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Existence of Ground States and KMS States for Approximately Inner Dynamics*

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Abstract. A strongly continuous one parameter group of $*$-automorphisms of a C^*-algebra with unit is said to be approximately inner if it can be approximated strongly by inner one parameter groups of $*$-automorphisms. It is shown that an approximately inner one parameter group of $*$-automorphisms has a ground state and, if there exists a trace state, a KMS state for all inverse temperatures. It follows that quantum lattice systems have ground states and KMS states. Conditions that a strongly continuous one parameter group of $*$-automorphisms of a UHF algebra be approximately inner are given in terms of the unbounded derivation which generates the automorphism group.

Introduction

Suppose $\{\alpha_t; -\infty < t < \infty\}$ is a strongly continuous one parameter group of $*$-automorphisms of a C^*-algebra $\mathfrak{A}$ with unit, where by strongly continuous we mean $\|\alpha_t(A) - A\| \to 0$ as $t \to 0$ for each $A \in \mathfrak{A}$. We say the group $\{\alpha_t\}$ is approximately inner if there exists a sequence $\{H_n\}$ of hermitian elements of $\mathfrak{A}$ such that

$$\|e^{itH_n} A e^{-itH_n} - \alpha_t(A)\| \to 0$$

as $n \to \infty$ for each $A \in \mathfrak{A}$ where for fixed A the convergence is uniform for t in a compact set. In this paper we will show that if $\{\alpha_t\}$ is approximately inner then there exists at least one ground state (Section 2) and there exist KMS states for all inverse temperatures β (Section 3) provided $\mathfrak{A}$ has a trace state. Since for quantum lattice systems the dynamics is given by approximately inner one parameter groups of $*$-automorphisms (see e.g. ([14], p. 193), [13] or [1]) it follows that quantum lattice systems have ground states and KMS states for all inverse temperatures β. Ruelle has shown the existence of ground states for quantum lattice systems in [15, Theorems 2(c) and 4].

In working with a strongly continuous one parameter group of $*$-automorphisms $\{\alpha_t\}$ it is often useful to introduce the unbounded derivation δ which generates the group. Suppose $\{\alpha_t\}$ is a strongly

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continuous group of $*$-automorphisms of a C^*-algebra $\mathfrak{A}$. The generator of the group $\{\alpha_t\}$ is a derivation δ given by

$$\delta(A) = \lim_{t\to 0} (\alpha_t(A) - A)/t$$

where the domain $\mathfrak{D}(\delta)$ of δ is the linear manifold of all $A \in \mathfrak{A}$ such that the above limit exists in the sense of norm convergence. It follows from semigroup theory (see [3] or [8]) and the fact the α_t are $*$-automorphisms that δ has the properties,

(i) $\mathfrak{D}(\delta)$ is a norm dense linear subset of $\mathfrak{A}$ and δ is a linear mapping of $\mathfrak{D}(\delta)$ into $\mathfrak{A}$.

(ii) $\mathfrak{D}(\delta)$ is an algebra and if $A, B \in \mathfrak{D}(\delta)$ then $AB \in \mathfrak{D}(\delta)$ and $\delta(AB) = \delta(A)\, B + A\delta(B)$.

(iii) $\mathfrak{D}(\delta)$ is a $*$-algebra and if $A \in \mathfrak{D}(\delta)$ then $A^* \in \mathfrak{D}(\delta)$ and $\delta(A^*) = \delta(A)^*$.

(iv) δ is closed, i.e., if $A_n \in \mathfrak{D}(\delta)$, $\|A_n - A\| \to 0$ and $\|\delta(A_n) - B\| \to 0$ as $n \to \infty$ then $A \in \mathfrak{D}(\delta)$ and $\delta(A) = B$.

Recently it was shown in [17] that if $\{\alpha_t\}$ is a strongly continuous one parameter group of $*$-automorphism of a UHF-algebra $\mathfrak{A}$ then there is an increasing sequence $M_1 \subset M_2 \subset \cdots$ of $(n_i \times n_i)$-matrix algebras so that $\mathfrak{A}_0 = \bigcup_{i=1}^{\infty} M_i$ is a norm dense $*$-subalgebra of $\mathfrak{A}$ and each element $A \in \mathfrak{A}_0$ is an analytic element for δ the generator of $\{\alpha_t\}$ [i.e., if $A \in \mathfrak{A}_0$ then $\alpha_t(A)$ can be extended to an analytic function which is holomorphic for $|\mathrm{Im}(t)| < r_0$ with $r_0 > 0$]. Furthermore, it was shown that there exists a sequence of hermitian elements $H_n \in \mathfrak{A}$ so that $i[H_n, A] = \delta(A)$ for all $A \in M_n$. It follows that if $A \in \mathfrak{A}_0$ we have $\delta(A) = \lim_{n\to\infty} i[H_n, A]$. We will show in Section 4 that if $\mathfrak{A}_0$ is a core for δ then $\{\alpha_t\}$ is approximately inner.

We end the paper with the conjecture that all strongly continuous one parameter groups of $*$-automorphisms of UHF-algebras are approximately inner. It would follow from the truth of this conjecture that all strongly continuous one parameter groups of $*$-automorphisms of UHF-algebras have ground states and KMS states for all inverse temperatures β.

Existence of Ground States

We begin this section with the definition of a ground state on a C^*-algebra with respect to a one parameter group of $*$-automorphisms. This definition is essentially the spectral condition of quantum field theory (see ([20], Chapter 3) and [2]).

Definition 2.1. Suppose $\{\alpha_t\}$ is a one parameter group of $*$-automorphisms of a C^*-algebra $\mathfrak{A}$ with unit. We say ω is a ground state of

$\mathfrak{A}$ for the group $\{\alpha_t\}$ if ω is a state of $\mathfrak{A}$ with the property, if $A, B \in \mathfrak{A}$ then $\omega(A\alpha_t(B))$ is a continuous function of t and

$$\int h(t)\, \omega(A\alpha_t(B))\, dt = 0$$

for all continuous L^1-functions h whose Fourier transform

$$\tilde{h}(\lambda) = \frac{1}{\sqrt{2\pi}} \int e^{-it\lambda} h(t)\, dt$$

vanishes on the negative real axis $(-\infty, 0]$.

We remark that a ground state ω for $\{\alpha_t\}$ is necessarily α_t invariant. To see this note that if $A = A^* \in \mathfrak{A}$ then the function $h(t) = \omega(\alpha_t(A))$ defines a tempered distribution (see [4] or [18]) by the relation

$$T(f) = \int f(t)\, h(t)\, dt$$

for all f in Schwartz's space. Since ω is a ground state the Fourier transform $\tilde{T}$ of T has support on the positive real axis $[0, \infty)$. Since h is a real valued function T is real, i.e., $\overline{T(t)} = T(t)$ for all t, and, therefore we have $\overline{\tilde{T}(k)} = \tilde{T}(-k)$ for the Fourier transform. Hence, $\tilde{T}$ has support on the negative real axis $(-\infty, 0]$. Thus, $\tilde{T}$ has its support at the single point 0. From the theory of distributions (see [4] or [18]) it follows that $\tilde{T}$ is a finite sum of derivatives of δ-functions at zero, i.e., $\tilde{T}(k) = \sum_{n=0}^{m} a_n \delta^{(n)}(k)$ and hence $T(t) = (2\pi)^{-1/2} \sum_{n=0}^{m} a_n(-it)^n$. Since h is a bounded function we have $T(t) = h(t) = a_0/\sqrt{2\pi}$ a constant. Hence $\omega(\alpha_t(A)) = \omega(A)$ for all t and all hermitian $A \in \mathfrak{A}$. Hence, ω is α_t invariant.

The fact that ω is a ground state has the following implications for the $*$-representation induced by ω. Suppose ω is an α_t invariant state of $\mathfrak{A}$ and $(\pi, \mathfrak{H}, f_0)$ is a cyclic $*$-representation of $\mathfrak{A}$ induced by ω on a Hilbert space $\mathfrak{H}$ with cyclic vector f_0 so that $\omega(A) = (f_0, \pi(A) f_0)$ for all $A \in \mathfrak{A}$. Since ω is α_t invariant we may define unitary operators $U(t)$ on $\mathfrak{H}$ by the relations

$$U(t)\, \pi(A) f_0 = \pi(\alpha_t(A)) f_0$$

for all $A \in \mathfrak{A}$. One can easily check that the above relations uniquely define isometries $U(t)$ of $\mathfrak{H}$ into $\mathfrak{H}$. From the group property of α_t and the continuity of $\omega(A^*\alpha_t(A))$ in t for all $A \in \mathfrak{A}$ it follows that $t \to U(t)$ is a strongly continuous one parameter group of unitary operators with the additional properties,

$$U(t)\, \pi(A)\, U(t)^{-1} = \pi(\alpha_t(A))$$

and

$$U(t) f_0 = f_0$$

for all real t and all $A \in \mathfrak{A}$.

From Stone's theorem (see e.g. ([12], Chapter X)) it follows there is a self-adjoint operator H which generates the one parameter group $U(t)=e^{itH}$. Since $U(t)f_0=f_0$ for all real t we have $f_0=\mathfrak{D}(H)$, the domain of H, and $Hf_0=0$. Let $\{E(\lambda); -\infty<\lambda<\infty\}$ be the spectral resolution of H, i.e.,

$$H=\int dE(\lambda) \quad \text{and} \quad U(t)=\int e^{it\lambda}dE(\lambda).$$

For $A, B\in\mathfrak{A}$ and h an L^1-function we have

$$\begin{aligned}\int h(t)\,\omega(A\alpha_t(B))\,dt &= \int h(t)\,(f_0, \pi(A)\,U(t)\,\pi(B)\,f_0)\,dt\\ &= \int h(t)\int e^{it\lambda}(\pi(A^*)\,f_0\,, dE(\lambda)\pi(B)\,f_0)dt\\ &= \sqrt{2\pi}\int \tilde{h}(-\lambda)\,(\pi(A^*)\,f_0\,, dE(\lambda)\,\pi(B)\,f_0)\end{aligned}$$

where in the last equality we have carried out the t integration. We have ω is a ground state if and only if the above integral vanishes for all $A, B\in\mathfrak{A}$ provided $\tilde{h}$ vanishes on the negative real axis $(-\infty, 0]$. Since $\{\pi(A^*)f_0; A\in\mathfrak{A}\}$ and $\{\pi(B)f_0; B\in\mathfrak{A}\}$ are dense in $\mathfrak{H}$ we have the fact that ω is a ground state is equivalent to the fact that the spectral measure $E(\lambda)$ has its support on the positive real axis $[0,\infty)$. Hence, ω is a ground state if and only if H is positive, i.e., $H\geqq 0$.

If ω is a ground state and if we associate the self-adjoint operator H with the energy of a physical system then the vector f_0 is a vector of norm one which minimizes the energy (f, Hf) with $\|f\|=1$. This is the origin of the term "ground state" for the state $\omega(A)=(f_0, \pi(A)f_0)$ for all $A\in\mathfrak{A}$.

The following theorem may be useful in characterizing ground states in terms of unbounded derivations.

Theorem 2.2. *Suppose $\{\alpha_t\}$ is a strongly continuous one parameter group of $*$-automorphisms of a C^*-algebra $\mathfrak{A}$ with unit. Suppose δ is the generator of $\{\alpha_t\}$ and $\mathfrak{D}$ is a core for δ. Then, a state ω is a ground state for $\{\alpha_t\}$ if and only if*

$$-i\omega(A^*\delta(A))\geqq 0$$

for all $A\in\mathfrak{D}$.

Proof. First suppose ω is a ground state for $\{\alpha_t\}$. Let $(\pi, \mathfrak{H}, f_0)$ be a cyclic $*$-representation of $\mathfrak{A}$ induced by ω with cyclic vector $f_0\in\mathfrak{H}$ so that $\omega(A)=(f_0, \pi(A)f_0)$ for all $A\in\mathfrak{A}$. We have from the previous discussion that there is a strongly continuous one parameter group of unitary operators $U(t)=e^{itH}$ with $U(t)f_0=f_0$ and

$$U(t)\,\pi(A)\,U(t)^{-1}=\pi(\alpha_t(A))$$

for all real t and $A\in\mathfrak{A}$. Since ω is a ground state we have that the generator H of $\{U(t)\}$ is positive, i.e., $H\geqq 0$.

Now if $A \in \mathfrak{D}(\delta)$ we have

$$(it)^{-1}(U(t)-I)\,\pi(A)\,f_0 = (it)^{-1}\,\pi(\alpha_t(A)-A)\,f_0 \to -i\pi(\delta(A))\,f_0$$

as $t \to 0$. Hence, from Stone's theorem we have $\pi(A)\,f_0 \in \mathfrak{D}(H)$, the domain of H, and $H\pi(A)\,f_0 = -i\pi(\delta(A))\,f_0$. Now, if $A \in \mathfrak{D} \subset \mathfrak{D}(\delta)$ we have

$$\begin{aligned} -i\omega(A^*\delta(A)) &= -i(f_0, \pi(A^*)\,\pi(\delta(A))\,f_0) = (f_0, \pi(A^*)\,H\pi(A)\,f_0) \\ &= (\pi(A)\,f_0, H\pi(A)\,f_0) \geqq 0\,. \end{aligned}$$

Hence, if ω is a ground state for $\{\alpha_t\}$ then $-i\omega(A^*\delta(A)) \geqq 0$ for all $A \in \mathfrak{D} \subset \mathfrak{D}(\delta)$.

Next suppose ω is a state of $\mathfrak{A}$ so that $-i\omega(A^*\delta(A)) \geqq 0$ for all $A \in \mathfrak{D} \subset \mathfrak{D}(\delta)$ where $\mathfrak{D}$ is a core for δ. We first show $-i\omega(A^*\delta(A)) \geqq 0$ for all $A \in \mathfrak{D}(\delta)$. Since $\mathfrak{D}$ is a core for δ there is for each $A \in \mathfrak{D}(\delta)$ a sequence $\{A_n \in \mathfrak{D}\}$ so that $\|A_n - A\| \to 0$ and $\|\delta(A_n) - \delta(A)\| \to 0$ as $n \to \infty$. Since multiplication is jointly continuous we have $\|A_n^*\delta(A_n) - A^*\delta(A)\| \to 0$ as $n \to \infty$. Hence, if $A \in \mathfrak{D}(\delta)$ we have

$$-i\omega(A^*\delta(A)) = \lim_{n\to\infty} -i\omega(A_n^*\delta(A_n)) \geqq 0\,.$$

Hence, $-i\omega(A^*\delta(A)) \geqq 0$ for all $A \in \mathfrak{D}(\delta)$.

Next we will show ω is α_t invariant. Since $\alpha_t(I) = I$ for all real t it follows that $I \in \mathfrak{D}(\delta)$ and $\delta(I) = 0$. If $A \in \mathfrak{D}(\delta)$ and λ is a complex number we have $-i\omega((\lambda I + A)^*\,\delta(\lambda I + A)) \geqq 0$. Hence, we have $-i\omega(\bar{\lambda}\delta(A) + A^*\delta(A)) \geqq 0$ for all complex λ. Hence, $\omega(\delta(A)) = 0$ for all $A \in \mathfrak{D}(\delta)$. Since α_t maps $\mathfrak{D}(\delta)$ into $\mathfrak{D}(\delta)$ we have for all $A \in \mathfrak{D}(\delta)$

$$\frac{d}{dt}\,\omega(\alpha_t(A)) = \omega(\delta(\alpha_t(A))) = 0\,.$$

Hence, $\omega(\alpha_t(A)) = \omega(A)$ for real t and $A \in \mathfrak{D}(\delta)$. Since $\mathfrak{D}(\delta)$ is norm dense in $\mathfrak{A}$ we have ω is α_t invariant.

Let $(\pi, \mathfrak{H}, f_0)$ be the cyclic $*$-representation induced by ω and let $t \to U(t)$ be the strongly continuous one parameter unitary group defined by the relations

$$U(t)\,\pi(A)\,f_0 = \pi(\alpha_t(A))\,f_0$$

for t real and all $A \in \mathfrak{A}$. Let H be the generator of the group $\{U(t)\}$, i.e., $U(t) = e^{itH}$. To prove ω is a ground state we must show $H \geqq 0$. Suppose $A \in \mathfrak{D}(\delta)$. Then, we have

$$(it)^{-1}(U(t)-I)\,\pi(A)\,f_0 = (it)^{-1}\,\pi(\alpha_t(A)-A)\,f_0 \Rightarrow -i\pi(\delta(A))\,f_0$$

as $t \to 0$.

Hence, from Stone's theorem we have $\pi(A) f_0 \in \mathfrak{D}(H)$ and $H\pi(A) f_0 = -i\pi(\delta(A)) f_0$ for all $A \in \mathfrak{D}(\delta)$. We have for $A \in \mathfrak{D}(\delta)$

$$\begin{aligned}(\pi(A) f_0, H\pi(A) f_0) &= -i(\pi(A) f_0, \pi(\delta(A)) f_0)\\ &= -i(f_0, \pi(A^*\delta(A)) f_0)\\ &= -i\omega(A^*\delta(A)) \geqq 0 .\end{aligned}$$

Let H_1 be the closure of the restriction of H to $\{\pi(\mathfrak{D}(\delta)) f_0\}$, i.e., $H_1 = \overline{H \mid \{\pi(\mathfrak{D}(\delta)) f_0\}}$. From the above inequality we have H_1 is positive. We will show H is positive by showing $H_1 = H$.

Since H_1 is a restriction of H (i.e., $H_1 \subset H$) we have H_1^* is an extension of $H^* = H$. Hence, we have $H_1 \subset H \subset H_1^*$. We will show $H_1 = H_1^*$ thereby showing $H = H_1$.

We have $U(t)\{\pi(\mathfrak{D}(\delta)) f_0\} = \{\pi(\alpha_t(\mathfrak{D}(\delta)))\} f_0\} = \{\pi(\mathfrak{D}(\delta)) f_0\}$ since $\mathfrak{D}(\delta)$ is invariant under α_t. Since $\{\pi(\mathfrak{D}(\delta)) f_0\}$ is a dense linear manifold of $\mathfrak{D}(H_1)$ invariant under $U(t)$ it follows from Lemma 2 of [19] that H_1 is self-adjoint. Hence, $H = H_1$ is positive and ω is a ground state. This completes the proof of the theorem.

Theorem 2.3. *Suppose $\{\alpha_t\}$ is a strongly continuous one parameter group of $*$-automorphisms of a C^*-algebra $\mathfrak{A}$ with unit. Suppose $\{\alpha_t\}$ is approximately inner. Then, there exists a ground state ω for $\{\alpha_t\}$. The ground state need not be unique.*

Proof. Suppose the hypothesis of the theorem are satisfied. Since $\{\alpha_t\}$ is approximately inner there is a sequence of hermitian elements $\{H_n \in \mathfrak{A}\}$ so that $\|e^{itH_n} A e^{-itH_n} - \alpha_t(A)\| \to 0$ as $n \to \infty$ for all $A \in \mathfrak{A}$ where for fixed $A \in \mathfrak{A}$ the convergence is uniform on compact sets. By adding a multiple of the unit to H_n we can arrange it so that H_n is positive and zero is in the spectrum of H_n, i.e., $0 \in \sigma(H_n)$ and $H_n \geqq 0$ for $n = 1, 2, \ldots$. Since $0 \in \sigma(H_n)$ it follows from ([16] or ([10], p. 306) there is a state ω_n of $\mathfrak{A}$ so that $\omega_n(H_n) = \omega_n(H_n^2) = 0$. Since the state space of a C^*-algebra is compact in the weak $*$-topology there is a state ω which is a cluster point of the sequence $\{\omega_n\}$ in the weak $*$-topology. We will show ω is a ground state.

Suppose h is a continuous L^1-function whose Fourier transform $\tilde{h}$ vanishes on the negative real axis $(-\infty, 0]$. Suppose $A, B \in \mathfrak{A}$. We will show

$$\int h(t)\, \omega(A\alpha_t(B))\, dt = 0 .$$

Let

$$B_0 = \int h(t)\, \alpha_t(B)\, dt \quad \text{and} \quad B_n = \int h(t)\, e^{itH_n} A e^{-itH_n} dt .$$

Suppose $\varepsilon > 0$. Since $h \in L^1$ there is a constant c so that

$$2\|B\| \int_{|t|>c} |h(t)|\, dt < \varepsilon/2 .$$

Since $e^{itH_n}Be^{-itH_n}$ converges to $\alpha_t(B)$ uniformly for $|t|\leqq c$ there is an integer n_0 so that $\|h\|_1 \|e^{itH_n}Be^{-itH_n}-\alpha_t(B)\|<\varepsilon/2$ for $|t|\leqq c$ and $n\geqq n_0$ where $\|h\|_1$ is the L^1 norm of h. For $n\geqq n_0$ we have

$$\begin{aligned}\|B_n-B_0\| &\leqq \|\int h(t)\,(e^{itH_n}Be^{-itH_n}-\alpha_t(B))\,dt\|\\ &\leqq \int_{-c}^{+c} |h(t)|\;\|e^{itH_n}Be^{-itH_n}-\alpha_t(B)\|\,dt\\ &\quad+\int_{|t|>c} |h(t)|\;\|e^{itH_n}Be^{-itH_n}-\alpha_t(B)\|\,dt\\ &\leqq \int_{-c}^{c} |h(t)|\,(\|h\|_1)^{-1}\,(\varepsilon/2)\,dt+2\|B\|\int_{|t|>c}|h(t)|\,dt\\ &\leqq \varepsilon/2+\varepsilon/2=\varepsilon\,.\end{aligned}$$

Since ω is a cluster point of the sequence $\{\omega_n\}$ in the weak $*$-topology there is an integer $r\geqq n_0$ so that $|\omega_r(AB_0)-\omega(AB_0)|<\varepsilon$. Now, we have

$$\begin{aligned}\omega_r(AB_r) &= \int h(t)\,\omega_r(Ae^{itH_r}Be^{-itH_r})\,dt\\ &= \int h(t)\,\omega_r(Ae^{itH_r}B)\,dt\\ &= \sqrt{2\pi}\,\omega_r(A\tilde{h}(-H_r)\,B)=0\end{aligned}$$

where we have $\tilde{h}(-H_r)=0$ since the Fourier transform $\tilde{h}$ of h is a continuous function which vanishes on the negative real axis $(-\infty,0]$ and the spectrum of $-H_r$ is contained in this interval. Hence, we have

$$\begin{aligned}|\omega(AB_0)| &\leqq |\omega(AB_0)-\omega_r(AB_0)|+|\omega_r(AB_0)-\omega_r(AB_r)|+|\omega_r(AB_r)|\\ &\leqq \varepsilon+\|A(B_0-B_r)\|+0\\ &\leqq \varepsilon+\|A\|\,\varepsilon\,.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary we have

$$\omega(AB_0)=\int h(t)\,\omega(A\alpha_t(B))\,dt=0\,.$$

Hence, ω is a ground state. This completes the proof of the theorem.

Remark. Since for quantum lattice systems the dynamics is given by an approximately inner one parameter group of $*$-automorphisms (see e.g. ([14], p. 193), [13] and [1]), it follows that quantum lattice systems have ground states. Ruelle has shown the existence of ground states for quantum lattice systems in [15, Theorems 2(c) and 4]. We thank the referee for pointing out this reference to us.

Existence of KMS States

We begin this section with the definition of KMS states (see [14, 13] and [6]).

Definition 3.1. Suppose $\{\alpha_t\}$ is a one parameter group of $*$-automorphisms of a C^*-algebra $\mathfrak{A}$ with unit. We say ω is a KMS state for $\{\alpha_t\}$ of inverse temperature $\beta>0$ if for each $A, B\in\mathfrak{A}$ there exists an analytic function F which is holomorphic for $0<\mathrm{Im}(z)<\beta$ and continuous for $0\leqq \mathrm{Im}(z)\leqq\beta$ so that

$$\omega(A\alpha_t(B))=F(t) \quad \text{and} \quad \omega(\alpha_t(A)\,B)=F(t+i\beta)$$

for all real t.

As in the case of ground states it follows that if ω is a KMS state for $\{\alpha_t\}$ then ω is α_t invariant. It is thought that KMS states describe physical systems in thermal equilibrium where the dynamics is given by the $*$-automorphism group $\{\alpha_t\}$ (see [14]).

Theorem 3.2. *Suppose $\{\alpha_t\}$ is a strongly continuous one parameter group of $*$-automorphisms of a C^*-algebra $\mathfrak{A}$ with unit. Suppose $\{\alpha_t\}$ is approximately inner. Furthermore, suppose $\mathfrak{A}$ has one or more trace states τ [i.e., $\tau(AB)=\tau(BA)$ for all $A, B\in\mathfrak{A}$]. Then, there exists at least one* KMS *state ω_β for all inverse temperatures $\beta>0$.*

Proof. Suppose the hypothesis of the theorem is satisfied and π is a trace state of $\mathfrak{A}$ and $\beta>0$. Since $\{\alpha_t\}$ is approximately inner there is a sequence of hermitian elements $\{H_n\in\mathfrak{A}\}$ so that $\|e^{itH_n}Ae^{-itH_n}-\alpha_t(A)\|\to 0$ as $n\to\infty$ for each fixed $A\in\mathfrak{A}$ where the convergence is uniform for t in a compact set. Let $\omega_n(A)=\tau(e^{-\beta H_n}A)/\tau(e^{-\beta H_n})$ for all $A\in\mathfrak{A}$. A straight forward computation shows that the ω_n are states of $\mathfrak{A}$ which satisfy the KMS condition for the automorphism groups $\alpha_t^{(n)}(A)=e^{itH_n}Ae^{-itH_n}$ for $A\in\mathfrak{A}$. Since the state space of a C^*-algebra with unit is compact in the weak $*$-topology there is a state ω which is a cluster point of the sequence $\{\omega_n\}$ in the weak $*$-topology. We will show ω satisfies the KMS condition.

Suppose $A, B\in\mathfrak{A}$. Let $\mathfrak{A}_0$ be the smallest C^*-subalgebra of $\mathfrak{A}$ containing $\{A, \alpha_t(B), H_n;\ -\infty<t<\infty,\ n=1,2,\ldots\}$. Since $\alpha_t(B)$ is norm continuous in t, $\mathfrak{A}_n$ is norm separable. Hence, there is a subsequence $\{\omega_{n(k)}\}$ of the sequence $\{\omega_n\}$ which converges weakly to ω on $\mathfrak{A}_0$ as $k\to\infty$. Let

$$F_k(z)=\omega_{(n)(k)}(Ae^{izH_{n(k)}}Be^{-izH_{n(k)}})\,.$$

We have F_k is an entire analytic function which is bounded in the strip $0\leqq\mathrm{Im}(z)\leqq\beta$. Since analytic functions are harmonic we can express $F_k(z)$ for $0\leqq\mathrm{Im}(z)\leqq\beta$ in terms of $F_k(z)$ on the lines $\mathrm{Im}(z)=0$ and $\mathrm{Im}(z)=\beta$, i.e.,

$$F_k(z)=\int K_1(t,z)\,f_{1k}(t)\,dt+\int K_2(t,z)\,f_{2k}(t)\,dt$$

for $0 < \mathrm{Im}(z) < \beta$ where

$$f_{1k}(t) = F_k(t) = \omega_{n(k)}(Ae^{itH_{n(k)}}Be^{-itH_{n(k)}})$$
$$f_{2k}(t) = F_k(t+i\beta) = \omega_{n(k)}(e^{itH_{n(k)}}Ae^{-itH_{n(k)}}\beta)$$

for all real t (here in the second equation we have used the fact that $\omega_{n(k)}$ satisfies the KMS condition). The functions K_1 and K_2 are positive L^1-functions of t for each fixed z and $\|K_1(\cdot, z)\|_1 + \|K_2(\cdot, z)\|_1 = 1$ for all $0 < \mathrm{Im}(z) < \beta$ where $\|K_1(\cdot, z)\|_1$ *is the* L^1 norm of the function $h(t) = K_1(t, z)$, $i = 1, 2$ (see [7], section 18.2).

We wish to thank Professor R. Herman for pointing out to us this integral representation of a function harmonic in the strip and there by greatly simplifying the proof of this theorem.

We have

$$\begin{aligned}|\omega(A\alpha_t(B)) - f_{1k}(t)| &\leqq |\omega(A\alpha_t(B)) - \omega_{n(k)}(A\alpha_t(B))| \\ &\quad + |\omega_{n(k)}(A\alpha_t(B)) - \omega_{n(k)}(Ae^{itH_{n(k)}}Be^{-itH_{n(k)}})| \\ &\leqq |\omega(A\alpha_t(B)) - \omega_{n(k)}(A\alpha_t(B))| \\ &\quad + \|A\| \, \|\alpha_t(B) - e^{itH_{n(k)}}Be^{-itH_{n(k)}}\| \,.\end{aligned}$$

Since $\{A\alpha_t(B)\}$ is norm compact for t in a compact set and $\omega_{n(k)}$ converges weakly to ω on $\mathfrak{A}_0$ we have $|\omega(A\alpha_t(B)) - \omega_{n(k)}(A\alpha_t(B))|$ tends to zero uniformly on compact sets as $k \to \infty$. Since the second term in the above inequality tends to zero uniformly on compact sets we have $f_{1k}(t)$ tends to $\omega(A\alpha_t(B))$ uniformly for t in a compact set. A similar calculation shows that $f_{2k}(t)$ tends to $\omega(\alpha_t(B)A)$ uniformly for t in a compact set. It follows from the integral representation of $F_k(z)$, $0 < \mathrm{Im}(z) < \beta$, and the facts that f_{1k} and f_{2k} are uniformly bounded [in fact, $|f_{1k}(t)| \leqq \|A\| \, \|B\|$ and $|f_{2k}(t)| \leqq \|A\| \, \|B\|$ for all real t and $k = 1, 2, \ldots$] and $\|K_1(\cdot, z)\|_1 + \|K_2(\cdot, z)\|_1 = 1$, that $F_k(z)$ converges to an analytic function $F(z)$ which is holomorphic for $0 < \mathrm{Im}(z) < \beta$ and bounded and continuous for $0 \leqq \mathrm{Im}(z) \leqq \beta$ and the convergence is uniform on compact subsets of the strip $0 \leqq \mathrm{Im}(z) \leqq \beta$.

Since $F(t) = \omega(A\alpha_t(B))$ and $F(t+i\beta) = \omega(\alpha_t(B)A)$ it follows that ω satisfies the KMS condition. This completes the proof of the theorem.

Remark. Since for quantum lattice systems the dynamics is given by an approximately inner one parameter group of $*$-automorphisms (see ([14], p. 193), [13] and [1]) and since the C^*-algebra describing quantum lattice systems have trace states it follows that these systems have KMS states for all inverse temperatures $\beta > 0$. Actually, we have KMS states exist for all inverse temperatures both positive and negative since the automorphism group $\{\alpha'_t = \alpha_{-t}\}$ is approximately inner if and only if $\{\alpha_t\}$ is approximately inner. It is the usual convention to define KMS states only for positive temperatures.

Unbounded Derivations of UHF Algebras

A uniformly hyperfinite (UHF) algebra is a C^*-algebra $\mathfrak{A}$ which contains an increasing sequence $M_1 \subset M_2 \subset \cdots$ of $(n_i \times n_i)$-matrix algebras whose union $\mathfrak{A}_0 = \bigcup_{i=1}^{\infty} M_i$ is a norm dense $*$-subalgebra of $\mathfrak{A}$. UHF algebras were introduced and studied by Glimm [5].

Suppose $\{\alpha_t\}$ is a strongly continuous one parameter group of $*$-automorphisms of a UHF algebra $\mathfrak{A}$ and δ is the derivation which generates $\{\alpha_t\}$. An element $A \in \mathfrak{A}$ is said to be an analytic element for δ or $\{\alpha_t\}$ if the function $t \to \alpha_t(A)$ can be extended to an analytic function in the strip $|\mathrm{Im}(z)| < r$ with $z = t + iy$ and $r > 0$. It follows from Nelson's paper [11] that an element $A \in \mathfrak{A}$ is an analytic element if and only if $A \in \mathfrak{D}(\delta)$, $\delta(A) \in \mathfrak{D}(\delta)$, $\delta(\delta(A)) = \delta^2(A) \in \mathfrak{D}(\delta), \ldots$ and

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} \|\delta^n(A)\| < \infty$$

for all $0 \leqq s \leqq r$ with $r > 0$.

Recently, it was shown in [17] that if $\{\alpha_t\}$ is a strongly continuous one parameter group of $*$-automorphisms of a UHF algebra $\mathfrak{A}$ and δ is the derivation which generates $\{\alpha_t\}$, then there exists an increasing sequence $M_1 \subset M_2 \subset \cdots \subset \mathfrak{A}$ of $(n_1 \times n_1)$-matrix algebras whose union $\mathfrak{A}_0 = \bigcup_{i=1}^{\infty} M_i$ is a norm dense $*$-subalgebra of $\mathfrak{A}$ and furthermore, each element $A \in \mathfrak{A}_0$ (i.e., $A \in M_n$ for some integer n) is an analytic element for δ. Furthermore, for each matrix algebra M_n there is an hermitian element $H_n \in \mathfrak{A}$ so that $\delta(A) = i[H_n, A]$ for all $A \in M_n$. It follows that if $A \in \mathfrak{A}_0$ we have $\delta(A) = \lim_{n \to \infty} i[H_n, A]$. We will show that if $\mathfrak{A}_0$ is a core for δ then $\{\alpha_t\}$ is approximately inner. First we consider the question of when a $*$-derivation of $\mathfrak{A}_0$ into $\mathfrak{A}$ uniquely defines a one parameter group of $*$-automorphisms.

Theorem 4.1. *Suppose $\mathfrak{A}$ is a* UHF *algebra and $M_1 \subset M_2 \subset \cdots \subset \mathfrak{A}$ is an increasing sequence of $(n_i \times n_i)$-matrix algebras whose union $\mathfrak{A}_0 = \bigcup_{i=1} M_i$ is a norm dense $*$-subalgebra of $\mathfrak{A}$. Suppose δ is a $*$-derivation of $\mathfrak{A}_0$ into $\mathfrak{A}$, i.e., δ is a linear mapping of $\mathfrak{A}_0$ into $\mathfrak{A}$ with the properties*

(i) $\delta(AB) = \delta(A)B + A\delta(B)$ *for* $A, B \in \mathfrak{A}_0$.

(ii) $\delta(A^*) = \delta(A)^*$ *for* $A \in \mathfrak{A}_0$.

Then, δ is closable, i.e., there is a unique closed derivation $\bar{\delta}$ with domain $\mathfrak{D}(\bar{\delta}) \supset \mathfrak{A}_0$ so that $\bar{\delta}(A) = \delta(A)$ for all $A \in \mathfrak{A}_0$ and for all $A \in \mathfrak{D}(\bar{\delta})$ there is a sequence $A_n \in \mathfrak{A}_0$ so that $\|A_n - A\| \to 0$ and $\|\bar{\delta}(A) - \delta(A_n)\| \to 0$ as $n \to \infty$. Furthermore, $\bar{\delta}$ is the generator of a strongly continuous one *parameter group of $*$-automorphisms of $\mathfrak{A}$ if and only if the only norm continuous linear functionals $\varrho_\pm \in \mathfrak{A}^*$ satisfying the equations*

$$\varrho_+(A+\delta(A))=0$$
$$\varrho_-(A-\delta(A))=0$$

for all $A\in\mathfrak{A}_0$ are the zero functionals $\varrho_+=\varrho_-=0$.

Proof. Suppose δ is a $*$-derivation of $\mathfrak{A}_0$ into $\mathfrak{A}$. First we show δ is closable. To show δ is closable it is sufficient to show that if $A_n\in\mathfrak{A}_0$ and $\|\delta(A_n)-B\|\to 0$ and $\|A_n\|\to 0$ as $n\to\infty$ then $B=0$.

Let τ be the unique trace state of $\mathfrak{A}$ i.e., $\tau(AB)=\tau(BA)$ for all $A, B\in\mathfrak{A}$ and τ is a state of $\mathfrak{A}$. We will show that $\tau(\delta(A))=0$ for all $A\in\mathfrak{A}_0$. Let $\{e_{ij}^{(n)};\, i,j=1,\dots,m(n)\}$ be a family of matrix units which span $M_n\subset\mathfrak{A}_0$. Let H_n be the hermitian element of $\mathfrak{A}$ given by

$$H_n=(-i/m(n))\sum_{i,j=1}^{m(n)}\delta(e_{ij}^{(n)})\,e_{ji}^{(n)}.$$

A straight forward computation shows $\delta(A)=i[H_n,A]$ for all $A\in M_n$. Hence, we have for $A\in M_n$ $\tau(\delta(A))=i\tau([H_n,A])=0$. Since $\mathfrak{A}_0=\bigcup_{n=1}^{\infty} M_n$ we have $\tau(\delta(A))=0$ for all $A\in\mathfrak{A}_0$.

We define an inner product $(A,B)=\tau(A^*B)$ on $\mathfrak{A}$. Let $\mathfrak{H}$ be the Hilbert space obtained by completing $\mathfrak{A}$ with respect to this inner product. We consider $\mathfrak{A}_0\subset\mathfrak{A}\subset\mathfrak{H}$ as dense subsets of $\mathfrak{H}$. Consider the linear operator Γ from $\mathfrak{A}_0$ into $\mathfrak{H}$ given by $\Gamma A=i\delta(A)$ for $A\in\mathfrak{A}_0$. We have Γ is hermitian since for $A,B\in\mathfrak{A}_0$ we have

$$(A,\Gamma B)=\tau(A^*i\delta(B))=i\tau(A^*\delta(B))$$
$$(\Gamma A,B)=\tau((i\delta(A))^*B)=-i\tau(\delta(A^*)B)$$

and

$$(A,\Gamma B)-(\Gamma A,B)=i\tau(A^*\delta(B)+\delta(A^*)B)=i\tau(\delta(A^*B))=0.$$

Since Γ is hermitian the hermitian adjoint Γ^* is densely defined and, therefore, Γ is closable (see [12], p. 305 and 306). Hence, if $A_n\to 0$ and $\Gamma A_n\to F\in\mathfrak{H}$ in the Hilbert space topology we have $F=0$. Now suppose $A_n\in\mathfrak{A}_0$, $\|\delta(A_n)-B\|\to 0$ and $\|A_n\|\to 0$ as $n\to\infty$ with $B\in\mathfrak{A}$, then $A_n\to 0$ and $\Gamma A_n\to iB$ in the norm topology of $\mathfrak{H}$. Hence, $B=0$. Hence, δ is closable.

Let $\bar\delta$ be the closure of δ and let $\mathfrak{D}(\bar\delta)$ be the domain of $\bar\delta$. We will show $\bar\delta$ is a $*$-derivation of $\mathfrak{D}(\bar\delta)$ into $\mathfrak{A}$. Suppose $A\in\mathfrak{D}(\bar\delta)$. Then there is a sequence $\{A_n\in\mathfrak{A}_0\}$ so that $\|A_n-A\|\to 0$ and $\|\delta(A_n)-\bar\delta(A)\|\to 0$ as $n\to\infty$. Hence, $\|A_n^*-A^*\|\to 0$ and $\|\delta(A_n^*)-\delta(A)^*\|\to 0$ as $n\to\infty$. Since $\bar\delta$ is closed we have $A^*\in\mathfrak{D}(\bar\delta)$

and $\bar{\delta}(A^*)=\bar{\delta}(A)^*$. Next, suppose $A, B \in \mathfrak{D}(\bar{\delta})$. Then there are sequences $\{A_n, B_n \in \mathfrak{A}_0\}$ so that $\|A_n - A\| \to 0$, $\|B_n - B\| \to 0$, $\|\delta(A_n) - \bar{\delta}(A)\| \to 0$ and $\|\delta(B_n) - \bar{\delta}(B)\| \to 0$ as $n \to \infty$. Hence, we have $\|A_n B_n - AB\| \to 0$ and $\|\delta(A_n B_n) - (\bar{\delta}(A)B + A\bar{\delta}(B))\| \to 0$ as $n \to \infty$. Hence, $AB \in \mathfrak{D}(\bar{\delta})$ and $\bar{\delta}(AB) = \bar{\delta}(A)B + A\bar{\delta}(B)$. Hence, $\bar{\delta}$ is a closed $*$-derivation of $\mathfrak{D}(\bar{\delta})$ into $\mathfrak{A}$.

Next, we will show $\bar{\delta}$ is the generator of a strongly continuous one parameter group of $*$-automorphisms of $\mathfrak{A}$ if and only if the only norm continuous linear functionals $\varrho_{\pm}$ satisfying the equations $\varrho_+(A+\delta(A))=0$ and $\varrho_-(A-\delta(A))=0$ for all $A \in \mathfrak{A}_0$ are the zero functionals.

First, suppose $\bar{\delta}$ is the generator of a strongly continuous one parameter group of $*$-automorphisms of $\mathfrak{A}$. Suppose ϱ_+ is a norm continuous linear functional on $\mathfrak{A}$ and $\varrho_+(A+\delta(A))=0$ for all $A \in \mathfrak{A}_0$. Since $\mathfrak{A}_0$ is a core for $\bar{\delta}$ we have for all $A \in \mathfrak{D}(\bar{\delta})$ there is a sequence $\{A_n \in \mathfrak{A}_0\}$ so that $\|A_n - A\| \to 0$ and $\|\delta(A_n) - \bar{\delta}(A)\| \to 0$ as $n \to \infty$. Hence, we have $\varrho_+(A+\bar{\delta}(A)) = \lim_{n\to\infty} \varrho_+(A_n + \delta(A_n)) = 0$. Hence, $\varrho_+(A+\bar{\delta}(A))=0$ for all $A \in \mathfrak{D}(\bar{\delta})$. For $A \in \mathfrak{D}(\bar{\delta})$ we have

$$\frac{d}{dt}\varrho_+(\alpha_t(A)) = \varrho_+(\delta(\alpha_t(A))) = -\varrho_+(\alpha_t(A)).$$

Hence, $\varrho_+(\alpha_t(A)) = e^{-t}\varrho_+(A)$ for all $A \in \mathfrak{D}(\bar{\delta})$, and all real t. Since $|\varrho_+(\alpha_t(A))| \leqq \|\varrho_+\| \, \|\alpha_t(A)\| = \|\varrho_t\| \, \|A\|$ for all real t and e^{-t} grows without bound as $t \to -\infty$ we have $\varrho_+(A)=0$ for all $A \in \mathfrak{D}(\bar{\delta})$. Since ϱ_+ is continuous and $\mathfrak{D}(\bar{\delta})$ is dense in $\mathfrak{A}$ it follows $\varrho_+=0$. A similar argument shows $\varrho_-=0$. Hence, if $\bar{\delta}$ is the generator of a strongly continuous one parameter group of $*$-automorphisms the functionals $\varrho_{\pm}$ are necessarily zero.

Now, suppose δ is a $*$-derivation of $\mathfrak{A}_0$ into $\mathfrak{A}$ with the property that the only norm continuous linear functionals $\varrho_{\pm}$ on $\mathfrak{A}$ satisfying the equations $\varrho_+(A+\delta(A))=0$ and $\varrho_-(A-\delta(A))=0$ for all $A \in \mathfrak{A}_0$ are the zero functionals. We will show that $\bar{\delta}$ the closure of δ is the generator of a strongly continuous one parameter group of $*$-automorphisms of $\mathfrak{A}$.

It follows from semi-group theory (see [3] or [8]) that $\bar{\delta}$ is the generator of strongly continuous contraction semi-group if and only if the mapping $A \to \lambda A - \bar{\delta}(A)$ from $\mathfrak{D}(\bar{\delta})$ into $\mathfrak{A}$ is one to one and has range all of $\mathfrak{A}$ and the norm of the inverse mapping (consider as a linear mapping of the Banach space $\mathfrak{A}$ into itself) satisfies the relation $\|(\lambda-\bar{\delta})^{-1}\| \leqq \lambda^{-1}$ for all $\lambda > 0$. If $\bar{\delta}$ is the generator of a one parameter group of $*$-automorphisms (i.e., the automorphisms α_t exist for both positive and negative t), then both $\bar{\delta}$ and $-\bar{\delta}$ are generators of contraction semi-groups. Hence, $\bar{\delta}$ generates a one parameter group of $*$-automorphisms if $A \to \lambda A \pm \bar{\delta}(A)$ are one to one mappings of $\mathfrak{D}(\bar{\delta})$ onto $\mathfrak{A}$ and $\|(\lambda \pm \bar{\delta})^{-1}\| \leqq \lambda^{-1}$ for all $\lambda > 0$.

A straight forward computation shows that these conditions on $\bar{\delta}$ are equivalent to the conditions $\|A+\lambda\bar{\delta}(A)\| \geqq \|A\|$ for all $A \in \mathfrak{D}(\bar{\delta})$ and the range of the maps $A \to A+\lambda\delta(A)$ of $\mathfrak{D}(\bar{\delta})$ into $\mathfrak{A}$ is all of $\mathfrak{A}$ for real $\lambda \neq 0$.

We will begin by showing $\|A+\lambda\delta(A)\| \geqq \|A\|$ for all $A \in \mathfrak{A}_0$ and all real λ. Suppose $A \in \mathfrak{A}_0$ and λ is real. There is an integer n so that $A \in M_n$ and there is an hermitian element $H_n \in \mathfrak{A}$ so that $\delta(B) = i[H_n, B]$ for all $B \in M_n$. Since $A \in M_n$ there is a state ω of M_n so that $\omega(A^*A) = \|A^*A\| = \|A\|^2$. It follows from the Hahn-Banach theorem that ω has an extension which we also denote by ω to a state on all of $\mathfrak{A}$. Now, we have

$$\begin{aligned}\omega&\big((A+\lambda\delta(A))^*(A+\lambda\delta(A))\big)\\ &= \omega(A^*A) + \lambda\omega(\delta(A^*A)) + |\lambda|^2\,\omega(\delta(A)^*\,\delta(A))\\ &= \omega(A^*A) + i\lambda\omega([H_n, A^*A]) + |\lambda|^2\,\omega(\delta(A)^*\,\delta(A))\end{aligned}$$

Since $D = \|A\|^2 I - A^*A \geqq 0$ and $\omega(D) = 0$ it follows from the generalized Schwarz inequality that

$$|\omega(BD)|^2 = |\omega(BD^{1/2}D^{1/2})|^2 \leqq \omega(BDB^*)\,\omega(D) = 0$$
$$|\omega(DB)|^2 = |\omega(D^{1/2}D^{1/2}B)|^2 \leqq \omega(D)\,\omega(B^*DB) = 0$$

for all $B \in \mathfrak{A}$. Hence, we have

$$\omega([H_n, A^*A]) = -\omega([H_n, D]) = \omega(DH_n) - \omega(H_nD) = 0 - 0 = 0\,.$$

Since $\omega([H_n, A^*A]) = 0$ we have

$$\begin{aligned}\omega\big((A+\lambda\delta(A))^*(A+\lambda\delta(A))\big) &= \omega(A^*A) + |\lambda|^2\,\omega(\delta(A)^*\,\delta(A))\\ &= \|A\|^2 + |\lambda|^2\,\omega(\delta(A)^*\,\delta(A))\\ &\geqq \|A\|^2\,.\end{aligned}$$

Since $\omega(B^*B) \leqq \|B\|^2$ for all $B \in \mathfrak{A}$ we have $\|A+\lambda\delta(A)\| \geqq \|A\|$ for all $A \in \mathfrak{A}_0$ and λ real. Since $\mathfrak{A}_0$ is a core for $\bar{\delta}$ it follows that $\|A+\lambda\bar{\delta}(A)\| \geqq \|A\|$ for all $A \in \mathfrak{D}(\bar{\delta})$ and λ real. It follows that if $(I+\lambda\bar{\delta})^{-1}$ exists then $\|(I+\lambda\bar{\delta})^{-1}\| \leqq 1$ for all real λ.

Since $\|A+\lambda\bar{\delta}(A)\| \geqq \|A\|$ for all $A \in \mathfrak{D}(\bar{\delta})$ and λ real and since $\bar{\delta}$ is closed a straight forward computation shows the range of the map $A \to A+\lambda\bar{\delta}(A)$, $A \in \mathfrak{D}(\delta)$ is norm closed for λ real and $\lambda \neq 0$. If the range of this mapping is not all of $\mathfrak{A}$ then it follows from the Hahn-Banach theorem there is a non zero norm continuous linear functional ϱ_λ on $\mathfrak{A}$ so that $\varrho_\lambda(A+\lambda\bar{\delta}(A)) = 0$ for all $A \in \mathfrak{D}(\bar{\delta})$. By assumption we have the only norm continuous solutions to the equations $\varrho_+\,(A+\delta(A)) = 0$ and $\varrho_-(A-\delta(A)) = 0$ for all $A \in \mathfrak{A}_0$ are the functionals $\varrho_+ = \varrho_- = 0$. Hence, it follows that the mappings $A \to A \pm \bar{\delta}(A)$, $A \in \mathfrak{D}(\bar{\delta})$ have range all of $\mathfrak{A}$. Since these mappings are norm increasing we have $(I \pm \bar{\delta})^{-1}$

exist and $\|(I\pm\bar{\delta})^{-1}\|\leqq 1$. From the resolvent equation $A^{-1}-B^{-1}=A^{-1}(B-A)B^{-1}$ and solving for $B=(I+A^{-1}(B-A))^{-1}A^{-1}$ [valid when $\|A^{-1}(B-A)\|<1$] we have

$$(\lambda+\bar{\delta})^{-1}=(I+(\lambda-1)(I+\bar{\delta})^{-1})^{-1}(I+\bar{\delta})^{-1}$$
$$(\lambda-\bar{\delta})^{-1}=(I+(\lambda-1)(I-\bar{\delta})^{-1})^{-1}(I-\bar{\delta})^{-1}$$

for all λ so that $|\lambda-1|<1$ since then $\|(\lambda-1)(I\pm\bar{\delta})^{-1}\|\leqq|\lambda-1|<1$ and then $I+(\lambda-1)(I\pm\bar{\delta})^{-1}$ is invertable. Hence, we have $(\lambda\pm\bar{\delta})^{-1}$ exists for $0<\lambda<2$. Since $\|A+\lambda\bar{\delta}(A)\|\geqq\|A\|$ for all $A\in\mathfrak{D}(\bar{\delta})$ and λ real it follows $\|\lambda A\pm\bar{\delta}(A)\|\geqq\lambda\|A\|$ for all $\lambda>0$ and $A\in\mathfrak{D}(\bar{\delta})$. Hence, $\|(\lambda\pm\bar{\delta})^{-1}\|\leqq\lambda^{-1}$ for $0<\lambda<2$. Using the resolvent equation again we have

$$(\lambda'+\bar{\delta})^{-1}=(I+(\lambda'-\lambda)(\lambda+\bar{\delta})^{-1})^{-1}(\lambda+\bar{\delta})^{-1}$$
$$(\lambda'-\bar{\delta})^{-1}=(I+(\lambda'-\lambda)(\lambda-\bar{\delta})^{-1})^{-1}(\lambda-\bar{\delta})^{-1}$$

provided $\|(\lambda'-\lambda)(\lambda\pm\bar{\delta})^{-1}\|\leqq|(\lambda'-\lambda)/\lambda|<1$. Setting $\lambda=2-\varepsilon$ the above equations show that $(\lambda'\pm\bar{\delta})^{-1}$ exist for $|(\lambda'-(2-\varepsilon))/(2-\varepsilon)|<1$ or $0<\lambda'<4-2\varepsilon$. Hence, $(\lambda\pm\bar{\delta})^{-1}$ exists for $0<\lambda<4$ and $\|(\lambda+\bar{\delta})^{-1}\|\leqq\lambda^{-1}$. Continuing in this manner we find $(\lambda\pm\bar{\delta})^{-1}$ exists for all $\lambda>0$ and $\|(\lambda\pm\bar{\delta})^{-1}\|\leqq\lambda^{-1}$. Hence, from the general theory of semi-groups $\bar{\delta}$ and $-\bar{\delta}$ are generators of contraction semi-groups or equivalently $\bar{\delta}$ is the generator of a strongly continuous group of contractions $\{\alpha_t\}$.

We complete the proof of the theorem by showing $\{\alpha_t\}$ is a group of $*$-automorphisms. Let $\beta_t(A)=\alpha_t(A^*)^*$ for all $A\in\mathfrak{A}$ and all real t. We have $\{\beta_t\}$ is a strongly continuous group of contractions of $\mathfrak{A}$ into $\mathfrak{A}$. The generator of $\{\beta_t\}$ is the operator $\delta_1(A)=\delta(A^*)^*=\delta(A)$ for all $A\in\mathfrak{D}(\bar{\delta})$. Hence, $\alpha_t=\beta_t$ for all real t and, hence, $\alpha_t(A^*)=\alpha_t(A)^*$ for all $A\in\mathfrak{A}$.

Next, suppose $A,B\in\mathfrak{D}(\bar{\delta})$ and let $C(t)=\alpha_{-t}(\alpha_t(A)\,\alpha_t(B))$. Since α_t maps $\mathfrak{D}(\bar{\delta})$ into $\mathfrak{D}(\bar{\delta})$ and since $\bar{\delta}$ is a $*$-derivation $\alpha_t(A)\,\alpha_t(B)\in\mathfrak{D}(\bar{\delta})$. A straight forward calculation then shows $dC(t)/dt=0$ for all real t where the derivative exists in the sense of norm convergence. Hence $\alpha_{-t}(\alpha_t(A)\,\alpha_t(B))=AB$ and $\alpha_t(AB)=\alpha_t(A)\,\alpha_t(B)$ for all $A,B\in\mathfrak{D}(\bar{\delta})$. Since the α_t are contractions and $\mathfrak{D}(\bar{\delta})$ is dense in $\mathfrak{A}$ we have $\alpha_t(AB)=\alpha_t(A)\,\alpha_t(B)$ for all $A,B\in\mathfrak{A}$. Hence, $\{\alpha_t\}$ is a strongly continuous one parameter group of $*$-automorphisms of $\mathfrak{A}$. This completes the proof of the theorem.

Remark. Theorem 4.1 shows that $\bar{\delta}$ is the generator of a strongly continuous one parameter group of $*$-automorphisms of $\mathfrak{A}$ if and only if the sets $S_\pm=\{A\pm\delta(A);\ A\in\mathfrak{A}_0\}$ are norm dense in $\mathfrak{A}$. The result remains true if the sets $S_\pm$ are replaced by the sets $S'_\pm=\{\lambda A\pm\delta(A);\ A\in\mathfrak{A}_0\}$ with the real part of λ not equal to zero.

If T is a densely defined hermitian operator on a Hilbert space $\mathfrak{H}$ then T has a self-adjoint extension T_1 if and only if the dimension of $\mathfrak{D}_+$ equals the dimension of $\mathfrak{D}_-$ where $\mathfrak{D}_+=\{\mathrm{Range}(T+iI)\}^\perp$ and

$\mathfrak{D}_- = \{\text{Range}(T - iI)\}^\perp$. It would be interesting to know under what conditions a $*$-derivation δ of $\mathfrak{A}_0$ into $\mathfrak{A}$ has an extension δ_1 which is the generator of a strongly continuous one parameter group of $*$-automorphisms.

Theorem 4.2. *Suppose* $\mathfrak{A}$ *is a* UHF *algebra and* $M_1 \subset M_2 \subset \cdots \subset \mathfrak{A}$ *is an increasing sequence of* $(n_i \times n_i)$*-matrix algebras whose union* $\mathfrak{A}_0 = \bigcup_{i=1}^{\infty} M_i$ *is a norm dense* $*$*-subalgebra of* $\mathfrak{A}$. *Suppose* δ *is a* $*$*-derivation of* $\mathfrak{A}_0$ *into* $\mathfrak{A}$ *whose closure* $\bar{\delta}$ *is the generator of a strongly continuous one parameter group of* $*$*-automorphisms* $\{\alpha_t\}$. *Then, the automorphism group* $\{\alpha_t\}$ *is approximately inner.*

Proof. Suppose the hypothesis of the theorem is true. Let $\{e_{ij}^{(n)};\ i,j = 1, \ldots, m(n)\}$ be a family of matrix units which span M_n and let

$$H_n = (-i/m(n)) \sum_{1,1=1}^{m(n)} \delta(e_{ij}^{(n)})\, e_{ji}^{(n)}.$$

Let $\delta_n(A) = i[H_n, A]$ for all $A \in \mathfrak{A}$. We have δ_n is an inner $*$-derivation of $\mathfrak{A}$ into $\mathfrak{A}$ and $\delta_n(A) = \delta(A)$ for all $A \in M_n$. We will show that $(I - \delta_n)^{-1}$ converges strongly to $(I - \bar{\delta})^{-1}$ as $n \to \infty$.

Since $\bar{\delta}$ is the generator of $\{\alpha_t\}$ we have $(I - \bar{\delta})^{-1}$ exists. In fact, we have

$$(I - \bar{\delta})^{-1}(A) = \int_0^\infty e^{-t} \alpha_t(A)\, dt$$

for all $A \in \mathfrak{A}$. Hence, the range of the map $A \to A - \bar{\delta}(A)$, $A \in \mathfrak{D}(\bar{\delta})$, is all of $\mathfrak{A}$. Since $\mathfrak{A}_0$ is a core for $\bar{\delta}$ (i.e., $\bar{\delta}$ is the closure of its restriction to $\mathfrak{A}_0$) the set $S_- = \{A - \bar{\delta}(A);\ A \in \mathfrak{A}_0\}$ is norm dense in $\mathfrak{A}$. Suppose $A \in S_-$. We have $A = B - \bar{\delta}(B)$ with $B \in \mathfrak{A}_0$ and

$$\begin{aligned}
&\|(I - \delta_n)^{-1}(A) - (I - \bar{\delta})^{-1}(A)\| \\
&\quad = \|\{(I - \delta_n)^{-1}(\bar{\delta} - \delta_n)(I - \bar{\delta})^{-1}\}(A)\| \\
&\quad = \|\{(I - \delta_n)^{-1}(\bar{\delta} - \delta_n)(I - \bar{\delta})^{-1}(I - \bar{\delta})\}(B)\| \\
&\quad = \|(I - \delta_n)^{-1}(\delta(B) - \delta_n(B))\| \\
&\quad \leqq \|\delta(B) - \delta_n(B)\| \to 0
\end{aligned}$$

as $n \to \infty$ since $\|(I - \delta_n)^{-1}\| \leqq 1$ and $B \in \mathfrak{A}_0$. Hence, for $A \in S_-$ $(I - \delta_n)^{-1}(A)$ converges in norm to $(I - \bar{\delta})^{-1}(A)$ as $n \to \infty$. Since $\|(I - \bar{\delta})^{-1}\| \leqq 1$ and $\|(I - \delta_n)^{-1}\| \leqq 1$ for all $n = 1, 2, \ldots$ and S_- is norm dense in $\mathfrak{A}$ we have $(I - \delta_n)^{-1}(A)$ converges to $(I - \bar{\delta})^{-1}(A)$ for all $A \in \mathfrak{A}$.

Since $(I - \delta_n)^{-1}$ converges strongly to $(I - \bar{\delta})^{-1}$ we have by the Trotter convergence theorem (see [21] or ([9], p. 502))

$$\begin{aligned}
\alpha_t(A) &= \{\exp(t\bar{\delta})\}(A) = \lim_{n \to \infty} \{\exp(t\delta_n)\}(A) \\
&= \lim_{n \to \infty} e^{itH_n} A e^{-ttH_n},
\end{aligned}$$

for all $A \in \mathfrak{A}$. Hence, $\{\alpha_t\}$ is approximately inner. This completes the proof of the theorem.

Conjecture. We conjecture that every strongly continuous one parameter group of $*$-automorphisms of a UHF algebra is approximately inner. The truth of this conjecture would show that every strongly continuous one parameter group of $*$-automorphisms of a UHF algebra has a ground state and a KMS state for all inverse temperatures β.

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