we have $f(\omega_a(\alpha)) = \varphi_a(\sigma(\alpha))$, $\forall \alpha \in \mathscr{D}[\omega_a]$, hence $f(\omega_a(\alpha)) \leqq a$ for φ_a projects $\mathscr{L}$ onto the sublattice $[\mathbf{0}, a]$. If $\beta \in \omega_a(\alpha)$, by the definition of f we get

$$\sigma(\beta) = f(\omega_a(\alpha)) \leqq a, \quad \text{whence} \quad \beta(a) = 1. \qquad \text{Q.E.D.}$$

Lemma 3. *If $a, b \in \mathscr{L}$, aCb, $\alpha \in \mathscr{D}[\omega_a]$, $\beta \in \omega_a(\alpha)$ then*
(i) $\alpha(b) = 1$ *implies* $\beta(b) = 1$;
(ii) $\beta(b) = \beta(a \wedge b)$.

Proof. (i) $\alpha(b) = 1$ implies $\sigma(\alpha) \leqq b$, hence $\varphi_a(\sigma(\alpha)) \leqq \varphi_a(b)$ for φ_a is monotone (as every residuated mapping). From aCb it follows that $a, a^\perp, b$ form a distributive triple, hence

$$\varphi_a(b) = (b \vee a^\perp) \wedge a = (b \wedge a) \vee (a^\perp \wedge a) = b \wedge a .$$

The commutativity of the diagram of Lemma 1 ensures that if $\beta \in \omega_a(\alpha)$ then $\sigma(\beta) = \varphi_a(\sigma(\alpha))$. Therefore $\sigma(\beta) \leqq \varphi_a(b) = b \wedge a \leqq b$, whence $\beta(b) = 1$.

(ii) We write $a\,C\,b$ in the form $b = (b \wedge a) \vee (b \wedge a^\perp)$ and remark that $(b \wedge a^\perp)^\perp = b^\perp \vee a \geqq b \wedge a$; thus $b \wedge a$ and $b \wedge a^\perp$ are orthogonal so that

$$\beta(b) = \beta(b \wedge a) + \beta(b \wedge a^\perp) = \beta(b \wedge a) + 1 - \beta(b^\perp \vee a) .$$

On account of Lemma 2 (ii) we have $\beta(b^\perp \vee a) = 1$ for $b^\perp \vee a \geqq a$; therefore $\beta(b) = \beta(b \wedge a)$. Q.E.D.

Theorem 1. *For every $a \in \mathscr{L}$, there exists at least one mapping $\Omega_a : \mathscr{S} \to \mathscr{S}$ such that*
(A) *Ω_a has domain $\mathscr{D}[\Omega_a] = \mathscr{D}[\omega_a]$,*
(B) *if $\alpha \in \mathscr{D}[\Omega_a]$, and $\alpha(a) = 1$ then $\Omega_a(\alpha) = \alpha$,*
(C) *if $\alpha \in \mathscr{D}[\Omega_a]$, and $\beta = \Omega_a(\alpha)$ then $\beta(a) = 1$,*
(D) *if $\alpha \in \mathscr{D}[\Omega_a]$, $\beta = \Omega_a(\alpha)$, $b \in \mathscr{L}$, $a\,C\,b$, and $\alpha(b) = 1$ then $\beta(b) = 1$,*
(E) *if $\alpha \in \mathscr{D}[\Omega_a]$, $\beta = \Omega_a(\alpha)$, $b \in \mathscr{L}$, and $a\,C\,b$ then $\beta(b) = \beta(a \wedge b)$,*
(F) *if $\alpha \in \mathscr{D}[\Omega_a]$, and $\beta = \Omega_a(\alpha)$ then $\sigma(\beta) = \varphi_a(\sigma(\alpha))$.*

Proof. We notice that $\mathscr{S}/\sigma$ is a partition of $\mathscr{S}$ (in disjoint classes) and we make use of the so called choice axiom (see, e.g., [9]): composing ω_a with any choice function which maps $\omega_a(\alpha) \in \mathscr{S}/\sigma$ into some single state belonging to $\omega_a(\alpha)$ we get, for every $a \in \mathscr{L}$, a mapping Ω_a of $\mathscr{S}$ into $\mathscr{S}$. By construction Ω_a fits (A). On account of Lemma 2(i), and remarking that $\alpha \in \{\alpha\}$, we are allowed to adopt a choice function which, whenever $\alpha(a) = 1$, maps $\omega_a(\alpha)$ into α itself: thus Ω_a satisfies (B). The Properties (C), (D), (E), (F) are then direct consequences of Lemmas 2 (ii), 3 (i), 3 (ii), 1 (i), respectively. Q.E.D.

3.

The previous theorem answers affirmatively the problem rised at the end of Section 1; the $(\mathscr{L}, \mathscr{S})$ structure (equipped with the Axioms 1 and 2) is sufficient to deduce, for every $a \in \mathscr{L}$, the existence of a mapping Ω_a of $\mathscr{S}$ into $\mathscr{S}$ which admits the following interpretation: if the initial state of the system is α then $\Omega_a(\alpha)$ is the final state of the system after an ideal, first-kind measurement of a which has given the yes answer. Let us examine briefly the Properties (A)–(F) of that theorem.

Since the condition $\sigma(\alpha) \not\perp a$ (occuring in $\mathscr{D}[\Omega_a]$) is equivalent to $\alpha(a) \neq 0$, we rephrase (A) by saying that Ω_a is defined for every state α which has a non vanishing probability $\alpha(a)$ of giving the yes answer of the yes-no experiment a. Then (B) and (C) correspond to the definition of first kind measurement [10]. The commutativity relation C previously used is fully equivalent to the physical notion of compatible experiments [3, 7, 8]. Thus the Property (D) corresponds to the commonly accepted definition of ideal measurement. When $\beta = \Omega_a(\alpha)$ and $a\,C\,b$ the Property (E) suggests for $\beta(b)$ the interpretation of conditional probability of compatible events; however, to guarantee that interpretation one should require, according to conventional probability theory, one further property, viz:

$$\text{(G) } \textit{if } a,\ b \in \mathscr{L},\ a \geqq b,\ \alpha \in \mathscr{D}[\Omega_a],\ \textit{and } \beta = \Omega_a(\alpha) \ \textit{then } \beta(b) = \frac{\alpha(b)}{\alpha(a)}.$$

The Property (F), which determines uniquely the support of $\beta = \Omega_a(\alpha)$ from the support of α, is the essential step to equip $\mathscr{L}$ with the covering property [6].

We come now to a comparison with the assumptions occuring in the Pool approach. The Properties (A), (B), and (C) coincide, respectively, with his Axioms II.1, II.2, and II.3. The Property (D) goes beyond the Pool axioms: actually he uses an alternative definition of ideal measurement resting on the Properties (E) and (G), which coincide, respectively, with his Axioms II.7 and II.6. However, these two axioms enter into the Pool approach only as instruments to prove a property weaker than (F), viz:

$$\text{if } \quad \alpha \in \mathscr{D}[\Omega_a], \quad \text{and} \quad \beta = \Omega_a(\alpha) \quad \text{then} \quad \sigma(\beta) \leqq \varphi_a(\sigma(\alpha)).$$

This inequality can also be proved by use of the characterization (D) of ideal measurement [4, 7] without any reference to (E) and (G); anyhow it is used by Pool only as a hint for adopting the stronger Property (F) which coincides with his Axiom II.8.

Thus we have to conclude that the $(\mathscr{L}, \mathscr{S})$ structure contains into itself the relevant properties of Pool's operations[3] with just one significant exception: the unicity of the ideal, first-kind measurement associated to $a \in \mathscr{L}$, (which is necessary to deduce the orthomodular lattice properties of $\mathscr{L}$ from the proposition-state-operation structure).

However, in our approach, the problem of the unicity of Ω_a has connections with the further hypotheses, about the atomicity of $\mathscr{L}$ and the pure states of $\mathscr{S}$, which are needed to introduce the covering property (called semimodularity in [6]). For the last we shall refer to the definition: $\mathscr{L}$ has the covering property if $\varphi_a(p)$ is an atom for every $a \in \mathscr{L}$ and

[3] The Axioms II.4 and II.5 of Pool evade the framework of the present analysis.

every atom $p \in \mathscr{L}$ such that $p \nleq a^{\perp}$. This definition is equivalent [11] to the one found in [8]. Then we have:

Theorem 2. *If, in the structure* $(\mathscr{L}, \mathscr{S})$, $\mathscr{L}$ *is atomic and* σ *determines a bijective mapping between the atoms of* $\mathscr{L}$ *and the pure states of* $\mathscr{S}$, *then*

(i) *the covering property of* $\mathscr{L}$ *implies that the restriction of* Ω_a *to the pure states of* $\mathscr{D}[\Omega_a]$ *is uniquely determined by a and transforms pure states into pure states;*

(ii) $\mathscr{L}$ *has the covering property and the restriction of* Ω_a *to the pure states of* $\mathscr{D}[\Omega_a]$ *is uniquely determined by a if* Ω_a *transforms pure states into pure states.*

Proof. (i) Due to the covering property $\varphi_a(\sigma(\alpha))$ is an atom; by (F) of Theorem 1 this atom is the support of $\beta = \Omega_a(\alpha)$. Hence this state is pure and uniquely determined by a and α.

(ii) By hypothesis, if $\alpha \in \mathscr{D}[\Omega_a]$ is pure, also $\Omega_a(\alpha)$ is pure, hence $\varphi_a(\sigma(\alpha))$ is an atom, by (F) of Theorem 1. This is equivalent to say that $\varphi_a(p)$ is an atom whenever $p \in \mathscr{L}$ is an atom such that $p \nleq a^{\perp}$. The uniqueness of $\beta = \Omega_a(\alpha)$ follows from $\sigma(\beta) = \varphi_a(\sigma(\alpha))$. Q.E.D.

The hypotheses of Theorem 2 about atomicity and pure states are equivalent to the ones adopted in [6] (see Axiom I.9 and Theorem II.1). We conclude that also the connections between covering property and pure operations are contained, in a natural way, into the $(\mathscr{L}, \mathscr{S})$ structure.

The role of the unicity of the ideal, first-kind measurement associated to $a \in \mathscr{L}$ has been particularly studied by Ochs [12]. He restricts $\mathscr{S}$ to the set $\mathscr{S}'$ of pure states, assumes $\mathscr{L}$ to be an atomic orthomodular lattice with a bijective mapping of $\mathscr{S}'$ onto the set of the atoms of $\mathscr{L}$, postulates the existence and the unicity of an ideal, first-kind measurement [i.e. the Properties (A), (B), (C), (D) of Theorem 1 with Ω_a replaced by τ_a], and proves the covering property [i.e. the Property (F) of Theorem 1 with Ω_a replaced by τ_a]. On account of Theorem 2, the restriction of Ω_a to the pure states coincides with τ_a. Thus, when dealing with pure states, the requirement of unicity of the ideal, first-kind measurement is equivalent to adopt our construction of Ω_a, made explicit by (F) of Theorem 1.

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Communicated by R. Haag

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