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Current Algebra Ward Identities in the Renormalized σ Model

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Abstract. The many-current Ward identities corresponding to the Gell-Mann current algebra are discussed in the renormalized σ model. The Ward identities are verified in the case of the SU(2) × SU(2) chiral symmetry. In the SU(3) × SU(3) case the uniqueness of the Adler-Bardeen anomaly is proved using the Wess-Zumino consistency conditions.

Introduction

In the framework of the renormalized σ model [1, 2] a multiplet of chiral currents was defined, the vector currents being conserved and the axial currents satisfying the P.C.A.C. [3] condition to any order of perturbation theory. On a heuristic basis the currents are normalized by fixing the field current commutation rules. These facts are best summarized by a system of Ward identities involving the time ordered product of one current operator and any number of quantized fields.

It remains to prove that these chiral current operators satisfy a suitable version of the commutation rules of the Gell-Mann current algebra[1]. In other words we have to extend to any order of perturbation theory the many-current Ward identities [4] which are valid in the semi-classical σ model [5] (i.e. in tree approximation). The time ordered products of any number of fields and current operators are defined [2] in the renormalized σ model up to the addition of seagull terms; that is terms which might be non zero only if the arguments of at least two currents coincide. In general the Ward identities will depend on such terms. Thus the problem is to see whether one can find a definition of the many-current T-products for which the Ward identities corresponding to the chiral current algebra are satisfied.

We shall begin studying the SU(2) × SU(2) σ model; in this case the current algebra Ward identities will be proved. Then we shall extend

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[1] A wide review of the principles and applications of current algebra is found in Ref. [3].

the discussion to the SU(3) × SU(3) case and we shall prove the uniqueness of the Adler-Bardeen [6] anomaly by applying the Wess-Zumino [7] consistency conditions.

The paper is organized as follows; in the first section we discuss the difficulties which come up beyond the tree approximation. In Section 2 the 1-current Ward identities are discussed in the SU(2) × SU(2) σ model. In Section 3 we prove the current algebra Ward identities in the SU(2) × SU(2) case. In Section 4 we extend the discussion to the SU(3) × SU(3) σ model and we give the results for the abelian case.

1. Beyond the Tree Approximation

For the sake of simplicity we shall first discuss the case of the SU(2) × SU(2) chiral symmetry.

Let us start by some general considerations and definitions. In the spirit of the functional method we shall consider a lagrangian depending on the quantized fields ψ (baryonic), σ, $\vec{\pi}$ (mesonic) and on the external vector and axial vector gauge fields $\vec{\omega}_\mu$ and $\vec{\alpha}_\mu$ which generate the corresponding currents. Under the isotopic spin (SU(2)) transformation ψ behaves as a 2 component spinor, σ as a scalar, $\vec{\pi}$, $\vec{\omega}_\mu$, $\vec{\alpha}_\mu$ as vectors. In this formalism the seagulls appear as direct couplings among external fields, and eventually quantized fields. In order to preserve the power behaviour of the Green functions at high euclidean momenta the dimension[2] of the seagull terms has to be lower than five.

In the tree approximation the SU(2)×SU(2) current algebra is easily verified. Indeed the tree approximation lagrangian is [5]:

$$\mathscr{L} = i\bar{\psi}\not{D}\psi - m\bar{\psi}\left(1 + \frac{1}{F}(i\vec{\pi}\cdot\vec{\tau}\gamma_5 + \sigma)\right)\psi + \frac{(D\pi)^2 + (D\sigma)^2}{2} - \frac{\mu^2}{2}(\pi^2 + \sigma^2)$$

$$-\frac{\delta^2}{2}\left[\sigma^2 + \frac{1}{F}(\sigma^2 + \pi^2)\,\sigma + \frac{1}{4F^2}(\pi^2 + \sigma^2)^2\right] \tag{1}$$

where $\vec{\tau}$ are the Pauli matrices and the covariant derivatives D are given by

$$\begin{aligned} D_\mu\psi &= \partial_\mu\psi - \frac{i}{2}(\vec{\omega}_\mu + \gamma_5\vec{\alpha}_\mu)\cdot\vec{\tau}\psi \\ D_\mu\vec{\pi} &= \partial_\mu\vec{\pi} + \vec{\omega}_\mu \wedge \vec{\pi} + \vec{\alpha}_\mu(\sigma + F) \\ D_\mu\sigma &= \partial_\mu\sigma - \vec{\alpha}_\mu\cdot\vec{\pi}\,. \end{aligned} \tag{2}$$

[2] The dimension of the meson fields ($\sigma, \vec{\pi}, \vec{\alpha}_\mu, \vec{\omega}_\mu$) is 1, that of ψ is 3/2, a derivative increases the dimension of 1.

A chiral SU(2) × SU(2) infinitesimal transformation is:

$$\begin{aligned}
&\psi \to \left(1 - \frac{i}{2}(\vec{v} + \vec{a}\gamma_5)\cdot\vec{\tau}\right)\psi \\
&\vec{\pi} \to \vec{\pi} + \vec{v}\wedge\vec{\pi} + \vec{a}(\sigma + F) \\
&\sigma \to \sigma - \vec{a}\cdot\vec{\pi} \\
&\vec{\omega}_\mu \to \vec{\omega}_\mu - \partial_\mu\vec{v} + \vec{v}\wedge\vec{\omega}_\mu + \vec{a}\wedge\vec{\alpha}_\mu \\
&\vec{\alpha}_\mu \to \vec{\alpha}_\mu - \partial_\mu\vec{a} + \vec{v}\wedge\vec{\alpha}_\mu + \vec{a}\wedge\vec{\omega}_\mu
\end{aligned} \tag{3}$$

here $\vec{v}$ and $\vec{a}$ are infinitesimal parameters. The corresponding variation of $\mathscr{L}$ is:

$$\delta\mathscr{L} = -c_0\vec{a}\cdot\vec{\pi} \tag{4}$$

where $c_0 = F\mu^2$. If we add to the lagrangian the source terms

$$\mathscr{L}_S = \vec{J}_\pi\cdot\vec{\pi} + J_\sigma\sigma + \bar{\eta}\psi + \bar{\psi}\eta \tag{5}$$

and we vary the classical sources according to:

$$\begin{aligned}
&\vec{J}_\pi \to \vec{J}_\pi + \vec{v}\wedge\vec{J}_\pi + \vec{a}J_\sigma \\
&J_\sigma \to J_\sigma - \vec{a}\cdot\vec{J}_\pi \\
&\eta \to \left(1 - \frac{i}{2}(\vec{v} - \vec{a}\gamma_5)\cdot\vec{\tau}\right)\eta
\end{aligned} \tag{6}$$

we get: $\delta(\mathscr{L} + \mathscr{L}_S) = -c_0\vec{a}\cdot\vec{\pi} + F\vec{a}\cdot\vec{J}_\pi$. The vacuum functional generating the T-ordered Green function is:

$$S[\vec{J}_\pi, J_\sigma, \eta, \bar{\eta}, \vec{\omega}_\mu, \vec{\alpha}_\mu] = \langle e^{i\int\mathscr{L}_S dx}\rangle_+ \equiv \langle X\rangle_+ \,. \tag{7}$$

It can be shown by simple combinatorial arguments that the variation of S corresponding to the transformation of the classical fields and sources given in Eqs. (3) and (6) is:

$$\delta S = i\int dx \langle\delta(\mathscr{L} + \mathscr{L}_S)\, e^{i\int\mathscr{L}_S dx}\rangle_+ \tag{8}$$

which is equivalent to the current algebra Ward identities[3]

$$\begin{aligned}
&\partial_\mu\frac{\vec{\delta}}{\delta\omega_\mu}S + \vec{\omega}_\mu\wedge\frac{\vec{\delta}}{\delta\omega_\mu}S + \vec{\alpha}_\mu\wedge\frac{\vec{\delta}}{\delta\alpha_\mu}S + \frac{i}{2}\left[\bar{\eta}\vec{\tau}\frac{\delta}{\delta\bar{\eta}}S - S\frac{\delta}{\delta\eta}\vec{\tau}\eta\right] \\
&\qquad + \vec{J}_\pi\wedge\frac{\vec{\delta}}{\delta J_\pi}S = 0 \\
&\partial_\mu\frac{\vec{\delta}}{\delta\alpha_\mu}S + \vec{\omega}_\mu\wedge\frac{\vec{\delta}}{\delta\alpha_\mu}S + \vec{\alpha}_\mu\wedge\frac{\vec{\delta}}{\delta\omega_\mu}S + \frac{i}{2}\left[\bar{\eta}\vec{\tau}\gamma_5\frac{\delta}{\delta\bar{\eta}}S + S\frac{\delta}{\delta\eta}\vec{\tau}\gamma_5\eta\right] \\
&\qquad + (J_\sigma + c_0)\frac{\vec{\delta}}{\delta J_\pi}S + \vec{J}_\pi\left(\frac{\delta}{\delta J_\sigma} + iF\right)S = 0\,.
\end{aligned} \tag{9}$$

[3] Since η and $\bar{\eta}$ are anticommuting objects the derivative with respect to η operates from the left.

This is nothing but the Noether theorem written in terms of Green functions.

Beyond the tree approximation Eq. (8) is still valid. Indeed Eq. (8) is a consequence of the equivalence theorem which has recently been proved in the framework of B.P.H. [8] renormalized field theory by Lam [9]. However the symmetry properties of the lagrangian are much less evident. The terms of the lagrangian appear as vertices in the Green functions of Eq. (8)[4]. Thus in the B.P.H. renormalization scheme a subtraction prescription has to be attached to each term of the lagrangian. Following Zimmermann's method [10] we shall assign to the vertex v a degree δ_v not lower than its dimension and we shall compute the superficial divergence of a diagram by: $D=4-n-\frac{3}{2}f+\Sigma_v(\delta_v-4)$; n is the number of both classical and quantized boson legs and f the number of the fermion legs. D is strictly related to the number of subtractions. Then we have to distinguish, in the expression of $\delta\mathscr{L}$, terms of different degrees. Of course there would be no problem if to each vertex were always assigned a degree equal to its dimension. This is however not the case, in particular the mass terms of the free lagrangian have degree 4 which is always greater than their dimension. Indeed the derivative with respect to the mass of a renormalized Green function (e.g. the propagator) has the same power behaviour at high momenta as the Green function itself. In other words the addition of a mass vertex does not change the power counting as it should happen if its degree were lower than 4. This is the potential source of anomalies in Ward identities.

Rouet [11] proved that in the case in which there is an invariant mass term the naive Ward identities remain true beyond the tree approximation. This is the case of the σ model without fermions [11, 12]. In our case the vector Ward identity remains valid at the quantum level because each term of the lagrangian is isotopic spin invariant.

In the following we shall assign degree 4 to all terms in the lagrangian[5]. In this case the coefficients of the lagrangian are directly given by the

[4] In general in the left-hand side of Eq. (8) there also appear the vertices corresponding to the free lagrangian.

[5] Our Lagrangian $\mathscr{L}$ includes in particular terms which depend on the external fields to which dimension 1 has been assigned.

The renormalization prescription for the proper Feynman diagrams containing external fields is defined by the expression of their superficial divergence D given above (with $\delta_v=4$).

It follows that in such a theory the Zimmermann identity which relates N-products of different degrees is still valid and it takes automatically care of terms contributes by subgraphs containing current vertices.

Technically speaking, this is a way of taking into account superrenormalizable vertices which identically amounts to applying the prescription given by Gomez, Lowenstein and Zimmermann (see Ref. [11]): in these author notation one should introduce a parameter s and multiply each external field by s.

value of the proper (amputated and 1-particle irreducible) diagrams at the origin in momentum space. More precisely each term in the lagrangian: $O_i = \Pi_j(\partial)^{\mu_j}\varphi_j$ depending on the fields $\varphi_j's$ and on their derivatives has a coefficient proportional to $\langle \Pi_i(i\partial_{p_i})^{\nu_i}\tilde{\varphi}_i(p_i)\rangle_+^{PROP}|_{p_i=0}$ where $\langle \Pi_i\tilde{\varphi}_i(p_i)\rangle_+^{PROP}$ is the Fourier transform of the proper Green function $\langle \Pi_i\varphi_i(x_i)\rangle_+^{PROP}$. This happens because in Zimmermann's scheme the superficially divergent diagrams are subtracted at the origin of momentum space and the value at this point of a proper superficially divergent Green function is equal to the elementary diagram (that without loops).

2. The 1-Current Ward Identity

Now we discuss the 1-current axial Ward identity in the SU(2) × SU(2) σ model.

In the abelian case the Ward identity in Eq. (9) has already been proved in the limit $\vec{\alpha}_p = \vec{\omega}_\mu = 0$. We shall repeat the discussion in the chiral SU(2) case.

The lagrangian is

$$\begin{aligned}\mathscr{L}_{\text{eff}} = {} & i(1+a)\,\overline{\psi}\partial\!\!\!/\psi - (m+A)\,\overline{\psi}\left(1 + \frac{1}{F}(i\vec{\pi}\cdot\vec{\tau}\gamma_5 + \sigma(1+d))\right)\psi \\ & + \frac{1+b}{2}((\partial\pi)^2 + (\partial\sigma)^2) + \frac{f}{2}(\partial\sigma)^2 - (\mu^2+B)\frac{\pi^2+\sigma^2}{2} \\ & - \frac{\delta^2+C}{2}\left[\sigma^2 + \frac{\sigma(\sigma^2+\pi^2)}{F} + \frac{(\pi^2+\sigma^2)^2}{4F^2}\right] \\ & - D\left[\frac{2\sigma^3}{F} + \frac{3}{F^2}\sigma^2(\pi^2+\sigma^2)\right] - \frac{E}{F^2}\sigma^4 .\end{aligned} \tag{10}$$

The variation of the action corresponding to the transformation of the fields given in Eq. (3) is for $\vec{v} = 0$:

$$\begin{aligned}\int \delta\mathscr{L}_{\text{eff}}\,dx = {} & \int dx\,\vec{a} \\ & \cdot\left\{N_4\left[-\partial_\mu\left(\frac{1+a}{2}\overline{\psi}\gamma_\mu\gamma_5\vec{\tau}\psi + (1+b)((\sigma+F)\,\partial_\mu\vec{\pi} - \vec{\pi}\partial_\mu\sigma)\right) + f\vec{\pi}\Box\sigma\right.\right. \\ & \left.\frac{d(m+A)}{F}\overline{\psi}(\vec{\pi} + i\sigma\gamma_5\vec{\tau})\,\psi + \frac{6D}{F^2}\vec{\pi}\sigma(\pi^2+\sigma^2) + \frac{4E}{F^2}\vec{\pi}\sigma^3\right] \\ & \left. + N_3[\vec{S}_\pi] - N_4[\vec{S}_\pi] - F(\mu^2+B)\,\vec{\pi}\right\}.\end{aligned} \tag{11}$$

$$S_\pi = -\left[(\delta^2+C)\left(\vec{\pi}\sigma + \frac{\vec{\pi}(\pi^2+\sigma^2)}{2F}\right) + \frac{6D}{F}\vec{\pi}\sigma^2 + i(m+A)\,\overline{\psi}\gamma_5\vec{\tau}\psi\right]$$

is proportional to the proper source of the $\vec{\pi}$ field. By the equivalence theorem [9] we have:

$$\int dx \langle (\delta \mathscr{L}_{\text{eff}} + \delta \mathscr{L}_S) X \rangle_+ = 0 . \tag{12}$$

$\mathscr{L}_S$ and X are defined in Section 1 and the time ordered product $\langle \, \rangle_+$ is defined by $\mathscr{L}_{\text{eff}}$. Using the Zimmermann identity which relates N products of different degree [10] we have:

$$\begin{aligned}
\langle (N_3[\vec{S}_\pi] - N_4[\vec{S}_\pi]) X \rangle_+ = \frac{F}{3} \Big\{ & \frac{i}{64} \partial_{p_\nu} \langle \tilde{\vec{\pi}}(-2p) \cdot \tilde{\bar{\psi}}(p) \vec{\tau} \gamma_\nu \gamma_5 \tilde{\psi}(p) \rangle_+^{PROP} |_{p=0} \\
& \cdot \langle N_4[\partial_\mu (\bar{\psi} \gamma_\mu \gamma_5 \vec{\tau} \psi)] X \rangle_+ \\
& + \tfrac{1}{8} (\langle (\tilde{\pi}(0))^2 \tilde{\bar{\psi}}(0) \tilde{\psi}(0) \rangle_+^{PROP} \langle N_4[\bar{\psi} \vec{\pi} \psi] X \rangle_+ \\
& - i \langle \tilde{\sigma}(0) \tilde{\vec{\pi}}(0) \cdot \tilde{\bar{\psi}}(0) \gamma_5 \vec{\tau} \tilde{\psi}(0) \rangle_+^{PROP} \langle N_4[\bar{\psi} i \sigma \gamma_5 \vec{\tau} \psi] X \rangle_+) \\
& - \tfrac{1}{8} (\partial_{p_\nu} \partial_{p_\nu} \langle \tilde{\vec{\pi}}(p) \cdot \tilde{\vec{\pi}}(-p) \tilde{\sigma}(0) \rangle_+^{PROP} |_{p=0} \langle N_4[\sigma \Box \vec{\pi}] X \rangle_+ \\
& + \partial_{k_\nu} \partial_{k_\nu} \langle \tilde{\vec{\pi}}(0) \cdot \tilde{\vec{\pi}}(-k) \tilde{\sigma}(k) \rangle_+^{PROP} |_{k=0} \langle N_4[\vec{\pi} \Box \sigma] X \rangle_+) \\
& - \tfrac{1}{4} \partial_{p_\nu} \partial_{k_\nu} \langle \tilde{\vec{\pi}}(-k-p) \cdot \tilde{\vec{\pi}}(p) \tilde{\sigma}(k) \rangle_+^{PROP} |_{p=k=0} \langle N_4[(\partial_\mu \sigma) \partial_\mu \vec{\pi}] X \rangle_+ \\
& + \tfrac{1}{6} (\langle (\tilde{\pi}(0))^4 \tilde{\sigma}(0) \rangle_+^{PROP} \langle N_4[\vec{\pi} \pi^2 \sigma] X \rangle_+ \\
& + \langle (\tilde{\pi}(0))^2 (\tilde{\sigma}(0))^3 \rangle_+^{PROP} \langle N_4[\vec{\pi} \sigma^3] X \rangle_+) \Big\} .
\end{aligned} \tag{13}$$

Taking into account Bose statistics we have up to second order in the momenta: $\langle \vec{\pi}(p) \cdot \vec{\pi}(-k-p) \tilde{\sigma}(k) \rangle_+^{PROP} = \frac{3}{F} [\lambda + \xi k^2 + \zeta p_\mu (k+p)_\mu]$. We further define: $\frac{iF}{96} \partial_{p_\mu} \langle \vec{\pi}(-2p) \cdot \bar{\psi}(p) \vec{\tau} \gamma_\nu \gamma_5 \psi(p) \rangle_+^{PROP} |_{p=0} = \chi$. We determine the parameters d, f, D, and E by the conditions:

$$\begin{aligned}
\frac{d(m+A)}{F} &= -\frac{F}{24} \langle (\pi(0))^2 \tilde{\bar{\psi}}(0) \tilde{\psi}(0) \rangle_+^{PROP} \\
\frac{6D}{F^2} &= -\frac{F}{18} \langle (\pi(0))^4 \tilde{\sigma}(0) \rangle_+^{PROP} \qquad f = \xi \\
\frac{6D}{F^2} + \frac{4E}{F^2} &= -\frac{F}{18} \langle (\pi(0))^2 (\tilde{\sigma}(0))^3 \rangle_+^{PROP}
\end{aligned} \tag{14}$$

and a, b, A, B, and C by the physical normalization conditions (1, 2).

By Eqs. (11), (12), (13) and Eq. (14) the 1-current Ward identity is proved if:

$$\langle \tilde{\sigma}(0) \tilde{\vec{\pi}}(0) \cdot \tilde{\bar{\psi}}(0) \gamma_5 \vec{\tau} \tilde{\psi}(0) \rangle_+^{PROP} = i \langle (\tilde{\pi}(0))^2 \tilde{\bar{\psi}}(0) \tilde{\psi}(0) \rangle_+^{PROP} \tag{15}$$

Eq. (12) then reduces to:

$$\int dx\vec{a}\cdot\left\langle \partial_\mu\left\{N_3\left(\frac{1+a+\chi}{2}\,\overline{\psi}\gamma_\mu\gamma_5\vec{\tau}\psi+(1+b-\zeta)\,\sigma\partial_\mu\vec{\pi}-(1+b)\,\vec{\pi}\partial_\mu\sigma\right)\right.\right.$$

$$\left.\left. +F(1+b)\,\partial_\mu\vec{\pi}\right\}X\right\rangle_+$$

$$=\int dx\vec{a}\cdot\partial_\mu\langle\vec{j}_{a,\mu}X\rangle_+$$

$$=\int dx\vec{a}\cdot\left\langle\left\{\vec{J}_\pi(\sigma+F)-(c+J_\sigma)\,\vec{\pi}-\frac{i}{2}\,(\overline{\eta}\gamma_5\vec{\tau}\psi+\overline{\psi}\gamma_5\vec{\tau}\eta)\right\}X\right\rangle_+ \qquad (16)$$

$$=\int dx\,\vec{a}\cdot\left\{\vec{J}_\pi\left(\frac{\delta}{i\delta J_\sigma}+F\right)S-(c+J_\sigma)\frac{\vec{\delta}}{i\delta J_\pi}\,S\right.$$

$$\left.-\frac{i}{2}\left(\overline{\eta}\gamma_5\vec{\tau}\,\frac{\delta}{i\delta\overline{\eta}}\,S+S\,\frac{\delta}{i\delta\eta}\,\gamma_5\vec{\tau}\eta\right)\right\}$$

where $c=F(\mu^2+B)$. This is the axial Ward identity (Eq. (9)) in the limit $\vec{\alpha}_\mu=\vec{\omega}_\mu=0$. A recurrence argument shows that Eq. (15) holds identically [2]. The details are given in the Appendix.

In conclusion we see that in the theory defined by the effective lagrangian:

$$\mathscr{L}_{\text{eff}}^{(1)}=\mathscr{L}_{\text{eff}}+\vec{\alpha}_\mu\cdot\vec{j}_{a,\mu}+\vec{\omega}_\mu\cdot\vec{j}_{v,\mu} \qquad (17)$$

where $\vec{j}_{v,\mu}=N_3\left[\frac{1+\text{a}}{2}\,\overline{\psi}\gamma_\mu\vec{\tau}\psi+(1+b)\,\vec{\pi}\wedge\partial_\mu\vec{\pi}\right]$ the 1-current Ward identities are verified.

In principle this method can be followed to prove the complete Ward identities, however we prefer to follow a simpler method based on the extended use of the functional techniques.

Before going into details we shall examine some properties of our deduction of the Ward identities which remain valid for non zero external fields. Let $\mathscr{L}_{\text{eff}}^{(\Sigma)}$ be obtained from $\mathscr{L}_{\text{eff}}^{(1)}$ by adding seagull terms of dimension lower than 5. The variation of the corresponding action due to an axial transformation of the fields has the form $\int dx\,\vec{a}\cdot\vec{\delta}\mathscr{L}_{\text{eff}}^{(\Sigma)} = \int dx\,\vec{a}\cdot\{N_4[\vec{\delta}_1\mathscr{L}_{\text{eff}}^{(\Sigma)}]+N_3[\vec{\delta}_2\mathscr{L}_{\text{eff}}^{(\Sigma)}]-c\vec{\pi}\}$ the symbol $\vec{\delta}_2$ means the variation generated by the transformation $\vec{\pi}\to\vec{\pi}+\vec{a}F$. Using the Zimmermann identity in much the same way as before we can write

$$\int dx\,\vec{a}\cdot\langle[\vec{\delta}\mathscr{L}_{\text{eff}}^{(\Sigma)}+c\vec{\pi}]\,X\rangle_+=\int dx\,\vec{a}\cdot\langle\{\Sigma_i\,c_iN_4[Q_i]\}\,X\rangle_+\,. \qquad (18)$$

The Q_i's are linearly independent monomials of dimension lower than 5. Since we subtract the divergent proper diagrams at the origin in momentum space to any $Q_i = \Pi_j(\partial)^{\nu_j}\,\varphi_j^{(q)}\Pi_m(\partial)^{\mu_m}\,\varphi_m^{(e)}$ there corresponds a monomial:

$$\tilde{X}_i(0) = i^{\Sigma_j \nu_j + \Sigma_m \mu_m}\Pi_j(\partial_{p_j})^{\nu_j}\hat{\tilde{\varphi}}^{(q)}(p_j)\,\Pi_m(\partial_{k_m})^{\mu_m}\frac{\delta}{\delta\tilde{\varphi}_m^{(e)}(k_m)}\Bigg|_{\substack{p=q=0\\ \varphi^{(e)}=0}}$$

such that: $\langle N_4[Q_i]\,X_j(0)\rangle_+^{PROP} = \delta_{ij}$.
Here $\hat{\tilde{\varphi}}_j^{(q)}(p)$ is a Fourier transformed, amputated, quantized field; $\varphi_m^{(e)}$ is a classical field and the functional derivative acts on the whole T product. Thus if:

$$\int dx\,\vec{a}\cdot\langle(\vec{\delta}\mathscr{L}_{\text{eff}}^{(\Sigma)} + c\vec{\pi})\,\tilde{X}_i(0)\rangle_+^{PROP} = 0 \tag{19}$$

for any $\tilde{X}_i(0)$ corresponding to a Q_i a dimension lower than 5 all the c_i's are zero and the Ward identities are satisfied.

3. Proof of the Current Algebra Ward Identities

Before discussing the many-current Ward identities we recall the definition of the proper functional [13]. Let us consider a lagrangian depending on the quantized fields φ_i's whose classical sources are J_i's and on the external fields β_j's, if $S[J,\beta]$ is the vacuum functional of the theory $Z[J,\beta] = -i\log S[J,\beta]$ is the functional generating the connected Green functions. We define: $\frac{\delta}{\delta J_i}Z[J,\beta] = \varphi_i[J,\beta] = \varphi_i$; φ_i is expressed as a formal series in J and β. Inverting the series we have: $J_i = J_i[\varphi,\beta]$. The Legendre transformation $W[\varphi,\beta] = Z[J[\varphi,\beta],\beta] - \int dx \Sigma_i \varphi_i J_i[\varphi,\beta]$ gives the functional generating the proper Green functions. Thus if S satisfies Eq. (9), W satisfies a corresponding system of Ward identities whose axial part is:

$$\begin{aligned}\partial_\mu \frac{\vec{\delta}}{\delta\alpha_\mu}W + \vec{\omega}_\mu \wedge \frac{\vec{\delta}}{\delta\alpha_\mu}W + \vec{\alpha}_\mu \wedge \frac{\vec{\delta}}{\delta\omega_\mu}W \\ + \frac{i}{2}\left[\overline{\psi}\vec{\tau}\gamma_5\frac{\delta}{\delta\overline{\psi}}W + W\frac{\delta}{\delta\psi}\vec{\tau}\gamma_5\psi\right] \\ + (\sigma+F)\frac{\vec{\delta}}{\delta\pi}W - \vec{\pi}\frac{\delta}{\delta\sigma}W = 0\,.\end{aligned} \tag{20}$$

(Here, as in Ref. [2], the definition of W has been changed by adding the term $\int dx\,c\sigma$). In the following we shall write Eq. (20) in the form:

$$\partial_\mu \vec{\delta}_{\alpha_\mu} W + [\vec{\mathscr{Y}} + \vec{\mathscr{X}} + F\vec{\delta}_\pi]\,W = 0 \tag{21}$$

where $\vec{\mathscr{Y}}W = \vec{\omega}_\mu \wedge \frac{\vec{\delta}}{\delta\alpha_\mu}W + \vec{\alpha}_\mu \wedge \frac{\vec{\delta}}{\delta\omega_\mu}W$ and

$$\vec{\mathscr{X}}W = \frac{i}{2}\left[\overline{\psi}\vec{\tau}\gamma_5\frac{\delta}{\delta\overline{\psi}}W + W\frac{\delta}{\delta\psi}\vec{\tau}\gamma_5\psi\right] + \sigma\frac{\vec{\delta}}{\delta\pi}W - \vec{\pi}\frac{\delta}{\delta\sigma}W.$$

Now we can translate into the functional language the results of Section 2. Let W_1 be the proper functional associated with $\mathscr{L}^{(1)}_{\text{eff}}$. W_1 is a formal power series in the fields:

$$W[\varphi] = \sum_{1}^{\infty}{}_n \frac{1}{n!} \int \prod_{1}^{n}{}_i\, dx_i \prod_{1}^{n}{}_i\, \varphi(x_i)\, \Gamma(x_1, \ldots, x_n)$$
$$= \sum_{1}^{\infty}{}_n \frac{(2\pi)^4}{n!} \int \prod_{1}^{n}{}_i\, dk_i \prod_{1}^{n}{}_i\, \tilde{\varphi}(k_i)\, \delta\left(\sum_{1}^{n}{}_i\, k_i\right) \tilde{\Gamma}\left(k_1, \ldots, k_{n-1}, -\sum_{1}^{n-1}{}_i\, k_i\right).$$

We can write $W_1 = \sum_{0}^{\infty}{}_n W_1^{(n)}$, where $W_1^{(n)}$ has degree n in the external fields $\vec{\alpha}_\mu$ and $\vec{\omega}_\mu$. The 1-current Ward identity which has been proved in Section 2 is:

$$\partial_\mu \vec{\delta}_{\alpha_\mu} W_1^{(1)} + [\vec{\mathscr{Y}} + \vec{\mathscr{X}} + F \vec{\delta}_\pi]\, W_1^{(0)} = 0\,. \tag{22}$$

The proper functional of the theory modified by adding seagull terms to $\mathscr{L}^{(1)}_{\text{eff}}$ is:

$$W = W_1 + \Sigma = \sum_{0}^{\infty}{}_n W^{(n)} \tag{23}$$

and it is:

$$W_1^{(0)} = W^{(0)} \quad \text{and} \quad W_1^{(1)} = W^{(1)}\,, \tag{24}$$

Σ will be determined up to local terms $\Delta\Sigma$ of dimension lower than 5 satisfying Eq. (21). It is easy to see that $\Delta\Sigma$ can depend on the external gauge fields only.

It is useful at this stage to introduce an operator which takes out from a given functional the local terms of dimension up to D. This is:

$$T_D W[\varphi] = (2\pi)^4\, \Sigma_n \frac{1}{n!} \int \prod_{1}^{n}{}_i\, dk_i \prod_{1}^{n}{}_i\, \tilde{\varphi}(k_i) \delta\left(\sum_{1}^{n}{}_i\, k_i\right) t_{D - \sum_1^n {}_i\, d\varphi_i}\, \tilde{\Gamma}(k) \tag{25}$$

where $d\varphi_j$ is the dimension of the field φ_j and $t_N \tilde{\Gamma}$ is the sum of the first N terms of the Taylor series of $\tilde{\Gamma}$ about the point $k=0$:

$$t_N \tilde{\Gamma}(k) = \tilde{\Gamma}(0) + \sum_{1}^{n-1}{}_i\, k_i^\mu [\partial^\mu_{k_i} \tilde{\Gamma}]_{k=0} + \cdots$$
$$+ \frac{1}{N!} \sum_{1}^{n-1}{}_{(i_1, \ldots, i_N)} k_{i_1}^{\mu_1} \ldots k_{i_N}^{\mu_N} [\partial^{\mu_1}_{k_{i_1}} \ldots \partial^{\mu_N}_{k_{i_N}} \tilde{\Gamma}]_{k=0}\,.$$

$T_D W[\varphi]$ is a local functional, that is, it can be written in the form:

$$T_D W[\varphi] = \int dx\, \mathscr{W}(x) \tag{26}$$

where $\mathscr{W}(x)$ is a function of the fields and of their derivatives at point x. We also define: $T_D \frac{\delta}{\delta\varphi(x)} W \equiv \frac{\delta}{\delta\varphi(x)} T_{D+d\varphi} W$. It follows that $T_D \varphi(x) \frac{\delta}{\delta\varphi(x)} W = \varphi(x) \frac{\delta}{\delta\varphi(x)} T_D W$ and $T_D \partial^\mu \frac{\delta}{\delta\varphi(x)} W = \partial^\mu \frac{\delta}{\delta\varphi(x)} T_{D+d\varphi-1} W$. At the end of Section 2 we showed that Eq. (21) is equivalent to its truncated version:

$$T_4[\partial_\mu \vec{\delta}_{\alpha_\mu} + \vec{\mathscr{Y}} + \vec{\mathscr{X}} + F\vec{\delta}_\pi] W = 0 . \tag{27}$$

By Eq. (22) we have:

$$\begin{aligned} T_{D-1} \int dx(\vec{\mathscr{X}} + F\vec{\delta}_\pi) W_1^{(0)} &= T_{D-1} \int dx(\vec{\mathscr{X}} T_{D-1} + F\vec{\delta}_\pi T_D) W_1^{(0)} \\ &= T_{D-1} \int dx(\vec{\mathscr{X}} + F\vec{\delta}_\pi) T_D W_1^{(0)} = 0 . \end{aligned} \tag{28}$$

The functional $T_D W_1^{(0)}$ satisfies the integrand axial Ward identity up to terms of dimension $D-1$. Putting $T_D W_1^{(0)} = \int dx \mathscr{W}_1^{(0)}(x)$; $\mathscr{W}_{1,D}^{(0)}(x)$ is a function of the quantized fields and of their derivatives. Let $\mathscr{W}_{1,D}^{(C)}(x)$ be obtained from $\mathscr{W}_{1,D}^{(0)}(x)$ by turning the derivatives of the fields into the corresponding covariant derivatives [Eq. (2)]. Let us then define:

$$W_{1,D} \equiv \sum_0^\infty {}_n W_{1,D}^{(n)} \equiv \mathscr{C} T_D W_1^{(0)}{}_D = \int dx \mathscr{W}_{1,D}^{(C)}(x) . \tag{29}$$

Taking into account Eq. (28) and recalling that the n^{th} covariant derivative of the field φ gives terms of dimension $n + d\varphi$ and $n + d\varphi - 1$ (coming for instance from $D_\mu \vec{\pi}$) we see that $W_{1,D}$ satisfies the n-current Ward identity up to terms of dimension $D-n$, that is:

$$T_{D-n}(\partial_\mu \vec{\delta}_{\alpha_\mu} W_{1,D}^{(n)} + (\vec{\mathscr{X}} + \vec{\mathscr{Y}} + F\vec{\delta}_\pi) W_{1,D}^{(n-1)}) = 0 . \tag{30}$$

Thus if we put:

$$T_{D-1}(W_1^{(1)} - W_{1,D}^{(1)}) = \Theta_1 \tag{31}$$

we get $\partial_\mu \vec{\delta}_{\alpha_\mu} \Theta_1 = 0$ by comparing Eqs. (22) and (30) and recalling that $T_D(W_1^{(0)} - W_{1,D}^{(0)}) = 0$ by definition. This implies that Θ_1 depends on the gauge fields through the antisymmetric tensor $\vec{\alpha}_{\mu\nu} = \partial_\mu \vec{\alpha}_\nu - \partial_\nu \vec{\alpha}_\mu$ and $\vec{\omega}_{\mu\nu} = \partial_\mu \vec{\omega}_\nu - \partial_\nu \vec{\omega}_\mu$. Thus we can conclude that Θ_1 contains neither terms of dimension 4 nor terms of dimension 5 which depend on the $\vec{\pi}$ field; hence:

$$T_4(\vec{\mathscr{X}} + \vec{\mathscr{Y}} + F\vec{\delta}_\pi) \Theta_1 = 0 . \tag{32}$$

Now we can prove the 2-current Ward identity. By Eqs. (23), (24), (30, (31) and by Eq. (32) we get:

$$\begin{aligned} &T_4(\partial_\mu \vec{\delta}_{\alpha_\mu} W^{(2)} + (\vec{\mathscr{Y}} + \vec{\mathscr{X}} + F\vec{\delta}_\pi) W^{(1)}) \\ &\quad = T_4\{\partial_\mu \vec{\delta}_{\alpha_\mu}(W^{(2)} - W_{1,D}^{(2)}) + (\vec{\mathscr{X}} + \vec{\mathscr{Y}} + F\vec{\delta}_\pi)(W^{(1)} - W_{1,D}^{(1)})\} \\ &\quad = \partial_\mu \vec{\delta}_{\alpha_\mu} T_4(\Sigma^{(2)} - W_{1,D}^{(2)}) . \end{aligned} \tag{33}$$

(From the subtraction rules it follows that $T_4 W^{(n)} = T_4 \Sigma^{(n)}$ for $n \geqq 2$). If we choose 2-current seagulls such that:

$$T_4 \Sigma^{(2)} = T_4 W^{(2)}_{1,D} + T_4 s^{(2)} \tag{34}$$

where

$$\partial_\mu \delta_{\alpha_\mu} T_4 s^{(2)} = 0 \tag{35}$$

the 2-current Ward identity is satisfied.

To prove the n-current Ward identity we shall iterate essentially the same procedure.

Let W_2 be the proper functional corresponding to $\mathscr{L}^{(1)}_{\text{eff}}$ modified by the 2-current seagulls. We have just proved that:

$$\partial_\mu \vec{\delta}_{\alpha_\mu} W_2^{(2)} + (\vec{\mathscr{Y}} + \vec{\mathscr{X}} + F\vec{\delta}_\pi)\, W_2^{(1)} = 0 \tag{36}$$

and we know that:

$$\begin{aligned} W^{(i)} &= W_2^{(i)} \qquad \text{for } i = 0, 1, 2 \\ W_2^{(i)} &= W_1^{(i)} \qquad \text{for } i = 0, 1\,. \end{aligned} \tag{37}$$

We define:

$$W_{2,D} \equiv \sum_0^\infty{}_n\, W^{(n)}_{2,D} = W_{1,D} + \mathscr{C}\Theta_1\,. \tag{38}$$

Here $\mathscr{C}$ transforms $\vec{\alpha}_{\mu\nu}$ and $\vec{\omega}_{\mu\nu}$ into the corresponding chiral covariant antisymmetric tensors $\vec{A}_{\mu\nu} = \vec{\alpha}_{\mu\nu} + \vec{\omega}_\mu \wedge \vec{\alpha}_\nu + \vec{\alpha}_\mu \wedge \vec{\omega}_\nu$ and $\vec{\Omega}_{\mu\nu} = \vec{\omega}_{\mu\nu} + \vec{\omega}_\mu \wedge \vec{\omega}_\nu + \vec{\alpha}_\mu \wedge \vec{\alpha}_\nu$. By Eqs. (30) and (36) we see that the functional $W_2^{(1)} - W^{(1)}_{1,D}$ satisfies the integrated Ward identity up to terms of dimension $D-2$; taking into account Eqs. (31) and (37) we see that the same is true for Θ_1. It follows that $\mathscr{C}\Theta_1$ satisfies the n-current Ward identity up to terms of dimension $D-n$; thus by Eq. (30) we get:

$$T_{D-n}(\partial_\mu \vec{\delta}_{\alpha_\mu} W^{(n)}_{2,D} + (\vec{\mathscr{X}} + \vec{\mathscr{Y}} + F\vec{\delta}_\pi)\, W^{(n-1)}_{2,D}) = 0\,. \tag{39}$$

Now we define

$$T_{D-2}(W_2^{(2)} - W^{(2)}_{2,D}) = \Theta_2\,. \tag{40}$$

Since by Eqs. (30), (37) and by Eq. (38) we know that $T_{D-1}(W_2^{(1)} - W^{(1)}_{2,D}) = 0$ comparing the Ward identities in Eq. (36) and in Eq. (39) yields $\partial_\mu \vec{\delta}_{\alpha_\mu} \Theta_2 = 0$. In much the same way as for Θ_1 we can say that:

i) Θ_2 does not contain terms of dimension lower than 4.

ii) the terms of Θ_2 of dimension 4 are given by:

$$T_4 \Theta_2 = T_4(W_2^{(2)} - W^{(2)}_{2,D}) = T_4(W^{(2)}_{1,D} + s^{(2)} + W^{(2)}_{2,D}) = T_4 s^{(2)}\,. \tag{41}$$

This is a consequence of Eqs. (34), (38), (40) and of the fact that $\mathscr{C}\Theta_1$ does not contain 2-current terms of dimension 4.

iii) The only possible term of $T_5 \Theta_2$ which depends on the $\vec{\pi}$ field is:

$$\lambda \int dx\, \vec{\pi} \cdot (\vec{\alpha}_{\mu\nu} \wedge \vec{\omega}_{\mu\nu})\,. \tag{42}$$

Now we discuss the 3-current Ward identity. Considering the terms of dimensions lower than 5 and taking into account Eqs. (23), (39), (41), and (42) we obtain:

$$\begin{aligned} &T_4(\partial_\mu \vec{\delta}_{\alpha_\mu} W^{(3)} + (\vec{\mathscr{X}} + \vec{\mathscr{Y}} + F\vec{\delta}_\pi)\, W^{(2)}) = \partial_\mu \vec{\delta}_{\alpha_\mu} T_4(\Sigma^{(3)} - W_{2,D}^{(3)}) \\ &\quad + T_4(\vec{\mathscr{X}} + \vec{\mathscr{Y}} + F\vec{\delta}_\pi)\,(W_2^{(2)} - W_{2,D}^{(2)}) = \partial_\mu \vec{\delta}_{\alpha_\mu} T_4(\Sigma^{(3)} - W_{2,D}^{(3)}) \\ &\quad + (\vec{\mathscr{X}} + \vec{\mathscr{Y}})\, T_4 s^{(2)} + F\lambda \vec{\alpha}_{\mu\nu} \wedge \vec{\omega}_{\mu\nu}\,. \end{aligned} \tag{43}$$

The functional $\hat{s} = \hat{s}^{(2)} + \hat{s}^{(3)} + \hat{s}^{(4)} = \frac{F\lambda}{2} \int dx (A_{\mu\nu})^2$ satisfies the Ward identity $(\partial_\mu \vec{\delta}_{\alpha_\mu} + \vec{\mathscr{Y}})\, \hat{s} = -\lambda F \vec{A}_{\mu\nu} \wedge \vec{\Omega}^2_{\mu\nu}$, which is at the 2-current level: $\partial_\mu \vec{\delta}_{\alpha_\mu} \hat{s}^{(2)} = 0$ and at the 3-current level is: $(\partial_\mu \vec{\delta}_{\alpha_\mu} \hat{s}^{(3)} + \vec{\mathscr{Y}} \hat{s}^{(2)}) = -\lambda F \vec{\alpha}_{\mu\nu} \wedge \vec{\omega}_{\mu\nu}$. Since $\hat{s}^{(2)}$ satisfies Eq. (35) we can choose $s^{(2)} = \hat{s}^{(2)}$ and $T_4(\Sigma^{(3)} - W_{2,D}^{(3)}) = \hat{s}^{(3)}$. Comparing with Eq. (43) we see that the 3-current Ward identity is satisfied for this choice of the seagull terms.
The proof of the 4 and 5-current Ward identity proceeds by a pedestrian iteration of the above procedure. We call W_3 the proper functional of $\mathscr{L}^{(1)}_{\text{eff}}$ equipped with the seagull terms: $T_4(W_{1,D}^{(2)} + W_{2,D}^{(2)}) + \hat{s}$. Up to 3-current terms W_3 coincides with W and satisfies the Ward identity. We define: $W_{3,D} = W_{2,D} + \mathscr{C}\Theta_2$ and $\Theta_3 = T_{D-3}(W_3^{(3)} - W_{3,D}^{(3)})$. Since $W_{3,D}$ satisfies the n-current Ward identity up to terms of dimension $D-n$, we have: $\partial_\mu \vec{\delta}_{\alpha_\mu} \Theta_3 = 0$. We conclude that Θ_3 contains no term of dimension lower than 6. The 4-current Ward identity up to terms of dimension 4 is:

$$T_4[\partial_\mu \vec{\delta}_{\alpha_\mu} W^{(4)} + (\vec{\mathscr{X}} + \vec{\mathscr{Y}} + F\vec{\delta}_\pi)\, W^{(3)}] = \partial_\mu \vec{\delta}_{\alpha_\mu} T_4(\Sigma^{(4)} - W_{3,D}^{(4)})\,. \tag{44}$$

Adding $T_4 W_{3,D}^{(4)}$ to the seagulls we define a theory in which the chiral Ward identity is satisfied up to the 4-current level. Repeating once more the same procedure one proves that the 5-current Ward identity is satisfied without the addition of any other seagull. The Ward identities with more than 5 currents are verified since, for $n > 5$,

$$T_4[\partial_\mu \vec{\delta}_{\alpha_\mu} W^{(n)} + (\vec{\mathscr{X}} + \vec{\mathscr{Y}} + F\vec{\delta}_\pi)\, W^{(n-1)}] = 0 \tag{45}$$

for dimensional reasons.

Thus we have proved the chiral current algebra Ward identities in the SU(2) × SU(2) σ-model.

4. The SU(3) × SU(3) Case

Now we see how the results of Section 3 can be extended to the case of other chiral groups; say for example: SU(3) × SU(3).

Let us assume that the 1-current axial Ward identity can be proved in the same way as in the SU(2) case[6], then the 2-current Ward identity can be proved using the method that leads to Eq. (36). However, passing from the 2-current to the 3-current Ward identity, a substantial difference appears: the terms of $T_5 \Theta_2$ which depend on the $\vec{\pi}$ field [see Eq. (42)] are now different:

$$H = \lambda \int dx \pi^i \{\lambda_0 f^{ijk} \alpha^j_{\mu\nu} \omega^k_{\mu\nu} + d^{ijk} (\lambda_1 \alpha^j_{\mu\nu} \omega^k_{\mu\nu} + \varepsilon^{\mu\nu\varrho\sigma}(\lambda_2 \omega^j_{\mu\nu} \omega^k_{\varrho\sigma} + \lambda_3 \alpha^j_{\mu\nu} \alpha^k_{\varrho\sigma}))\} \tag{46}$$

where f^{ijk} and d^{ijk} are the usual SU(3) symbols.

In the same way as in Section 3 we can prove that:

$$T_4(\partial_\mu \delta^i_{\alpha_\mu} W^{(3)} + (\mathscr{Y}^i + \mathscr{X}^i + F\delta^i_\pi) W^{(2)}) = G^{(2)i} \tag{47}$$

where:

$$G^{(2)i} = F d^{ijk} (\lambda_1 \alpha^j_{\mu\nu} \omega^k_{\mu\nu} + \varepsilon^{\mu\nu\varrho\sigma}(\lambda_2 \omega^j_{\mu\nu} \omega^k_{\varrho\sigma} + \lambda_3 \alpha^j_{\mu\nu} \alpha^k_{\varrho\sigma})) \tag{48}$$

$\mathscr{Y}^i$ and $\mathscr{X}^i$ are the functional differential operators corresponding to $\vec{\mathscr{Y}}$ and $\vec{\mathscr{X}}$ in the SU(2) case. Since $G^{(2)}$ does not depend on the quantized fields we can remove the operator T_4 form Eq. (47). As a consequence the 3-current axial Ward identity is in general anomalous, the anomaly being given in Eq. (48).

Now we consider the $n+1$-current Ward identity for $n>2$. Since the possible n-current anomalies $G^{(n)}$ are Lorentz invariant and have dimension lower than 5 only $G^{(3)}$ and $G^{(4)}$ can be non zero and they can only depend on the external gauge fields.

The axial Ward identity thus reads:

$$[\partial_\mu \delta^i_{\alpha_\mu} + \mathscr{Y}^i + \mathscr{X}^i + F\delta^i_\pi]\, W = G^i[\omega_\mu, \alpha_\mu] \tag{49}$$

where $G = G^{(2)} + G^{(3)} + G^{(4)}$ and $G^{(2)}$ is given by Eq. (48). Furthermore owing to the above statements G has to satisfy the Wess-Zumino consistency conditions [7]. By applying them to $G[\omega_\mu, \alpha_\mu]$ it can be seen that $\lambda_1 = 0$ and G has to be proportional to the Adler-Bardeen anomaly [6].

In the case of an abelian chiral group it can be shown that the anomaly is of the form:

$$\lambda_1 \alpha_{\mu\nu} \omega_{\mu\nu} + \varepsilon^{\mu\nu\varrho\sigma}(\lambda_2 \omega_{\mu\nu} \omega_{\varrho\sigma} + \lambda_3 \alpha_{\mu\nu} \alpha_{\varrho\sigma}). \tag{50}$$

A Comment. It is interesting to note that in principle our method can also be applied to the case in which the gauge fields are quantized. In this case the validity of the Ward identities can be proved by using our method order by order in $\hbar$ (in the loop number). A similar technique

[6] The proof is longer because the number of possible terms in the SU(3) lagrangian is greater than in the SU(2) case.

has been used by Piguet [14] in the case of the abelian Higgs-Kibble model.

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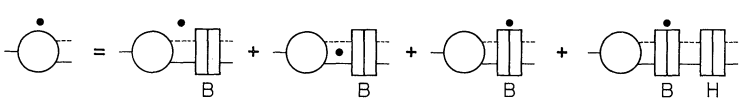
Appendix

Let $G_{j_1,\ldots,j_n}(p_1,p_2;q_1,\ldots,q_n)$ be a connected Green function; the variables p_1,p_2 are associated with fermion legs, $q_1,\ldots,q_n$ are associated with boson legs whose nature (either π or σ) is given by the indices $j_1,\ldots,j_n$. The connected Green function $\dot{G}_{j_1,\ldots,j_n}(p_1,p_2,q_1,\ldots,q_n)$ has one amputated π leg of zero momentum besides those explicitely indicated. The variables associated with amputated legs are underlined.

$H_{i,j}(p_1,p_2;p_3,p_4)$ is the function corresponding to the Feynman diagrams which are 1-particle irreducible with respect to cuts between the points (1,3) and (2,4). $B_{i,j}(\underline{p}_1,\underline{p}_2;\underline{p}_3,\underline{p}_4)$ and $\dot{B}_{i,j}(\underline{p}_1,\underline{p}_2;\underline{p}_3,\underline{p}_4)$ correspond to completely amputated Feynman diagrams which are 2-particle irreducible with respect to cuts separating the points (1,3) or (2,4) from the other ones. We denote by Γ the proper (1-particle irreducible) functions.

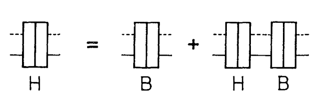
By a simple analysis of the topological structure of the diagrams contributing to $\dot{\Gamma}_i(\underline{o};\underline{p};-\underline{p})$ we have:

$$\begin{aligned}\dot{\Gamma}_i(\underline{o},\underline{p};-\underline{p}) = \int dq\,\Gamma_j(\underline{o},\underline{q};-\underline{q})\,[\Delta_j(q)\,\dot{\Gamma}_{j,Rj}(;\underline{q},-\underline{q})\,H_{Rj,i}(-q,\underline{p};q,-\underline{p})\\ +S(q)\,\dot{\Gamma}(-\underline{q},\underline{q};)\,H_{j,i}(-q,\underline{p};q,-\underline{p})+\Delta_j(q)\,S(q)\,(\dot{B}_{j,i}(-\underline{q},\underline{p};\underline{q},-\underline{p})\\ +\int dr\,\dot{B}_{j,k}(-\underline{q},\underline{r};\underline{q},-\underline{r})\,H_{k,i}(-r,\underline{p};r,-\underline{p}))]\,, \qquad \text{(A1)}\end{aligned}$$



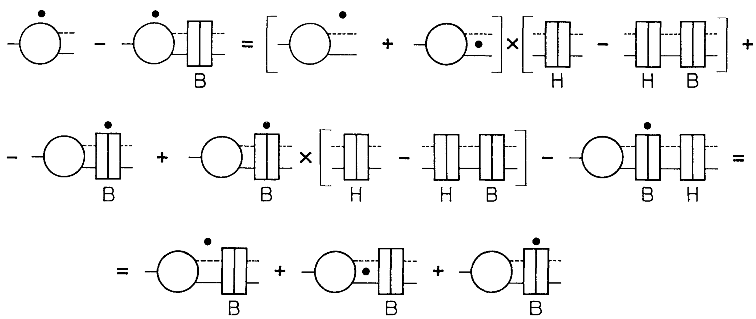
and also:

$$\begin{aligned}&H_{i,j}(-\underline{q},\underline{p};\underline{q},-\underline{p})\\ &\quad=B_{i,j}(-\underline{q},\underline{p};\underline{q},-\underline{p})+\int dr H_{i,k}(-\underline{q},r;\underline{q},-r)\,B_{k,j}(-\underline{r},\underline{p};\underline{r},-\underline{p})\end{aligned} \qquad \text{(A2)}$$



where $G_{\ldots R\pi\ldots}=G_{\ldots\sigma\ldots}$ and $G_{\ldots R\sigma\ldots}=-G_{\ldots\pi\ldots}$, $\Delta_j(q)$, and $S(q)$ are the boson and fermion propagators and the summation over repeated indices has been omitted.

From Eqs. (A1) and (A2), we have:



$$\dot{\Gamma}_i(\underline{\varrho},\underline{p};-\underline{p})=\int dq[\dot{\Gamma}_j(\underline{\varrho},\underline{q};-\underline{q})\,\Delta_j(q)\,S(q)\,B_{j,i}(-\underline{q},\underline{p};\underline{q},-\underline{p})$$
$$+\Gamma_j(\underline{\varrho},\underline{q};-\underline{q})\,\Delta_{Rj}(q)\,S(q)\,(\dot{\Gamma}_{j,Rj}(;\underline{q},-\underline{q})\,\Delta_{Rj}(q)\,B_{Rj,j}(-\underline{q},\underline{p};-\underline{q};\underline{p}) \quad \text{(A3)}$$
$$+\dot{\Gamma}(-\underline{q},\underline{q};)\,S(q)\,B_{j,i}(-\underline{q},\underline{p};\underline{q};-\underline{p})+\dot{B}_{j,i}(-\underline{q},\underline{p};\underline{q},-\underline{p}))]\,.$$

Now consider the expansion of $\dot{\Gamma}_i(\underline{\varrho},\underline{p};-\underline{p})$ up to n loops, assuming that the integrated axial Ward identity [2] is proved up to $n-1$ loops. The Green functions on the right-hand side of Eq. (A3) satisfy the following equations:

$$F\dot{\Gamma}_j(\underline{\varrho},\underline{q};-\underline{q})=\frac{i}{2}\{\gamma_5,\Gamma_j(\underline{\varrho},\underline{q};-\underline{q})\}+\Gamma_{Rj}(\underline{\varrho}\underline{q};-\underline{q})\,, \qquad \text{(A4)}$$

$$F\dot{B}_{i,j}(-\underline{q},\underline{p};\underline{q},-\underline{p})=\frac{i}{2}\{\gamma_5,B_{i,j}(-\underline{q},\underline{p};\underline{q},-\underline{p})\}$$
$$+B_{Ri,j}(-\underline{q},\underline{p};\underline{q},-\underline{p})+B_{i,Rj}(-\underline{q},\underline{p};\underline{q};\underline{q},\underline{p})\,, \qquad \text{(A5)}$$

$$F\dot{\Gamma}_{j,Rj}(;\underline{q},-\underline{q})=\Delta_j^{-1}(q)-\Delta_{Rj}^{-1}(q)\,, \qquad \text{(A6)}$$

$$F\dot{\Gamma}(-\underline{q},\underline{q};)=-\frac{i}{2}\{\gamma_5,S^{-1}(q)\}\,. \qquad \text{(A7)}$$

Taking into account Eqs. (A4–A7), Eq. (A3) becomes:

$$\dot{\Gamma}_i(\underline{\varrho},\underline{p};-\underline{p})=\int dq\left[\frac{i}{2}\{\gamma_5,\Gamma_j(\underline{\varrho},\underline{q};-\underline{q})\,\Delta_j(q)\,S(q)\,B_{j,i}(-\underline{q},\underline{p};\underline{q},-\underline{p})\}\right.$$
$$\left.+\Gamma_j(\underline{\varrho},\underline{q};-\underline{q})\,\Delta_j(q)\,S(q)\,B_{j,Ri}(-\underline{q},\underline{p};\underline{q},-p)\right]. \qquad \text{(A8)}$$

Here the divergent parts of the two integrands cancel since the integral in Eq. (A3) is not divergent.

From Eq. (A8) we have:

$$F\,\mathrm{Tr}(\dot{\Gamma}_\pi(\underline{o},\underline{o};\underline{o}))$$
$$=\int dq\,\mathrm{Tr}(\Gamma_j(\underline{o},\underline{q};-\underline{q})\,\Delta_j(\underline{q})\,S(\underline{q})\,(iB_{j,\pi}(-\underline{q},\underline{o};\underline{q},\underline{o})\,\gamma_5+B_{j,\sigma}(-\underline{q},\underline{o};\underline{q},\underline{o})))$$
$$=-iF\,\mathrm{Tr}(\gamma_5\dot{\Gamma}_\sigma(\underline{o},\underline{o};\underline{o}))\,.$$

Thus:
$$\langle\tilde{\sigma}(0)\,\tilde{\vec{\pi}}(0)\cdot\tilde{\bar{\psi}}(0)\,\gamma_5\vec{\tau}\tilde{\psi}(0)\rangle_+^{PROP}=i\langle(\tilde{\pi}(0))^2\,\tilde{\bar{\psi}}(0)\,\tilde{\psi}(0)\rangle_+^{PROP}$$

or, using the symbols of Ref. [2], $c_4=c_5$ to n loops. Thus the axial Ward identity is true to n loops.

Another recursive proof of Eq. (15) can be given by using power counting arguments [15].

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