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Decay Properties and Borel Summability for the Schwinger Functions in $P(\Phi)_2$ Theories

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Abstract. For the truncated Schwinger functions of the $P(\Phi)_2$ field theories, we show strong decrease in the separation of points. This shows uniqueness of theories with P of degree four. We also extend the domain of analyticity in the coupling constant. For theories with P of degree four, the combination of these two results gives Borel summability.

Introduction

In this paper, we consider the two dimensional Euclidean boson field theories and we give bounds on the truncated Schwinger functions which have the decay properties expected from perturbation theory and introduced in statistical mechanics in [3]. We use methods known from statistical mechanics [9] to obtain these bounds.

We first formulate the bound and give then some applicati^ns. The Schwinger functions for Euclidean field theories in two dimensions in a finite (space-time) volume Λ are defined as the moments of the normalized measure

$$e^{-\lambda V(\Lambda)} d\mu_{m^2} / \int e^{-\lambda V(\Lambda)} d\mu_{m^2},$$

where $d\mu_{m^2}$ is the Gaussian measure on $\mathscr{S}'(\mathbb{R}^2)$ with mean zero and covariance $(-\Delta + m^2)^{-1}$, and

$$V(\Lambda) = \int_{\Lambda} d^2x : P(\Phi) : (x).$$

Here, P is a lower bounded polynomial, and Wick ordering $: \, :$ is with respect to the free theory defined by $d\mu_{m^2}$.

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Let $S_F(\lambda)=\int F e^{-\lambda V(\Lambda)} d\mu_{m^2}$, where $F=\prod_{j=1}^{n} F_j$, and

$$F_j=\int dy \prod_{l=1}^{k(j)} :\Phi_l^{n_l^{(j)}}:(y_l)\, w_j(y_1,\ldots,y_{k(j)}),$$

with $supp\, w_j \subset \Delta_j \times \cdots \times \Delta_j$, Δ_j a unit square in $\mathbb{R}^2$ centered at a lattice point of $\mathbb{Z}^2$, and $w_j \in L_p$, for some p. The degree of F_j is defined to be $\sum_{l=1}^{k(j)} n_l^{(j)}$. The truncated functions are then defined by

$$S_F^T=-\sum_{p=1}^{n}\frac{1}{p}\sum_{R_1,\ldots R_p}^{P(\{1,\ldots,n\})}\prod_{j=1}^{p}\left(-S_{\prod_{i\in R_j} F_i}(\lambda)\Big/S_{F=1}(\lambda)\right),$$

where Σ^P is the sum over all partitions of $\{1,\ldots n\}$ into p non empty sets $R_1,\ldots R_p$. Our main technical estimate is

Theorem A: *There is an $\varepsilon>0$, and constants K_1, K_2, K_3 depending on $p>1$ such that for $|\lambda|<\varepsilon$, $\operatorname{Re}\lambda>0$, and m sufficiently large,*

$$|S_F^T| \leqq K_1 K_2^{\Sigma \deg F_j} \prod_{j=1}^{n} \|w_j\|_p \prod_{j=1}^{k}\Big(\sum_{i,\Delta_i=\Delta_j'} \deg F_i\Big)!^{1/2} \cdot n!\, e^{-K_3 d(\Delta_1',\ldots\Delta_k')},$$

where $\Delta_1',\ldots\Delta_k'$ are the distinct squares in $\cup\Delta_i$ and $d(\Delta_1',\ldots\Delta_k')$ is the length of the shortest tree connecting the centers of these squares.

This theorem is given as Theorem 6 and 8 below. Theorem A states that the "strong decrease property" of Duneau, Iagolnitzer, and Souillard [3b] holds in $P(\Phi)_2$ models. Our improved bounds as compared to [5] come from using methods from statistical mechanics which exhibit in a better way the cancellations between the numerator and the denominator (see also Section 3 for a more detailed explanation). In the same way one proves (cf. Theorem 7 below):

Theorem B: *Let $F=F_1$. Then, for $|\lambda|<\varepsilon$, $\operatorname{Re}\lambda>0$, and m sufficiently large,*

$$\left|\frac{d^n}{d\lambda^n}\left(\frac{S_F(\lambda)}{S_{F=1}(\lambda)}\right)\right| \leqq C_1 C_2^n n!^{d/2},$$

where d is the degree of the interaction polynomial P.

Theorem B implies that the Φ^4 theories ($\deg P=4$) are uniquely defined by the Taylor series of the Schwinger functions at $\lambda=0$, since they are analytic in $\operatorname{Re}\lambda>0$, $|\lambda|<\varepsilon$ and C^∞ on the imaginary axis with bounds preserved (cf. Hardy [7], p. 194). This means that among all the theories which are analytic in the same region, the one constructed by Glimm, Jaffe, and Spencer [5], is the unique one corresponding to conventional perturbation theory.

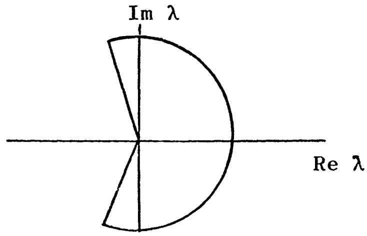

Fig. 1

In Chapter II, we extend the domain of analyticity and the bounds for $P(\Phi)_2$ in λ to a region of the form of Fig. 1. Let $\mathscr{D}_p^k$ be the set of functions f which, for some $\varepsilon > 0$, are analytic in the region

$$\{(z_1, \ldots, z_{2k}) \in \mathbb{C}^2, |\arg z_i| < \varepsilon, i = 1, \ldots, 2k\}$$

and for which

$$\|f\|_{p,\varepsilon} = \sup_{0 \leq \varepsilon' \leq \varepsilon} \left(\int_{-\infty}^{+\infty} dx_1 \ldots dx_{2k} |f(x_1 e^{i\varepsilon'}, \ldots, x_{2k} e^{i\varepsilon'})|^p \right)^{1/p} < \infty .$$

Then we show the

Theorem C: *If* $w_j \in \mathscr{D}_p^{k(j)}, j = 1, \ldots n$ *and* $\deg P = 4$, *one has Borel summability of the Schwinger functions* $S_F(\lambda)/S_{F=1}(\lambda)$ *at* $\lambda = 0$.

Note that such a result has been obtained for Φ^4 theories with cutoffs by Rosen and Simon [8]. Also, Dimock showed in [1] $S_F(\lambda)/S_{F=1}(\lambda)$ to be C^∞ at $\lambda = 0$ for all lower semi-bounded polynomials P, and that the derivatives can be identified with truncated functions, which is a useful input for the proof of Theorem B.

Theorem D: *The "pressure"* $\lim_{\Lambda \to \mathbb{R}^2} \frac{1}{|\Lambda|} \log S_{F=1}(\lambda)$ *has the same summability properties as* $S_F(\lambda)/S_{F=1}(\lambda)$.

Chapter I
Bounds on Truncated Schwinger Functions
(Mayer Expansion)

1. A Reordering of the Γ-Expansion of [5]

We define a reordering of the Γ-expansion of Glimm, Jaffe, and Spencer [5] which will turn out to be well adapted for the cancellations we want to carry out in the truncated Schwinger functions.

In this section, we use some notations of [5] without repeating the definitions. We apply the Γ-expansion to the theory inside a box Y

(a union of unit lattice squares over $\mathbb{Z}^2 \subset \mathbb{R}^2$) with an interaction inside Λ, with $Y \supset \Lambda$, $\partial Y \cap \partial \Lambda = \emptyset$ and Dirichlet boundary conditions on ∂Y. We shall obtain bounds independent of Y and Λ. As the theory is "regular at infinity" [5, p. 209], proving uniform bounds for all bounded Y and $\Lambda \subset Y$ implies that the same bounds hold in the infinite volume limit.

We change notation with respect to the introduction and we will come back to the original notation in Section 6. Define F_j as in the introduction. For Ω a union of lattice squares, define

$$F(\Omega) = \prod_{j:\Delta_j \subset \Omega} F_j ,$$

with the conventions $F(\emptyset) = 1$, empty products $= 1$. Let

$$S_{F(\Omega),\Lambda} = \int F(\Omega)\, e^{-\lambda V(\Lambda)} d\mu_{C_Y} ,$$

note that

$$S_{F(\phi),\Lambda} = \int e^{-\lambda V(\Lambda)} d\mu_{C_Y} .$$

The basic formula of [5] is

$$S_{F(\Omega),\Lambda} = \sum_{\Gamma \subset Y^*} \int_0^1 \prod_{b \in \Gamma} ds_b \frac{d}{ds_b} F(\Omega)\, e^{-\lambda V(\Lambda)} d\mu_{C_Y(\{s_b\})}$$
$$=: \sum_{\Gamma \subset Y^*} S_{F(\Omega),\Lambda,\Gamma} ,$$

where Y^* is the set of lattice lines of $Y \backslash \partial Y$. Two lattice squares are called *connected* if they have a boundary segment in common. Given $\Gamma \subset Y^*$, the lattice lines Y_Γ, where $Y_\Gamma := \partial Y \cup (Y^* \backslash \Gamma)$ define a partition of Y into connected sets $\hat{X}_1, \ldots, \hat{X}_{p(\Gamma)}$. Let $X_i = \hat{X}_i - \partial(\hat{X}_i)$, so Γ defines a partition into connected sets $X_1, \ldots, X_{p(\Gamma)}$. Note that $\partial X_i \cap \Gamma = \emptyset$ and therefore the X_i define a partition $\Gamma = \cup \Gamma_i$, $\Gamma_i = \Gamma \cap X_i$ of Γ. By definition of $S_{F(\Omega),\Lambda,\Gamma}$, there are Dirichlet boundary conditions on ∂X_i, so that

$$S_{F(\Omega),\Lambda,\Gamma} = \prod_{i=1}^{p(\Gamma)} S_{F(\Omega \cap X_i),\Lambda \cap X_i,\Gamma \cap X_i} . \tag{1}$$

The rearrangement consists now in collecting all terms in $\sum_\Gamma$ which give rise to the same partition of Y. Given X, connected, $\Gamma \subset Y^*$ is called *compatible* with X, (notation Γ/X) if $X \backslash X_\Gamma$ is connected. In other terms, if $\mathscr{P} = \{X_1, \ldots, X_p\}$ is a partition of Y into connected sets and Γ_i/X_i for all i, then $\Gamma = \cup \Gamma_i$ generates again $\mathscr{P}$. Therefore

$$\begin{aligned} S_{F(\Omega),\Lambda} &= \sum_{\Gamma \subset Y^*} \prod_{i=1}^{p(\Gamma)} S_{F(\Omega \cap X_i),\Lambda \cap X_i,\Gamma \cap X_i} \\ &= \sum_{X_1,\ldots X_r}^{P(Y)} \prod_{i=1}^{r} \sum_{\Gamma_i/X_i} S_{F(\Omega \cap X_i),\Lambda \cap X_i,\Gamma_i} , \end{aligned} \tag{2}$$

where $\Sigma^{P(Y)}$ denotes the sum over all partitions of Y into $r=1,\dots|Y|$ non empty connected sets (which are unions of lattice squares).

We now fix F and Λ, and we define

$$S'_\Omega(X)=\begin{cases}\sum\limits_{\Gamma/X} S_{F(\Omega\cap X),\Lambda\cap X,\Gamma}, & \text{if } X \text{ is connected}\\ 0 & , \text{ otherwise}.\end{cases}$$

Then

$$S_{F(\Omega),\Lambda}=\sum_{X_1,\dots X_r}^{P(Y)}\prod_{i=1}^{r} S'_\Omega(X_i),$$

and

$$\int e^{-\lambda V(\Lambda)}d\mu_{C_Y}=\sum_{X_1,\dots X_r}^{P(Y)}\prod_{i=1}^{r} S'_\phi(X_i).$$

Note that for $X\cap\Omega=\emptyset$, $S'_\Omega(X)=S'_\phi(X)$.

In contrast to [5], we also expand in sums over compatible contours for regions *not* intersecting $\Omega=supp\, F$, i.e. we keep the expansion in its primitive form and do not perform the resummation over the above terms.

2. The Functional Calculus for Power Series

It is our desire to perform cancellations in $S_{F(\Omega),\Lambda}/S_{F(\phi),\Lambda}$ by expanding the numerator and the denominator into power series and we then invert the series for the denominator. We apply the formalism of Ruelle [9] with the following conventions.

We number the (centers of the) unit squares in Y in the obvious manner by a subset $\bar{Y}$ of $\mathbb{Z}^2$. The "indices" in our series will be elements of the set $\hat{Y}$ of functions N from $\bar{Y}$ to the non negative integers, $N(i)\geqq 0$ for $i\in\bar{Y}$. The support $\tilde{N}$ of N is the union of the lattice squares associated to the $i\in\mathbb{Z}^2$ for which $N(i)>0$. We write $N\leqq 1$ if $N(i)\leqq 1$ for all i, we use the notation $|N|=\sum_{i\in\bar{Y}}N(i)$, and for $X\subset Y$ we define the characteristic function χ_X by $\chi_X(i)=1$ if i "in" X, 0 otherwise; $(\chi_X)^\sim=X$, $\chi_X\leqq 1$. Also $N!=\prod_i N(i)!$.

The functional calculus relates functions on $\hat{Y}$ as formal power series. If f,g are functions on $\hat{Y}$, then $f*g$ is defined by

$$(f*g)(N)=\sum_{N_1+N_2=N} f(N_1)\,g(N_2)\frac{N!}{N_1!\,N_2!}$$

and $f+g$ by linearity. The unit for the product is $1(N)=0$ unless $\tilde{N}=\emptyset$. Inverses, exponentials and logarithms are defined in the obvious fashion, and one defines $(D_N f)(M)=f(M+N)$; for $|N|=1$, D_N is a derivation with respect to the $*$-product.

3. The Series for the Normalized Schwinger Function

We combine the definitions of Sections 1 and 2 and we proceed to an expansion in those components of the partition $X_1, \ldots X_{p(\Gamma)}$ for which $|X_i| = 1$. Since a contour Γ/X has the property of factorizing the measure across ∂X, we see that what we are doing is actually an expansion in the smallest regions for which the measure factorizes. It will then turn out (as to be expected) that for the truncated function, these regions have to be connected and to contain all squares of Ω, and this will yield the desired decay properties (strong clustering of [3]) which allows for the bounds on derivatives with respect to λ.

We say that X is Ω-connected ($X \smile \Omega$) if $X \subset Y$ and if each connected component of X has a non trivial intersection[1] with Ω. Also $X \underset{\smile}{\supset} \Omega$ means $X \smile \Omega$ and $X \supset \Omega$. We rewrite Eq. (2)

$$\begin{aligned} S_{F(\Omega),\Lambda} &= \sum_{X_1 \ldots X_r}^{P(Y)} \prod_{j: X_j \smile \Omega} S'_{\Omega \cap X_j}(X_j) \prod_{j: \neg X_j \smile \Omega} S'_\phi(X_j) \\ &= \sum_{X \underset{\smile}{\supset} \Omega} \sum_{\substack{W_1 \ldots W_r \\ W_j \smile \Omega}}^{P(X)} \prod_{j=1}^{r} S'_{\Omega \cap W_j}(W_j) \sum_{X_1 \ldots X_s}^{P(Y \setminus X)} \prod_{j=1}^{s} S'_\phi(X_j) . \end{aligned}$$

Define now $Z(X) = \left(\prod_{\Delta \in X} S'_\phi(\Delta) \right)$, where the product ranges over the unit squares of X. By continuity, $Z(\Delta) \geqq \frac{1}{2}$ for $|\lambda| < \lambda_0$, $\mathrm{Re}\,\lambda \geqq 0$. Note that $Z(X \cup Y) = Z(X)\, Z(Y)$ if $X \cap Y = \emptyset$. Now

$$\begin{aligned} S_{F(\Omega),\Lambda} = &\sum_{X \underset{\smile}{\supset} \Omega} \sum_{\substack{W_1 \ldots W_r \\ W_j \smile \Omega}}^{P(X)} \prod_{j=1}^{r} S'_{\Omega \cap W_j}(W_j)\, Z(W_j)\, Z(W_j)^{-1} \\ &\cdot \sum_{X_1 \ldots X_S}^{P(Y \setminus X)} Z(Y \setminus X) \prod_{j: |X_j| > 1} S'_\phi(X_j)\, Z^{-1}(X_j) . \end{aligned}$$

This suggests the definition of two functions on $\hat{Y}$:

If $\Omega \neq \emptyset$, we let

$$\hat{S}_\Omega(N) = \begin{cases} \displaystyle\sum_{\substack{X_1, \ldots X_r \\ X_j \smile \Omega}}^{P(\tilde{N})} \prod_{j=1}^{r} S'_\Omega(X_j)\, Z^{-1}(X_j) , & \text{if } N \leqq 1 \text{ and } \tilde{N} \underset{\smile}{\supset} \Omega \\ 0 , & \text{otherwise} , \end{cases}$$

and if $\Omega = \emptyset$,

$$S_\phi(N) = \begin{cases} S'_\phi(\tilde{N})\, Z^{-1}(\tilde{N}) , & \text{if } N \leqq 1 \text{ and } \tilde{N} \text{ connected, } |\tilde{N}| \neq 1 \\ 0 , & \text{otherwise} . \end{cases}$$

[1] I.e. at least one unit square.

We observe that $S'_\Omega(\tilde{N})$ (Ω arbitrary) describes the fluctuations between a theory in which there is coupling from the free covariance (absence of boundary conditions) and a theory in which there is no coupling (Dirichlet data on lines in Γ^c, $\Gamma|\tilde{N}$). These fluctuations are small at large mass. It is therefore our aim to do an expansion in elementary squares with these fluctuations as coefficients. This is the purpose of dividing by $Z^{-1}(\tilde{N})$ in the above definition.

We note now that

$$\sum_{X_1\dots X_r}^{P(X)} = \sum_{r\geqq 1}\frac{1}{r!}\sum_{\substack{X_1\cup\dots\cup X_r=X\\ X_i \text{ is connected}\\ X_i\cap X_j=\phi \text{ if } i\neq j}},$$

and setting $S_\phi(0)=1$, we get

$$S_{F(\Omega),A}=Z(Y)\sum_{X\supseteq\Omega}\hat{S}_\Omega(\chi_X)\sum_{W\subset Y\setminus X}(\exp S_\phi)(\chi_W),$$

where exp is the $*$-exponential. Therefore

$$\frac{S_{F(\Omega),A}}{S_{F(\phi),A}}=\frac{\sum\limits_{X\supseteq\Omega}\left\{\hat{S}_\Omega(\chi_X)\sum\limits_{W\subset Y\setminus X}(\exp S_\phi)(\chi_W)\right\}}{\sum\limits_{W\subset Y}(\exp S_\phi)(\chi_W)}. \tag{3}$$

Let $T(N)=(\exp S_\phi)(N)$ if $N\leqq 1$, and 0 otherwise and let $D(N)=(\hat{S}_\Omega * T)(N)$ if $N\leqq 1$, and 0 otherwise. Then we define $\mathscr{S}_\Omega$ by writing

$$\frac{S_{F(\Omega),A}}{S_{F(\phi),A}}=\frac{\sum\limits_N D(N)\,N!^{-1}}{\sum\limits_N T(N)\,N!^{-1}}=\sum_N \mathscr{S}_\Omega(N)\,N!^{-1}, \tag{4}$$

so that $\mathscr{S}_\Omega=D*T^{-1}$. Note that the sum in (4) extends over all N and not only over N of the form χ_X; this is due to the inversion of the power series for $\sum\limits_N T(N)\,N!^{-1}$.

4. *Bounds on* $\mathscr{S}_\Omega(N)$

We rewrite Eqs. (3) and (4) as

$$\mathscr{S}_\Omega(N)=\sum_{\tilde{N}\supset X\supseteq\Omega}\hat{S}_\Omega(\chi_X)\,Q_X(N-\chi_X)\frac{N!}{(N-\chi_X)!}, \tag{5}$$

where the "quotient" Q is

$$Q_X(M)=\sum_{\substack{M_1+M_2=M\\ \tilde{M}_1\cap X=\phi\\ M_1\leqq 1}}T(M_1)\,T^{-1}(M_2)\frac{M!}{M_2!}. \tag{6}$$

The bound on $\mathscr{S}_\Omega(N)$ comes from a bound on Q_X.

Lemma 1. *There is a* $C_1 > 1$, *and for all* $K > 0$, *there is a* $\lambda_0 > 0$ *such that for* $|\lambda| < \lambda_0$, $\operatorname{Re}\lambda > 0$ *and m sufficiently large one has*

$$|Q_X(M)| \leqq C_1^{|X|+|M|} e^{-K|M|} M!\,.$$

Proof. The proof uses an induction on $|M| + |X|$ and the following recurrence relation, valid for $\Delta \subset X$, Δ a unit square

$$Q_X(M) = Q_{X\setminus\Delta}(M) - \sum_{\substack{\tilde{N}\cap X=\phi \\ \tilde{N}\subset(M-\chi_\Delta)^\sim}} S_\phi(N+\chi_\Delta)\, Q_{\tilde{N}\cup X}(M-\chi_{\Delta\cup\tilde{N}}) \frac{M!}{(M-\chi_{\Delta\cup\tilde{N}})!}\,. \tag{7}$$

Proof of the recurrence formula (7):

$$\begin{aligned} &Q_X(M) - Q_{X\setminus\Delta}(M) \\ &\quad = - \sum_{\substack{M_1+M_2=M \\ \tilde{M}_1\cap(X\setminus\Delta)=\phi \\ \tilde{M}_1\supset\Delta}} T(M_1)\, T^{-1}(M_2) \frac{M!}{M_2!} \\ &\quad = - \sum_{\substack{M_1+M_2=M-\chi_\Delta \\ \tilde{M}_1\cap X=\phi \\ M_1\leqq 1}} (D_{\chi_\Delta} \exp S_\phi)(M_1)\, T^{-1}(M_2) \frac{M!}{M_2!} \\ &\quad = - \sum_{\substack{M_1+M_2=M-\chi_\Delta \\ \tilde{M}_1\cap X=\phi \\ M_1\leqq 1}} \sum_{N_1+N_2=M_1} S_\phi(N_1+\chi_\Delta)\, T(N_2)\, T^{-1}(M_2) \frac{M!}{M_2!} \\ &\quad = - \sum_{\substack{\tilde{N}_1\cap X=\phi \\ \tilde{N}_1\subset(M-\chi_\Delta)^\sim}} S_\phi(N_1+\chi_\Delta) \cdot \sum_{\substack{N_2+M_2=M-\chi_{\Delta\cup\tilde{N}_1} \\ \tilde{N}_2\cap(X\cup\tilde{N}_1)=\phi}} T(N_2)\, T^{-1}(M_2) \frac{M!}{M_2!}\,, \end{aligned}$$

which proves (7).

Note that $|X\setminus\Delta| + |M| < |X| + |M|$ and $|X\cup\tilde{N}| + |M-\chi_{\Delta\cup\tilde{N}}| < |X| + |M|$, so that Eq. (7) allows for an induction on $|X| + |M|$. To start the induction, note that $Q_\phi(M) = 1(M)$. Suppose now the lemma is true for $|X'| + |M'| < |X| + |M|$. Then, by (7)

$$\begin{aligned} |Q_X(M)| &\leqq C_1^{|X|+|M|-1} \Big\{ e^{-K|M|} M! + \sum_{\substack{\tilde{N}\cap X=\phi \\ \tilde{N}\subset(M-\chi_\Delta)^\sim}} |S_\phi(N+\chi_\Delta)|\, e^{-K|M-\chi_{\Delta\cup\tilde{N}}|} M! \Big\} \\ &\leqq C_1^{|X|+|M|-1} e^{-K|M|} M! \Big\{ 1 + \sum_{\substack{Y\supset\Delta \\ |Y|\geqq 2}} |S'_\phi(Y)|\, e^{(K+a)|Y|} \Big\}\,, \end{aligned}$$

where $e^{a|Y|}$ bounds $Z(Y)$.

We now use the definition of $S'_\phi(Y)$. By the bound of Proposition 5.3. of Glimm, Jaffe, and Spencer [5, p. 218], we find

$$|S'_\phi(Y)| \leqq C_0 \exp(-(|Y|-1)\, K(m))\, 2^{4|Y|}\,.$$

Here, we have used that the number of bonds to make Y connected is at least $|Y|-1$. On the other hand, there are at most $2^{4|Y|}$ ways to place bonds in Y. The construction of $S_\phi(Y)$ ensures that $|Y|\geqq 2$. Therefore, if $|\lambda|$ is sufficiently small, $\operatorname{Re}\lambda>0$ and m sufficiently large, then

$$|S'_\phi(Y)|\leqq C_0\exp(-(|Y|/2)(K(m)-2\log 16))\leqq C_0\exp(-|Y|(K+\varepsilon))$$

for some $\varepsilon>a+8\log 4$. The bound on $Q_X(M)$ follows now by setting

$$C_1=1+C_0\sum_{\substack{Y\smile\Delta\\|Y|\geqq 2}} e^{-(\varepsilon-a)|Y|},$$

where the last sum is finite because the number of $Y\smile\Delta$ with $|Y|=k$ is bounded by 4^{8k}. This completes the proof of Lemma 1.

We have the following bound:

Lemma 2. *There is a* $C_2>1$ *and for all* $K>0$ *there is a* $\lambda_0>0$ *such that for* $|\lambda|<\lambda_0$, $\operatorname{Re}\lambda>0$ *and* m *sufficiently large one has*

$$|\hat{S}_\Omega(\chi_X)|\leqq\|w\|_p\,C_2^{\sum_{\Delta_i\in\Omega}n_i}\,e^{-K(|X\setminus\Omega|)}\prod_{\Delta_i\in\Omega}\sqrt{n_i!}$$

for some $p>1$. *Here* n_i *is the number of* Φ*'s in* $\int Fw$ *localized in* Δ_i.

Proof. By the theorem of the Appendix, we have that

$$\left|\int F(w)\,d\mu_C\right|\leqq\|w\|_p\,C^{\sum_{\Delta_i\in\Omega}n_i}\prod_{\Delta_i\in\Omega}\sqrt{n_i!}\,.$$

Now Corollary 9.6 of [5, p. 237] follows with $\Pi\, n_i!$ replaced by $\Pi\, n_i!^{1/2}$. We now follow the proof of [5, Lemma 10.1] which is the main input for the bound [5, Lemma 10.2].

(Lemma 10.1′: There exists a K_{13} independent of m_0 *and* n_i such that

$$\prod_\Delta N(\Delta)!^{1/2}\leqq e^{K_{13}|\Gamma|}C^{\sum n_i}\prod_\Delta(M(\Delta)!)^p\,\Pi\, n_i!^{1/2}$$

where $N(\Delta)$ is the number of Φ's in Δ coming from n_i and the applications of $e^{-\lambda V(\Delta)}$). Lemma 2 now follows as in [5, Proposition 5.3, p. 218, proved in § 10, p. 239].

By Eq. (5) and Lemmas 1 and 2 we get

$$\begin{aligned}|\mathscr{S}_\Omega(N)|&\leqq\sum_{\substack{X\supseteq\Omega\\ \tilde{N}\supset X}}\|w\|_p\,C_2^{\sum n_i}e^{-K|X|+K|\Omega|}\,\Pi\sqrt{n_i!}\\&\quad\cdot C_1^{|X|+|N-\chi X|}e^{-K|N-\chi X|}N!\\&=\|w\|_p\,\Pi\sqrt{n_i!}\,N!\,C_1^{|N|}C_2^{\sum n_i}e^{K|\Omega|}e^{-K|N|}\sum_{\substack{X\supseteq\Omega\\ \tilde{N}\supset X}}1\,.\end{aligned}$$

The last sum is bounded by $2^{|\tilde{N}|}$ and therefore we have shown that for all $K>0$,

$$|\mathscr{S}_\Omega(N)| \leqq \|w\|_p \Pi \sqrt{n_i!}\, C_3^{\Sigma n_i} e^{-K(|N|-|\Omega|)} N! \tag{8}$$

if $\mathrm{Re}\,\lambda>0$, $|\lambda|<\lambda_1$ for some $\lambda_1>0$, m sufficiently large.

5. *Bounds on* $S_{F(\Omega),\Lambda}/S_{F(\emptyset),\Lambda}$

These bounds are obtained by using Eq. (8) and the following support property of $\mathscr{S}_\Omega$.

Lemma 3. $\mathscr{S}_\Omega(N)=0$ *unless* $\tilde{N}\supseteq\Omega$.

Proof. We call X and Y *separated* ($X\leftrightarrow Y$) if they have no boundary segment in common. The formal power series have the properties:

P1) If for all $M\leftrightarrow N$ one has $F(M+N)=F(M)F(N)$ and $F(0)\neq 0$ then also $F^{-1}(M+N)=F^{-1}(M)F^{-1}(N)$,

P2) $Q_X(M)=1(M)$ if $X\cap\tilde{M}=\emptyset$,

P3) if $M\leftrightarrow N$, $X\cap\tilde{N}=\emptyset$ then $Q_X(M+N)=Q_X(M)1(N)$,

P4) if $M\leftrightarrow N$, then $Q_X(M+N)=Q_{X\cap\tilde{M}}(M)Q_{X\cap\tilde{N}}(N)$.

We leave the proof of P1) and the inspection of P2) to the reader. Proof of P3): $S_\phi(N)=0$ unless $\tilde{N}$ is connected. Therefore $\exp S_\phi$ and T factorize in the sense of P1), and thus so does T^{-1}. We rewrite $Q_X(M+N)$,

$$Q_X(M+N)=\sum_{\substack{M_1+M_2=M\\ N_1+N_2=N\\ \tilde{M}_1\cap X=\emptyset}} T(M_1)\,T(N_1)\,T^{-1}(M_2)\,T^{-1}(N_2)\frac{M!\,N!}{M_2!\,N_2!}$$
$$=Q_X(M)\,1(N)\,,\quad \text{which is P3)}\,.$$

P4) is now an easy consequence of P3).

We come back to Lemma 3. Let $\tilde{N}=\tilde{N}_1\cup\tilde{N}_2$, $\tilde{N}_2\leftrightarrow\Omega$, then by definition

$$\mathscr{S}_\Omega(N)=\sum_{X\supseteq\Omega}\hat{S}_\Omega(\chi_X)\,Q_X(N-\chi_X)\frac{N!}{(N-\chi_X)!}$$
$$=\sum_{X\supseteq\Omega}\hat{S}_\Omega(\chi_X)\,Q_X((N_1-\chi_X)+N_2)\frac{N!}{(N-\chi_X)!}$$
$$=\sum_{X\supseteq\Omega}\hat{S}_\Omega(\chi_X)\,Q_X(N_1-\chi_X)\,1(N_2)\frac{N_1!}{(N_1-\chi_X)!}\quad\text{(by P4))}$$

which is zero if $N_2\neq 0$. q.e.d.

The bound (8) and Lemma 3 are the input to

Theorem 4. $\left|\dfrac{S_{F(\Omega),\Lambda}}{S_{F(\phi),\Lambda}}\right| \leqq \|w\|_p \Pi\sqrt{n_i!}\, C_4^{\Sigma n_i}$.

Proof. By (8) and Lemma 3, we see that for $\lambda \geqq 0$,

$$\left|\frac{S_{F(\Omega),\Lambda}}{S_{F(\phi),\Lambda}}\right| = \left|\sum_N \mathscr{S}_\Omega(N)\, N!^{-1}\right| \leqq \sum_{N:\,\tilde{N} \supseteq \Omega} \|w\|_p \,\Pi \sqrt{n_i!}\, e^{-K(|N|-|\Omega|)} C_3^{\sum n_i},$$

since $S_{F(\phi),\Lambda} \geqq 1$ by Jensen's inequality.

But,

$$\sum_{\tilde{N} \supseteq \Omega} e^{-K|N|} = \sum_{X \supseteq \Omega} \sum_{\substack{N(i) \geqq 1 \\ \forall i \,\text{"}\in\text{"}\, X}} \prod_{i \,\text{"}\in\text{"}\, X} e^{-KN(i)}$$

$$= \sum_{X \supseteq \Omega} e^{-K'|X|}$$

for some $0 < K' < K$ if K is large enough. We use again the argument about the number of $X \supseteq \Omega$ with $|X| = k$ to obtain the bound on the expansion $\Sigma\, \mathscr{S}_\Omega(N)\, N!^{-1}$ for $\operatorname{Re}\lambda \geqq 0$ and $|\lambda| \leqq \lambda_0$. Now, unnormalized Schwinger functions of a finite volume are analytic in $\operatorname{Re}\lambda \geqq 0$, as can be seen by approximating through cylinder functions and using Vitali's theorem. Therefore the terms $\mathscr{S}_\Omega(N)$ are analytic in the region $\operatorname{Re}\lambda \geqq 0$, $|\lambda| < \lambda_0$ since $Z(\Delta)$ does not vanish in this region. Thus $\Sigma \mathscr{S}_\Omega(N)\, N!^{-1}$ is an analytic function which equals $S_{F(\Omega),\Lambda}/S_{F(\phi),\Lambda}$ on $\lambda \geqq 0$ so that $S_{F(\Omega),\Lambda}/S_{F(\phi),\Lambda}$ is actually analytic in $\operatorname{Re}\lambda \geqq 0, |\lambda| \leqq |\lambda_0|$ with bounds preserved, and independent of the volume Λ.

6. *The Truncated Functions*

Consider the functions of section 1, $F = F_1 \dots F_n$. For each subset $\{i_1, \dots, i_k\}$ of $\{1, \dots, n\}$, one can define a function $\mathscr{S}_{i_1,\dots i_k}(N)$ associated to $F' = F_{i_1} \dots F_{i_k}$ in the same way as $\mathscr{S}_\Omega(N)$ was associated to F, Eq. (4). Now write also $S_{\{i_1,\dots i_k\}}$ for $S_{F'(\Omega),\Lambda}/S_{F(\phi),\Lambda}$. Then the truncated function S_F^T is defined by

$$S_F^T = -\sum_{R_1 \dots R_p}^{P(\{1,\dots n\})} \frac{(-1)^p}{p} \prod_{j=1}^{p} S_{R_j},$$

where $\sum_{R_1 \dots R_p}^{P(\{1,\dots n\})}$ is the sum of partitions of $\{1, \dots n\}$ into $p = 1, \dots n$ non empty sets $R_j = \{i_{j_1} \dots i_{j_{r(j)}}\}$; $j = 1, \dots p$. We can rewrite this as

$$S_F^T = -\sum_{R_1 \dots R_p}^{P(\{1,\dots n\})} \frac{(-1)^p}{p} \sum_{N_1 + \dots + N_p = N} \frac{\mathscr{S}_{R_1}(N_1) \dots \mathscr{S}_{R_p}(N_p)}{\prod_{j=1}^{p} N_j!}.$$

We define therefore

$$\mathscr{S}_F^T(N) = -\sum_{R_1 \dots R_p}^{P(\{1,\dots n\})} \frac{(-1)^p}{p} (\mathscr{S}_{R_1} * \dots * \mathscr{S}_{R_p})(N) \tag{9}$$

so that

$$S_F^T = \sum_N \mathscr{S}_F^T(N)\, N!^{-1}.$$

In analogy to the earlier sections, we establish support properties and bounds on $\mathscr{S}_F^T$.

Lemma 5. *$\mathscr{S}_F^T(N)=0$ unless $\tilde{N}$ is connected and $\tilde{N}\supset\Omega=\bigcup_{i=1}^n \Delta_i$, where Δ_i is the support of F_i.*

Proof. Case 1: Suppose $\tilde{N}$ does not contain the square Δ_k. Then all terms in the sum (9) contain a factor of the form $\mathscr{S}_{R\cup k}(N_i)$ with $\tilde{N}_i\cap\Delta_k=\emptyset$ which is zero by Lemma 3. Therefore $\mathscr{S}_{1,\dots,n}^T(N)=0$ unless $\tilde{N}\supset\Omega$.

Case 2: Suppose $N=N_1+N_2$, $N_1\leftrightarrow N_2$, $\tilde{N}_1\supset\Omega_1$, $\tilde{N}_2\supset\Omega_2$. Then $\mathscr{S}_{\Omega_1\cup\Omega_2}(N_1+N_2)=\mathscr{S}_{\Omega_1}(N_1)\,\mathscr{S}_{\Omega_2}(N_2)$, because

$$\mathscr{S}_{\Omega_1\cup\Omega_2}(N_1+N_2)=\sum_{\substack{X_1\supset\Omega_1\\ X_2\supset\Omega_2}}\hat{S}_{\Omega_1}(\chi_{X_1})\,\hat{S}_{\Omega_2}(\chi_{X_2})\,\varrho_{X_1\cup X_2}((N_1-\chi_{X_1})+(N_2-\chi_{X_2}))$$

since $\hat{S}_\Omega$ factorizes in the sense of P1), p. 260. The assertion follows now from P4) and well-known properties of truncation sums (9).

In order to bound $\mathscr{S}_F^T$, we want to be more specific about what the functions F_i are in our case. We want to bound the n'th derivative with respect to λ of the Schwinger function $S(\lambda)$ defined by

$$\begin{aligned}S(\lambda)&=\frac{S_{F_1}(\lambda)}{S_{F=1}(\lambda)}\\&=\frac{\int d\mu_C e^{-\lambda V(\Lambda)}\int :\Phi^{l_1}:(y_1)\dots:\Phi^{l_k}:(y_k)\,w_1(y_1,\dots y_k)\,dy}{\int d\mu_C e^{-\lambda V(\Lambda)}}\,.\end{aligned}$$

By the explicit calculation of Dimock [1],

$$\frac{d^{n-1}}{d\lambda^{n-1}}S(\lambda)=\sum_{\Delta_2,\dots\Delta_n}S^T_{F_1\cdot F(\Delta_2)\dots F(\Delta_n)}\,,\tag{10}$$

where $F(\Delta)=\int_\Delta :P(\Phi):(x)\,dx$. For simplicity, we assume P is a monomial of degree $2d$, $d>1$[2]. Let the degree of F_1 in Φ be v and let w_1 have support in Δ_1.

Write $\Omega=\cup\Delta_i$ as a union of distinct lattice squares $\Delta_1',\dots\Delta_k'$, and let n_i be the number of $\Delta_j=\Delta_i'$. Let $d(\Delta_1',\dots\Delta_k')$ be the length of the shortest tree connecting the centers f the Δ_i'. Our main bound is then

Theorem 6. *With the above assumptions and definitions, there is for all $K_3>0$ a $\lambda_0>0$ such that for $|\lambda|<\lambda_0$, $\mathrm{Re}\,\lambda>0$, and m sufficiently large one has*

$$|S_F^T|\leqq\|w_1\|_p\,K_1K_2^n\prod_{i=1}^n(n_i!)^{d-1}\,e^{-K_3d(\Delta_1',\dots\Delta_k')}\,n!$$

for some p, $K_1,K_2>1$.

[2] The assumption that the F_i are monomials is for notational convenience only.

Proof. Let $\Delta_1' = \Delta_1$. Given a subset R of $\{1, \dots n\}$, we define $n_R'(\Delta)$ to be the number of $i \in R$ with $i \geqq 2$, and $\Delta_i = \Delta$.

The degree $n_R(\Delta_j')$ of the monomial $\prod_{i \in R} F_i$ in Δ_j' is $2dn_R'(\Delta_j')$ in general and $2dn_R'(\Delta_j') + v$ if $1 \in R$ and $j = 1$.

By Eqs. (8) and (9) and Lemma 3

$$|\mathcal{S}_F^T(N)| \leqq \|w_1\|_p \sum_{R_1 \dots R_p}^{P(\{1, \dots n\})} N! \frac{1}{p} \sum_{\substack{N_1 + \dots + N_p = N \\ \tilde{N}_i \supseteq (R_i)}} C_3^{2d(n-l)+v} e^{-K|N|} e^{Kn} \prod_{i=1}^{p} \prod_{\Delta \subset (R_i)} \sqrt{n_{R_i}(\Delta)!}\,, \tag{11}$$

where $(R_i) = \bigcup_{j \in R_i} \Delta_j$. Since the degree v of F_1 is fixed,

$$\prod_{i=1}^{n} \prod_{\Delta \subset (R_i)} \sqrt{n_{R_i}(\Delta)!} \leqq C(v)\, 2^{dn} \prod_{i=1}^{p} \prod_{\Delta \subset (R_i)} \sqrt{(2d\, n_{R_i}'(\Delta))!}$$

[by the inequality $\binom{a}{b} \leqq 2^a$]

$$\leqq C^n \prod_j |R_j|! \prod_{i=1}^{k} (|n_i|!)^{d-1}\,.$$

We have used

$$\prod_{j=1}^{p} \prod_{\Delta \subset (R_i)} n_{R_i}'(\Delta)! \leqq \begin{cases} \prod_{i=1}^{k} |n_i|! \\ \prod_{j=1}^{p} |R_j|! \end{cases}.$$

We next estimate

$$\sum_{R_1 \dots R_p}^{P(\{1, \dots n\})} \frac{1}{p} \prod_{j=1}^{p} |R_j|! \leqq \sum_{p=1}^{n} \frac{1}{p} \sum_{\substack{n_1 + \dots + n_p = n \\ n_i \geqq 1}} \frac{n!}{n_1! \dots n_p!} n_1! \dots n_p!$$

$$\leqq \sum_{p=1}^{n} \frac{1}{p} n! \binom{n-1}{p-1} \leqq n!\, 2^n\,.$$

Going back to (11), we get

$$\left|\sum_N \mathcal{S}_F^T(N)\, N!^{-1}\right| \leqq \sum_{\substack{\tilde{N} \supseteq \Omega \\ \tilde{N} \text{ connected}}} \|w_1\|_p\, C'^{n+1} \Pi n_i!^{d-1} n!\, e^{-K|N|}\,,$$

and summing over N as in the proof of Theorem 4,

$$\sum_{\substack{\tilde{N}\supseteq\Omega \\ \tilde{N} \text{ connected}}} e^{-K|N|} \leqq e^{-\frac{K}{2}d(\Delta_1', \ldots \Delta_k')} \sum_{\tilde{N}\supseteq\Omega} e^{-\frac{K}{2}|N|}$$

we complete the proof of the theorem.

Finally, we formulate the bound

Theorem 7. $\left|\frac{d^{n-1}}{d\lambda^{n-1}} S(\lambda)\right| \leqq \|w_1\|_p D_1 D_2^{n-1}(n-1)!^d$.

Proof. By the symmetry of $F_2, \ldots F_n$, [Eq. (10)],

$$\left|\frac{d^{n-1}}{d\lambda^{n-1}} S(\lambda)\right| = \left|\sum_{\Delta_2 \ldots \Delta_n} S^T_{F_1 \cdot F(\Delta_2) \ldots F(\Delta_n)}\right|$$

$$\leqq \sum_{k=1}^{n} \sum_{\substack{n_i \geqq 1 \\ n_1 + \ldots + n_k = n-1}} \frac{1}{(k-1)!} \frac{(n-1)!}{n_1! \ldots n_k!} \sum_{\Delta_1 \ldots \Delta_k} |S^T_{1 \ldots n}|$$

which is bounded by

$$\begin{aligned} &\sum_{k=1}^{n} \frac{1}{(k-1)!} \sum_{\substack{n_i \geqq 1 \\ n_1 + \ldots + n_k = n-1}} \frac{n!^2}{\prod n_i!} \|w_1\|_p K_1 K_2^n \prod_{i=1}^{k} (n_i!)^{d-1} \\ &\cdot \sum_{\Delta_1' \ldots \Delta_k'} e^{-K_3 d(\Delta_1', \ldots \Delta_k')} . \end{aligned} \tag{12}$$

By [3a, p. 196], the sum over Δ_i' is bounded by $K_4^k(k-1)!$, so (12) is less than

$$\|w_1\|_p K_1 K_2^n \sum_{k=1}^{n} K_4^k \sum_{\substack{n_i \geqq 1 \\ n_1 + \ldots + n_k = n-1}} n!^2 \prod_{i=1}^{k} (n_i!)^{d-2}$$

$$\leqq \|w_1\|_p K_1 K_5^n n!^d$$

from which the assertion follows by adapting the constants.

The following generalization of Theorem 6 is useful when considering truncated functions of general arguments. Let $S_F(\lambda)$ be defined as in the introduction with $k=1$, and let $n_i = \sum_{j: \Delta_j = \Delta_i'} (\deg F_j)$. Let S_F^T be the corresponding truncated function.

Theorem 8. *For all* $K_3 > 0$ *there is a* $\lambda_0 > 0$ *such that for* $|\lambda| < \lambda_0$, $\operatorname{Re}\lambda > 0$ *and m sufficiently large one has*

$$|S_F^T(w)| \leqq \prod_{i=1}^{n} \|w_i\|_p n! K_1 K_2^{\sum n_i} \prod_{i=1}^{k} (n_i!)^{1/2} e^{-K_3 d(\Delta_1', \ldots \Delta_k')},$$

for some $p, K_1, K_2 > 1$.

Proof. The proof is an easy adaptation of the proof of Theorem 6. Given a subset R of $\{1, \dots n\}$ one defines now

$$n_R(\Delta_j') = \sum_{\substack{i \in R \\ \Delta_i = \Delta_j'}} n_i ,$$

and Eq. (11) holds in this case. Now $\prod_{j=1}^{p} \prod_{\Delta \in (R_j)} \sqrt{n_{R_j}(\Delta)!} \leqq \prod_{i=1}^{k} (n_i!)^{1/2}$ and the remainder of the proof is as in Theorem 6.

Chapter II
Extending Analyticity Domains

We show that it is possible to extend the analyticity and the bounds of Chapter I for a $P(\Phi)_2$ interaction to a region of the shape of Fig. 1. This yields the Borel summability of the Taylor series of the Schwinger functions at $\lambda = 0$ (Hardy [7, Theorem 136, p. 192]). The extension rests on two basic identities related to dilatations and changes of mass. The use of these well-known identities for the present problem has been advocated by Simon.

We make the convention that Wick ordering is always with respect to the covariance occurring in the integral in question. This necessitates a Wick reordering formula, which we give now.

Let

$$A_{u,v}(\Phi^n) = \sum_{l=0}^{[n/2]} \Phi^{n-2l} \frac{n!}{2^l l! (n-2l)!} \left(\int \frac{d^2 k (u-v)}{(k^2+u)(k^2+v)} \right)^l . \tag{1}$$

It is easy to see that

$$:\Phi^n:_{m_0^2} = :A_{m_0^2, m_1^2}(\Phi^n):_{m_1^2} \tag{2}$$

and we extend A by linearity to polynomials.

Our two basic formulae are

Lemma 1. $\int \prod_{j=1}^{k} :\Phi^{n_j}:(x_j)\, W(x_1, \dots, x_k)\, e^{-\lambda \int d^2x :P(\Phi):(x) h(x)} d\mu_{m_1^2}$

$$= \left(\frac{m_2}{m_1}\right)^{2k} \int \prod_{j=1}^{k} :\Phi^{n_j}:(x_j)\, W\left(\frac{m_2}{m_1} x_1, \dots \frac{m_2}{m_1} x_k\right) dx_1 \dots dx_k \tag{3}$$

$$\cdot e^{-\lambda \left(\frac{m_2}{m_1}\right)^2 \int d^2x :P(\Phi):(x)\, h\left(\frac{m_2}{m_1} x\right)} d\mu_{m_2^2} .$$

Lemma 2. *For* $|b| < m_1^2, b \in \mathbb{R}$,

$$\lim_{h \to 1} \frac{e^{-\frac{b}{2}\int :\Phi^2:(x)h(x)\,dx} d\mu_{m_1^2}}{\int e^{-\frac{b}{2}\int :\Phi^2:(x)h(x)\,dx} d\mu_{m_1^2}} = d\mu_{m_1^2+b}\,. \tag{4}$$

These two lemmas are easy to prove and will be shown at the end of this chapter. It is worth noting that Lemma 2 means (we always write $h=1$)

$$\begin{aligned} &\frac{\int \prod_{j=1}^{k} :\Phi^{n_j}:(x_j)\, W(x_1, \ldots, x_k)\, e^{-\frac{b}{2}\int :\Phi^2:(x)\,dx} d\mu_{m_1^2}}{\int e^{-\frac{b}{2}\int :\Phi^2:(x)\,dx} d\mu_{m_1^2}} \\ &= \int \prod_{j=1}^{k} :A_{m_1^2, m_1^2+b}(\Phi^{n_j}):(x_j)\, W(x_1, \ldots, x_k)\, d\mu_{m_1^2+b}\,. \end{aligned} \tag{5}$$

Also, the left hand side of (5) is an analytic function of b, in $|b| < m_1^2 \alpha$ for a fixed $\alpha > 0$, as can be seen by the proof of Theorem 4.

Combining Lemmas 1 and 2, we get with $V(\Lambda) = \int_\Lambda :P(\Phi):(x)\,dx$:

$$\begin{aligned} f(\lambda, \beta) &= \lim_{\Lambda \to \infty} \frac{\int \prod_{j=1}^{k} :\Phi^{n_j}:(x_j)\, W(x)\,dx\, e^{-\frac{\lambda}{1+\beta}V(\Lambda)} d\mu_{m^2}}{\int e^{-\frac{\lambda}{1+\beta}V(\Lambda)} d\mu_{m^2}} \\ &= \lim_{\Lambda \to \infty} \frac{(1+\beta)^k \int \prod_{j=1}^{k} :\Phi^{n_j}:(x_j)\, W(\sqrt{1+\beta}\,x)\,dx\, e^{-\lambda V(\Lambda)} d\mu_{m^2(1+\beta)}}{\int e^{-\lambda V(\Lambda)} d\mu_{m^2(1+\beta)}} \\ &= g(\lambda, \beta) = \lim_{\Lambda \to \infty} \left(\int e^{-\lambda \tilde{V}(\Lambda)} e^{-\frac{\beta m^2}{2}\int :\Phi^2:(x)\,dx} d\mu_{m^2}\right)^{-1} (1+\beta)^k \\ &\quad \cdot \int \prod_{j=1}^{k} :A_{m^2, m^2(1+\beta)}(\Phi^{n_j}):(x_j)\, W(\sqrt{1+\beta}\,x)\,dx\, e^{-\lambda \tilde{V}(\Lambda)} e^{-\frac{\beta m^2}{2}\int :\Phi^2:(x)\,dx} d\mu_{m^2} \end{aligned} \tag{6}$$

where $\tilde{V}(\Lambda) = \int_\Lambda :A_{m^2, m^2(1+\beta)}(P):(x)\,dx$ and β being real and small.

We now want to extend β into the complex plane, and we identify f with the analytic continuation of the first expression in (6), g with the last.

If $|\beta| < \frac{1}{2}$, m^2 large and $|\lambda| < \varepsilon(m^2)$, $\mathrm{Re}\dfrac{\lambda}{1+\beta} > 0$, then, as in the proof of Theorem 4, $f(\lambda, \beta)$ is analytic in λ and β in the above region, and in this region one has, uniformly in λ and β, $\left|\dfrac{d^n}{d\lambda^n} f(\lambda, \beta)\right| \leqq \|W\|_p K_1 K_2^n (n!)^d$ where P has degree 2d, by Chapter I.

The discussion of $g(\lambda, \beta)$ is more delicate. We first restrict β such that the map $A_{m^2, m^2(1+\beta)}$ has analytic coefficients. This is the case for $|\beta| < 1$, cf. Eq. (1).

If $|\lambda|$, $|\beta|$ are small (say $< \varepsilon_1$) and $\mathrm{Re}\,\lambda \geqq 0$, $\mathrm{Re}\,\beta > 0$, then $g(\lambda, \beta)$ has a cluster expansion, cf. Theorem 4.

By the assumption of Theorem C (p. 5), W has analytic continuations into some angle and thus the terms in the expansion are analytic and therefore the expansion defines the analytic continuation of $g(\lambda, \beta)$ in $|\lambda| < \varepsilon_1$, $|\beta| < \varepsilon_1$, $\mathrm{Re}\,\lambda > 0$, $\mathrm{Re}\,\beta > 0$. Also we have again

$$\left|\frac{d^n}{d\lambda^n} g(\lambda, \beta)\right| < K_1 \|W(\sqrt{1+\beta}\cdot)\|_p K_2^n (n!)^d$$
$$\leqq K_1 \|W\|_{p,\delta} K_2^n (n!)^d, \qquad \text{for some } \delta > 0,$$

cf. p. 5[3].

By Eq. (6), the two functions $f(\lambda, \beta)$ and $g(\lambda, \beta)$ coincide in an open real subset in λ and β which is interior to their domains of analyticity. So they are analytic continuations of each other, and verify a bound of the form:

$$\left|\frac{d^n}{d\lambda^n} \bar{f}(\lambda, \beta)\right| < K_1' \|W\|_{p,\delta} K_2'^n (n!)^d, \tag{7}$$

where $\bar{f}$ is the common analytic continuation of f and g. Combining the domains of analyticity we see that $S_F(\lambda)$ extends to a function analytic in the region:

$$\mathscr{U} = \left\{\lambda \,|\, \lambda = \frac{\mu}{1+\beta}, \left(\left|\frac{\mu}{1+\beta}\right| < \varepsilon, \mathrm{Re}\frac{\mu}{1+\beta} > 0\right) \text{or } (\mathrm{Re}\,\mu > 0, \mathrm{Re}\,\beta > 0, \; |\mu| < \varepsilon, |\beta| < \varepsilon < 1)\right\}$$

so that $\mathscr{U} \supset \left\{\lambda, |\lambda| < \frac{\varepsilon}{2}, |\arg \lambda| < \frac{\pi}{2} + \mathrm{arctg}\frac{\varepsilon}{2}\right\}$, which is the region of Fig. 1. Theorem C is now an immediate consequence of Eq. (7), the form of $\mathscr{U}$ and Theorem 136 in Hardy [7, p. 192].

Proof of Lemma 1. Consider the cylinder function $F(\Phi(f_1), \ldots, \Phi(f_n))$ with $f_1, \ldots, f_n$ linearly independent functions in $\mathscr{S}(\mathbb{R}^2)$. Let $(\Lambda_{m^2})_{ij} = \langle f_i, C_{m^2} f_j\rangle$ with C_{m^2} the free covariance with mass m. Let $d\nu_{m^2}$ the measure on $\mathbb{R}^n$ given by

$$d\nu_{m^2}(z) = \frac{\exp\{-\frac{1}{2} z \Lambda_{m^2}^{-1} z\}\, dz}{\int \exp\{-\frac{1}{2} z \Lambda_{m^2}^{-1} z\}\, dz},$$

[3] We thank B. Simon for pointing out an error in the original manuscript at this step.

then $\int F(\Phi(f_1), \ldots, \Phi(f_n))\, d\mu_{m^2} = \int F(z)\, d\nu_{m^2}(z)$. We now dilate, then

$$\begin{aligned}(\Lambda_{m^2})_{ij} &= \int f_i(x) \frac{e^{ik(x-y)}}{k^2+m^2} f_j(y)\, dk\, dx dy \\ &= s^4 \int f_i(sx) \frac{e^{ik(x-y)}}{k^2+s^2m^2} f_j(sy)\, dk\, dx\, dy \\ &= s^4 \langle f_i^{(s)}, C_{s^2m^2} f_j^{(s)} \rangle = s^4 (\Lambda^{(s)}_{s^2m^2})_{ij} \quad \text{with} \quad f_i^{(s)}(x) = f_i(sx)\,.\end{aligned}$$

Thus $d\nu_{m^2}(z) = \dfrac{\exp\{-\frac{1}{2}s^{-2}z(\Lambda^{(s)}_{s^2m^2})^{-1}s^{-2}z\}\, ds^{-2}z}{\int \exp\{-\frac{1}{2}s^{-2}z(\Lambda^{(s)}_{s^2m^2})^{-1}s^{-2}z\}\, ds^{-2}z} = d\nu^{(s)}_{s^2m^2}(s^{-2}z)$ and

$$\begin{aligned}\int F(z)\, d\nu_{m^2}(z) &= \int F(z)\, d\nu^{(s)}_{s^2m^2}(s^{-2}z) = \int F(s^{+2}z)\, d\nu^{(s)}_{s^2m^2}(z) \\ &= \int F(s^{+2}\Phi(f_1^{(s)}), \ldots, s^{+2}\Phi(f_n^{(s)}))\, d\mu_{s^2m^2}\end{aligned}$$

since the $f_i^{(s)}$ are also linearly independent.

The general case follows by approximating Wick polynomials as in Dimock and Glimm [2]. We first replace $\Phi(y)$ by $\Phi_K(y) = \Phi(\chi_K(\cdot - y))$ with χ_K an approximate δ function, $\chi_K(x) = \frac{1}{K^2}\chi(K^{-1}x)$, $\chi(0) = 1$, $\chi(x) = 0$ for $|x| > 1$, $\int \chi(x)\, dx = 1$.

Then we replace integrals by Riemann sums $\int dx^2 \to \frac{1}{j^2} \sum_{x \in \frac{1}{j}\mathbb{Z}^2}$. In this way we can approximate our expressions by cylinder functions for which we gave the proof. The result comes from the fact that dilatations go through this approximation procedure.

Proof of Lemma 2. The proof relies solely on the Gaussian character. Think of C as $(-\Delta + m^2)^{-1}$, typically. There is a unique countable additive measure $d\mu_C$ such that $\int e^{i\Phi(f) + \frac{1}{2}(f, Cf)} d\mu_C = 1$ and this measure is Gaussian [Gelfand-Vilenkin, 4]. By Dimock and Glimm [2]

$$\begin{aligned}\int \Phi(g)\, e^{i\Phi(f) + \frac{1}{2}(f, Cf)} d\mu_C &= \int dx\, dy\, g(x)\, C(x, y) \int \frac{\partial}{\partial \Phi(y)} e^{i\Phi(f) + \frac{1}{2}(f, Cf)} d\mu_C \\ &= i(g, Cf)\,. \end{aligned} \tag{8}$$

The Gaussian character of $d\mu_C$ implies that for all measurable functions F,

$$\int F(\Phi)\, e^{i\Phi(f) + \frac{1}{2}(f, Cf)} d\mu_C = \int F(\Phi + \hat{f})\, d\mu_C\,,$$

where $(\Phi + \hat{f})(g) = \Phi(g) + i(Cf, g)$. Lemma 2 is proved if we can show for $C'^{-1} = C^{-1} + h$, h a symmetric function in $\mathbb{R}^2 \times \mathbb{R}^2$, that

$$\frac{\int e^{i\Phi(g) + \frac{1}{2}(g, C'g)} e^{-\frac{1}{2}\int \Phi(x)\Phi(y) h(x, y)\, dx\, dy} d\mu_C}{\int e^{-\frac{1}{2}\int \Phi(x)\Phi(y) h(x, y)\, dx\, dy} d\mu_C} = 1\,, \tag{9}$$

and by taking limits on h.

Write $g = C'^{-1} f = C^{-1} f + h*f, h*f(x) = \int h(x, y) f(y)\, dy$, and use the Gaussian character of $d\mu_C$ [Eq. (8)]. So the left hand side of (2) is:

$$\frac{\int e^{i\Phi(C^{-1}f) + i\Phi(h*f) + \frac{1}{2}(f, C^{-1}f) + \frac{1}{2}(h*f, f) - \frac{1}{2}\int \Phi(x)\Phi(y) h(x,y)\, dx\, dy}\, d\mu_C}{\int e^{-\frac{1}{2}\int \Phi(x)\Phi(y) h(x,y)\, dx\, dy}\, d\mu_C}$$

$$= \frac{\int e^{i(\widehat{\Phi + C^{-1}f})(h*f) + \frac{1}{2}(h*f, f) - \frac{1}{2}\int (\widehat{\Phi + C^{-1}f})(x)(\widehat{\Phi + C^{-1}f})(y) h(x,y)\, dx\, dy}\, d\mu_C}{\int e^{-\frac{1}{2}\int \Phi(x)\Phi(y) h(x,y)\, dx\, dy}\, d\mu_C}$$

$= 1$; this is obtained by evaluating the exponent.

Finally we give the

Proof of Theorem D. By [6, Chapter VI] $\lim_{\Lambda \to \mathbb{R}^2} \frac{1}{|\Lambda|} \log S_{F=1}(\lambda)$ converges. Let $F_\Delta = \int_\Delta : P(\Phi) : (x)\, dx$. The derivatives of the pressure are of the form $\frac{(-1)^k}{|\Lambda|} \sum_{\Delta_1, \dots, \Delta_k} S^T_{F_{\Delta_1}, \dots, F_{\Delta_k}}(\lambda)$, by Dimock [1]. Now by Theorem B, the sum over $\Delta_2, \dots, \Delta_k$ converges and the sum over Δ_1 yields a contribution bounded by something proportional to $|\Lambda|$ and hence the theorem is proved.

Appendix

We consider integrals of Wick monomials with respect to a Gaussian measure $d\mu_C$ with covariance $C \in \mathscr{C}$, the set of convex combinations of $(-\Delta + m_0^2)^{-1}$ with Dirichlet boundary conditions along Γ^c [5, p. 202 and p. 224].

We index $\mathbb{Z}^2$ by numbers $l = 1, 2, \dots$, and we write $\mathbb{R}^2 = \bigcup_l S_l$ where the S_l are unit lattice squares centered at points of $\mathbb{Z}^2 \subset \mathbb{R}^2$. Let $R = \int \prod_{j=1}^r : \Phi(x_i)^{p_i} : w(x_1, \dots, x_r)\, dx$, with w supported in $S_{l_1} \times \dots \times S_{l_r}$. Let $n_i = \sum_{j : l_j = i} p_j$, the degree of R in S_i. Let $m_0 \geqq 1$ and $q \geqq p' \cdot (\max p_i)$, $n = \Sigma n_i$, $p'^{-1} + p^{-1} = 1$.

Theorem. *For $C \in \mathscr{C}$ one has*

$$\left|\int R\, d\mu_C\right| \leqq \|w\|_p (C m_0^{-\frac{2}{q}})^n \prod_{i=1}^\infty n_i!^{1/2}.$$

Proof. The theorem is a slight improvement of Theorem 9.4 in [5, p. 236], replacing $\prod_{i=1}^\infty n_i!$ by $\prod_{i=1}^\infty n_i!^{1/2}$. The proof is identical to the one in [2], but we give a better bound on

$$\sum_{G \in \mathscr{V}(R)} \prod_l e^{-\alpha \operatorname{dist}(S_{l_1}, S_{l_2})}, \qquad (\mathrm{A}_1)$$

where $\mathscr{V}(R)$ is the sum over all vacuum graphs G (pairings of Φ's), $\prod_l$ is the product over all the bonds in the pairings, extending from square S_{l_1} to S_{l_2} and $\alpha \geqq \frac{m_0}{2} > 0$.

We show $(\mathrm{A}_1) \leqq \prod_i n_i!^{1/2} C^n$, which, inserted into the proof of Theorem 9.4 of [5, p. 236] proves our assertion.

Let now q_{kl} denote the number of bonds from square S_k to square S_l[4]. Then (A_1) can be bounded by

$$\Sigma'' \left\{ \prod_k \frac{n_k!}{\prod_l q_{kl}!} \right\} \left\{ \prod_{k>l} (q_{kl}!\, e^{-\alpha q_{kl} \operatorname{dist}(S_k, S_l)}) \right\} \prod_k \left(\frac{q_{kk}!}{\frac{q_{kk}}{2}!\, 2^{\frac{q_{kk}}{2}}} \right) \qquad (\mathrm{A}_2)$$

where Σ'' ranges over the set $\{q_{kl} \mid q_{kl} = q_{lk}, \sum_l q_{kl} = n_k, q_{kk} \text{ even}\}$, and the first factor comes from the number of ways one can choose the points which connect to a given point, and the other factors come from the number of pairings and the exponential decrease.

We use now

$$\frac{(2a)!}{a!\, 2^a} \leqq (2a)!^{1/2} \leqq a!\, 2^a$$

and $\Pi a_i! \leqq (\Sigma a_i)!$, and we symmetrize the product $\prod_{k>l}$ by taking square roots, to obtain the following bound for (A_2)

$$\Sigma'' \prod_k \left(\frac{n_k!}{\prod_l q_{kl}!} e^{-\frac{\alpha}{2} \Sigma q_{kl} \operatorname{dist}(S_k, S_l)} (n_k!)^{1/2} C_1^{n_k} \right). \qquad (\mathrm{A}_3)$$

The theorem is proved if we can show that (A_3) is bounded by $C_2^n \Pi (n_k!)^{1/2}$. Now (A_3) is certainly bounded by

$$\prod_k \left(\sum_{\sum_{l=0}^{\infty} q_l = n_k} \frac{n_k!}{\prod_l q_l!} e^{-\frac{\alpha}{2} \sum_l q_l \operatorname{dist}(S_k, S_l)} \right) C_1^{n_k} (n_k!)^{1/2}, \qquad (\mathrm{A}_4)$$

and the theorem follows if we can show that the ( ) in (A_4) is bounded by $C_3^{n_k}$.

Note now that there is for each k an arrangement of indices such that $\operatorname{dist}(S_k, S_l) \geqq \frac{1}{2}(l^{1/2} - 3)$, so that it suffices to bound

$$\sum_{\sum_{j=0}^{\infty} d_j = d} \frac{d!}{\Pi d_j!} e^{-\alpha' \Sigma d_i i^{1/2}}$$

[4] For $k = l$, q_{kk} is the number of bonds times 2.

which is equal to

$$\left(\sum_{i=0}^{\infty} e^{-\alpha' i^{1/2}}\right)^d = C_4^d$$

and this proves the theorem.

Remark. The combinatorial argument extends to any number ν of dimensions because there is an arrangement of indices such that $\operatorname{dist}(S_k, S_l) \geqq \mathcal{O}(l^{1/\nu})\ l = 1, 2, \ldots$.

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