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Infinite Volume Asymptotics in $P(\phi)_2$ Field Theory

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Abstract. We prove a number of asymptotic results in the $P(\phi)_2$ theory in the limit when the space cut-offs are removed, in particular the behavior of E_l and $Z_{t,l}$ as $t, l\to\infty$. Such results are used to study the question of orthogonality of infinite volume Euclidean measures $\mu_\infty(\lambda)$ for varying interaction constants λ.

1. Asymptotics

In this paper we consider any fixed real polynomial $P(y)$ with $P(0)=0$ which is bounded below, and the corresponding $P(\phi)_2$ quantum field theory in two-dimensional space-time [1]. The approximate, or cut-off, Hamiltonian is

$$H_l = H_0 + \lambda \int_{-l/2}^{l/2} :P(\phi(x)):\, dx \tag{1.1}$$

where H_0 is the usual free Hamiltonian of mass $m_0 > 0$, and $\lambda \geqq 0$ is the coupling constant. H_l has a simple eigenvalue E_l at the bottom of its spectrum, with corresponding eigenvector Ω_l, the (approximate) physical vacuum. The positive operator $H_l - E_l$ has no spectrum in some interval $(0, m_l)$ where $m_l > 0$. With Ω_0 denoting the bare vacuum in Fock space, it is known that $(\Omega_0, \Omega_l) \neq 0$. Thus $|(\Omega_0, \Omega_l)|^2 = \exp(-l\eta_l)$ defines η_l, where Ω_0 and Ω_l are both taken to have norm 1. The quantity

$$Z_{t,l} = e^{G_{t,l}} = (\Omega_0, e^{-tH_l}\Omega_0) \tag{1.2}$$

is the analogue of the partition function in classical statistical mechanics.

The following asymptotic results are known to hold for any $\lambda \geqq 0$ [2, 3].

Theorem 1. *There are functions $\alpha_\infty(\lambda)$ and $\beta_\infty(\lambda)$ such that*

i) $E_l = -\alpha_\infty l - \beta_\infty + o(1)$ *as* $l \to \infty$.

ii) $0 < A \leqq \eta_l \leqq B < \infty$ *as* $l \to \infty$.

iii) $G_{t,l} = \alpha_\infty t l + o(tl)$ *as* $t, l \to \infty$.

Further results are known when λ is restricted to a sufficiently small interval. Thus it has been shown by Glimm, Jaffe, and Spencer [4] that for all sufficiently small λ

$$\underline{m} \equiv \liminf_{l\to\infty} m_l > 0 .$$

We write $\lambda_c = \inf\{\lambda : \underline{m}(\lambda) = 0\}$.

Theorem 2 ([5]). *There is a positive* $\lambda_0 \leqq \lambda_c$ *such that for* $\lambda < \lambda_0$, $\tilde{m} < \underline{m}(\lambda)$, *and* $C > 0$ *one has*

i) $\eta_l = -\beta_\infty + o(1)$ *as* $l \to \infty$.

ii) $G_{t,l} = -tE_l - l\eta_l + O(e^{-\tilde{m}t})$ *as* $t \geqq Cl \to \infty$.

iii) $G_{t,l} = \alpha_\infty tl + \beta_\infty(t+l) + o(t+l)$ *as* $t, l \to \infty$.

Theorem 2 was useful in proving local L_1-convergence of the cut-off Euclidean fields to the physical (infinite volume) fields, thus it seems reasonable that further asymptotic properties should also be useful. In this direction it has been conjectured that E_l has an asymptotic expansion in descending powers of l as $l \to \infty$ [2]. This is indeed the case, but we can state a much stronger result.

Theorem 3. *For* $\lambda < \lambda_0$ *there is a* $\gamma_\infty = \gamma_\infty(\lambda)$ *such that for* $\tilde{m} < \underline{m}(\lambda)$

i) $E_l = -\alpha_\infty l - \beta_\infty + O(e^{-\tilde{m}l})$ *as* $l \to \infty$.

ii) $\eta_l = -\beta_\infty - \dfrac{\gamma_\infty}{l} + O(e^{-\tilde{m}l})$ *as* $l \to \infty$.

iii) $G_{t,l} = \alpha_\infty tl + \beta_\infty(t+l) + \gamma_\infty + O(le^{-\tilde{m}t} + te^{-\tilde{m}l})$ *as* $t, l \to \infty$.

The proof of this theorem will be given in Section 3 of this paper; meanwhile we make several remarks about Theorem 3, and conclude this section with an application to the asymptotic properties of the spectral measure of H_l in the limit $l \to \infty$. Section 2 of the paper concerns the application of Theorem 3 to the orthogonality between infinite volume Euclidean measures corresponding to different λ.

Remark 1. $(-E_l/l)$ and $(E_l + \alpha_\infty l)$ are known to be positive non-decreasing functions of l when $\lambda > 0$; hence $\alpha_\infty > 0$ and $\beta_\infty < 0$ in this case [2]. We do not know any corresponding monotonicity property of η_l, hence know nothing about the sign of γ_∞[1].

Remark 2. It seems likely that the introduction of λ_0 in Theorem 2 is only a technicality necessitated by the method of proof. We conjecture that in fact $\lambda_0 = \lambda_c$. This would follow if it were known, for example, that $\eta_l = -\beta_\infty + o(1)$ as $l \to \infty$ for all $\lambda < \lambda_c$, as explained in [5].

Remark 3. In view of the connection between the Euclidean $P(\phi)_2$ theory and certain two-dimensional Ising models [3], one can presumably prove results for Ising models analogous to Theorem 3 with the methods discussed in this paper.

[1] Second order perturbation theory suggests however, that η_l and $-l(\eta_l + \beta_\infty)$ are positive non-decreasing functions of l, so that $\gamma_\infty > 0$.

We define ϱ_l via the spectral theorem by the equation

$$(\Omega_0, \exp(-t(H_l - E_l))\,\Omega_0) = \int_0^\infty e^{-tq} d\varrho_l(q)\,. \tag{1.3}$$

$\varrho_l[0] = |(\Omega_0, \Omega_l)|^2 = \exp(-l\eta_l)$. A reasonable conjecture is that for any $q \geqq 0$, $\varrho_l[0,q]/\varrho_l[0]$ has a limit as $l \to \infty$; we have only been able to obtain the weaker result:

Theorem 4. *For $\lambda < \lambda_0$, $\tilde{m} < \underline{m}$, and any fixed q, $\varrho_l[0,q]/\varrho_l[0] = O(l^{q/\tilde{m}})$ as $l \to \infty$.*

Proof. We first note that

$$\exp(G_{t,l} + tE_l) = \int_0^\infty e^{-tq'} d\varrho_l(q') \geqq e^{-tq} \varrho_l[0,q] \tag{1.4}$$

so that

$$1 \leqq \varrho_l[0,q]/\varrho_l[0] \leqq \exp(G_{t,l} + tE_l + l\eta_l + tq)\,. \tag{1.5}$$

It follows from Theorem 3 that

$$G_{t,l} + tE_l + l\eta_l = O(l \exp(-m't) + t \exp(-m'l))$$

as $t, l \to \infty$ for any $m' < \underline{m}$; we choose $\tilde{m} < m' < \underline{m}$, let $t = \log l/\tilde{m}$, and then apply this estimate to (1.5). This yields

$$\varrho_l[0,q]/\varrho_l[0] = O\big(\exp(l^{1-(m'/\tilde{m})} + (q/\tilde{m}) \log l\big) = O(l^{q/\tilde{m}})$$

as desired. Q.E.D.

2. Orthogonality of Euclidean Measures

We recall some notation for the $P(\phi)_2$ theory. The basic measurable space is $\boldsymbol{\Omega} = \mathscr{D}'(\mathbb{R}^2)$ equipped with the σ-algebra $\mathscr{B}$ generated by functions on $\boldsymbol{\Omega}$ of the form $\phi_u: F \to F(u)$ where u is a test-function in $\mathscr{D}(\mathbb{R}^2)$. The free Euclidean measure is denoted μ_0. It is Gaussian with mean zero and covariance $\boldsymbol{E}_0(\phi_u \phi_v) = \iint u(-\Delta + m_0^2)^{-1}\, v\, dx dt$. The cut-off interacting measures are defined by $d\mu_l = X_l d\mu_0$, where

$$X_l(F) = (Z_{l,l})^{-1} \exp\left(-\lambda \int_{-l/2}^{l/2} \int_{-l/2}^{l/2} :P(F(x,t)):\, dx dt\right) \tag{2.1}$$

with $F \in \boldsymbol{\Omega} = \mathscr{D}'(\mathbb{R}^2)$. (In this discussion we take square cutoffs for the sake of convenience.) The quantity $Z_{l,l}$ is a normalisation constant, so determined that μ_l is a probability measure. It can be shown by the Feynman-Kac formula that this definition is consistent with the right hand side of (1.2), but this relationship will not concern us in the following.

We let $\mathscr{B}_l$ be the σ-algebra generated by only those ϕ_u for which u has support in the square $-l/2 < x, t < l/2$, and let $\boldsymbol{E}_l$ denote conditional expectation with respect to $\mathscr{B}_l$. With $\lambda < \lambda_0$ and l arbitrarily fixed, $\boldsymbol{E}_l(X_{l'})$ converges in $L_1(\boldsymbol{\Omega}, \mathscr{B}, \mu_0)$ as $l' \to \infty$. This yields the infinite volume Euclidean measure μ_∞ which is absolutely continuous with respect to μ_0 when both are restricted to $\mathscr{B}_l$[5]:

$$\frac{d(\mu_\infty | \mathscr{B}_l)}{d(\mu_0 | \mathscr{B}_l)} = \lim_{l' \to \infty} \boldsymbol{E}_l(X_{l'}) = Y_l \,. \tag{2.2}$$

μ_∞ and associated quantities depend on λ, of course.

The Euclidean version of Haag's theorem states the orthogonality (mutual singularity) of $\mu_\infty(\lambda)$ and $\mu_\infty(\lambda')$ for $\lambda \neq \lambda'$. Before discussing the relevance of Theorem 3 to this orthogonality question, we wish to point out that there is a very simple proposition of ergodic theory which, when applied to Euclidean field theory, yields the Euclidean version of Haag's theorem. This proposition is certainly not new [6] and its relevance to field theory has also been remarked by Fröhlich [7] and by J. Rosen and Simon [8].

Theorem 5. *If T is a measurable transformation on a measurable space $(\boldsymbol{\Omega}, \mathscr{B})$, measure preserving and ergodic with respect to two probability measures μ_1 and μ_2, then either $\mu_1 = \mu_2$ or else μ_1 and μ_2 are orthogonal.*

Proof. Suppose $\mu_1 \neq \mu_2$; then there is an $A \in \mathscr{B}$ such that $\mu_1(A) \neq \mu_2(A)$. Let

$$S_i = \left\{ \omega \in \boldsymbol{\Omega} : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_A(T^j \omega) = \mu_i(A) \right\}$$

for $i = 1, 2$ where χ_A is the indicator function of the set A. By Birkhoff's ergodic theorem $\mu_1(S_1) = \mu_2(S_2) = 1$. But S_1 and S_2 are disjoint, hence the orthogonality of μ_1 and μ_2. Q.E.D.

Remark 4. Theorem 4 applies to field theory by taking $\mu_1 = \mu_\infty(\lambda_1)$ and $\mu_2 = \mu_\infty(\lambda_2)$ with $\lambda_1 \neq \lambda_2$ and T a non-zero translation acting on $\mathscr{D}'$. The fact that T is measure preserving is nothing more than the Euclidean invariance of the theory, while its ergodicity with respect to $\mu_\infty(\lambda)$ for $\lambda < \lambda_0$ follows from the fact that T is mixing, and this follows from the existence of a uniform mass gap ($\underline{m} > 0$) [4, 7].

Remark 5. The results of Dimock [9] on the asymptotic nature of the perturbation series (expansion in powers of λ) of the Schwinger functions shows that $\mu_\infty(\lambda) \neq \mu_\infty(\lambda')$ at least for very small $\lambda \neq \lambda'$; thus Theorem 5 and Remark 4 show that in fact $\mu_\infty(\lambda)$ is orthogonal to $\mu_\infty(\lambda')$.

Proofs of the orthogonality of $\mu_\infty(\lambda)$ and $\mu_\infty(\lambda')$ have also been obtained by Schrader [10], Fröhlich [7], and J. Rosen and Simon [8] for various regions of coupling constant values. Although Schrader's results are not the most general of these, they are particularly interesting in that they relate orthogonality of measures to convexity properties of $\alpha_\infty(\lambda)$. It is known that this function is convex [11]; Schrader's result is

Theorem 6. *If $\lambda < \lambda' < \lambda_0$ and α_∞ is not affine (inhomogeneous linear) on the closed interval $[\lambda, \lambda']$, then $\mu_\infty(\lambda)$ and $\mu_\infty(\lambda')$ are orthogonal.*

Although it is probably true in these two-dimensional models that $\mu_\infty(\lambda)$ is orthogonal to $\mu_\infty(\lambda')$ for any $\lambda \neq \lambda'$ and that α_∞ is strictly convex for all λ, it nevertheless seems to us of interest to extend Schrader's result by strengthening the relation between orthogonality of measures ("distinctness of field theories") and convexity properties of "thermodynamic" parameters. In higher dimensional models, perhaps certain physical coupling constant values are singled out by the requirement of orthogonality, which means that such a theory would so to speak predict its own permissible interaction strengths. Thus the relationship alluded to may be of more than passing interest.

Theorem 7. *Suppose $\lambda < \lambda' < \lambda_0$. If α_∞ is affine on $[\lambda, \lambda']$ then β_∞ is convex on $[\lambda, \lambda']$. If α_∞ and β_∞ are both affine on $[\lambda, \lambda']$, then $\mu_\infty(\lambda) = \mu_\infty(\lambda')$.*

The proof of Theorem 7 depends on the notion of the Kakutani product of two finite measures defined on the same measurable space $(\boldsymbol{\Omega}, \mathscr{B})$. In the following δ is a fixed but arbitrary number, $0 < \delta < 1$.

Definition. The Kakutani product of two finite measures μ and μ' is

$$K(\mu, \mu') = \int_{\boldsymbol{\Omega}} \left(\frac{d\mu}{d\nu}\right)^{\delta} \left(\frac{d\mu'}{d\nu}\right)^{1-\delta} d\nu \tag{2.3}$$

where ν is any measure with respect to which both μ and μ' are absolutely continuous (the definition is independent of the choice of ν).

The Kakutani product has the following simple properties [12].

Proposition 8. *Suppose μ and μ' are probability measures. Then*

i) $0 \leqq K(\mu, \mu') \leqq 1$.

ii) $K(\mu, \mu') = 0$ *if and only if* μ *and* μ' *are orthogonal.*

iii) $K(\mu, \mu') = 1$ *if and only if* $\mu = \mu'$.

If $\mathscr{B}_1$ is a sub-σ-algebra of $\mathscr{B}$, we write $K(\mu, \mu' | \mathscr{B}_1)$ for the Kakutani product of the restricted measures $\mu | \mathscr{B}_1$ and $\mu' | \mathscr{B}_1$. The following technical lemma is useful [12]:

Lemma 9. *If $\mathscr{B}_1 \subset \mathscr{B}_2$ then $K(\mu, \mu' | \mathscr{B}_1) \geqq K(\mu, \mu' | \mathscr{B}_2)$. If $\{\mathscr{B}_l\}$ is an increasing family of σ-algebras generating $\mathscr{B}$ then $\lim_{l \to \infty} K(\mu, \mu' | \mathscr{B}_l) = K(\mu, \mu')$.*

Proof of Theorem 7. Assume temporarily that $K(\mu_\infty(\lambda), \mu_\infty(\lambda')) \geqq \lim_{l\to\infty} K(\mu_l(\lambda), \mu_l(\lambda'))$. Recalling the definition of μ_l [see (2.1) above] and $Z_{l,l} = \exp(G_{l,l})$, we see that

$$K(\mu_l(\lambda), \mu_l(\lambda')) = \exp[G_{l,l}(\delta\lambda + (1-\delta)\lambda') - \delta G_{l,l}(\lambda) - (1-\delta) G_{l,l}(\lambda')]. \tag{2.4}$$

Thus by Theorem 3

$$K(\mu_\infty(\lambda), \mu_\infty(\lambda')) \geqq \lim_{l\to\infty} \exp[a_\infty l^2 + 2b_\infty l + c_\infty] \tag{2.5}$$

where $a_\infty = \alpha_\infty(\delta\lambda + (1-\delta)\lambda') - \delta\alpha_\infty(\lambda) - (1-\delta)\alpha_\infty(\lambda')$, and b_∞ (resp. c_∞) is analogously defined in terms of β_∞ (resp. γ_∞). By Proposition 8(*i*) it is clear that $a_\infty = 0$ (i.e., α_∞ affine) implies $b_\infty \leqq 0$ (i.e., β_∞ convex), and that $a_\infty = b_\infty = 0$ implies $c_\infty \leqq 0$. But the latter case would imply by Proposition 8(ii) that $\mu_\infty(\lambda)$ is not orthogonal to $\mu_\infty(\lambda')$, and thus by Theorem 5 (and Remark 4) that $\mu_\infty(\lambda) = \mu_\infty(\lambda')$.

It remains to show $K(\mu_\infty(\lambda), \mu_\infty(\lambda')) \geqq \lim_{l\to\infty} K(\mu_l(\lambda), \mu_l(\lambda'))$. By the first part of Lemma 9, $K(\mu_l(\lambda), \mu_l(\lambda')) \leqq K(\mu_l(\lambda), \mu_l(\lambda') | \mathscr{B}_{l'})$ while by the second part of Lemma 9, $K(\mu_\infty(\lambda), \mu_\infty(\lambda')) = \lim_{l'\to\infty} K(\mu_\infty(\lambda), \mu_\infty(\lambda') | \mathscr{B}_{l'})$. It thus suffices to show that, for any fixed l', $\varepsilon_{l,l'} \to 0$ as $l \to \infty$ where

$$\varepsilon_{l,l'} = |K(\mu_l(\lambda), \mu_l(\lambda') | \mathscr{B}_{l'}) - K(\mu_\infty(\lambda), \mu_\infty(\lambda') | \mathscr{B}_{l'})|. \tag{2.6}$$

Letting $W_{l,l'} = \boldsymbol{E}_{l'}(X_l)$, we have

$$\begin{aligned} \varepsilon_{l,l'} &= |\textstyle\int W_{l,l'}(\lambda)^\delta W_{l,l'}(\lambda')^{1-\delta} d\mu_0 - \int Y_{l'}(\lambda)^\delta Y_{l'}(\lambda')^{1-\delta} d\mu_0| \\ &\leqq \textstyle\int |W_{l,l'}(\lambda)^\delta - Y_{l'}(\lambda)^\delta| W_{l,l'}(\lambda')^{1-\delta} d\mu_0 \\ &\quad + \textstyle\int Y_{l'}(\lambda)^\delta |W_{l,l'}(\lambda')^{1-\delta} - Y_{l'}(\lambda')^{1-\delta}| d\mu_0. \end{aligned} \tag{2.7}$$

By Hölder's inequality the first integral on the right hand side is bounded by $(\int |W_{l,l'}(\lambda)^\delta - Y_{l'}(\lambda)^\delta|^{1/\delta})^\delta$. Now for $a, b \geqq 0$ and $0 < \delta < 1$ one has $|a^\delta - b^\delta| \leqq |a-b|^\delta$, whence the above quantity is bounded by

$$(\textstyle\int |W_{l,l'}(\lambda) - Y_{l'}(\lambda)| d\mu_0)^\delta.$$

The local L_1-convergence results of [5] state that this quantity tends to zero as $l \to \infty$; the other term in the right hand side of (2.7) is handled similarly. Q.E.D.

Remark 6. We conjecture that $K(\mu_\infty(\lambda), \mu_\infty(\lambda')) = \lim_{l\to\infty} K(\mu_l(\lambda), \mu_l(\lambda'))$. If this were known, Theorems 6 and 7 could be immediately strengthened to yield that $\mu_\infty(\lambda)$ is orthogonal to $\mu_\infty(\lambda')$ unless α_∞ and β_∞ are both affine on $[\lambda, \lambda']$ in which case γ_∞ is also affine and $\mu_\infty(\lambda) = \mu_\infty(\lambda')$.

3. Proof of Theorem 3

Let $g(l) = -E_l - \alpha_\infty l - \beta_\infty$ and $h(l) = -l(\eta_l + \beta_\infty) + lg(l)$. Theorem 1(i) says that

$$\lim_{l\to\infty} g(l) = 0\,. \tag{3.1}$$

Theorem 2(ii) may be restated in the form

$$\begin{aligned} G_{t,l} = \alpha_\infty tl + \beta_\infty(t+l) + (t-l)\,g(l) + h(l) \\ + O(e^{-\tilde{m}t}) \quad \text{as} \quad t \geqq Cl \to \infty\,. \end{aligned} \tag{3.2}$$

This, together with Nelson's symmetry $G_{t,l} = G_{l,t}$ yields

$$\begin{aligned} h(t) - h(l) = (t-l)\,[g(t) + g(l)] + O(e^{-\tilde{m}l}) \\ \text{as} \quad l/C \geqq t \geqq Cl \to \infty\,. \end{aligned} \tag{3.3}$$

Let $\tau > 0$, and consider the above equation, when for the pair (t, l) one substitutes in turn $(l, l+\tau)$, $(l+\tau, l+2\tau)$, and $(l+2\tau, l)$. Adding these three equations, h cancels out, and for fixed τ one obtains

$$\tau[g(l+2\tau) - g(l+\tau)] - \tau[g(l+\tau) - g(l)] = 0(e^{-\tilde{m}l}) \quad \text{as} \quad l \to \infty\,. \tag{3.4}$$

We now replace l in (3.4) by $l, l+\tau, l+2\tau, \ldots$ and sum the resulting infinite series. In view of the vanishing of g at infinity, and observing

$$\sum_{j=0}^{\infty} e^{-\tilde{m}(l+j\tau)} = \frac{e^{-\tilde{m}l}}{1 - e^{-\tilde{m}\tau}} = O(e^{-\tilde{m}l})\,, \tag{3.5}$$

one obtains

$$\tau g(l+\tau) - \tau g(l) = O(e^{-\tilde{m}l})\,, \tag{3.6}$$

and then repeating the argument

$$\tau g(l) = O(e^{-\tilde{m}l})\,. \tag{3.7}$$

This proves Part (i) of Theorem 3. Returning to (3.3), we note that when $l \leqq t \leqq l+1$ it now asserts

$$h(t) - h(l) = O(e^{-\tilde{m}l})\,.$$

When $t \geqq l+1$ we write $t = l + n\tau$ with $\tau < 1$; then

$$\begin{aligned} h(l+j\tau+\tau) - h(l+j\tau) = \tau[g(l+j\tau+\tau) + g(l+j\tau)] \\ + O(e^{-\tilde{m}(l+j\tau)}) = O(e^{-\tilde{m}(l+j\tau)})\,. \end{aligned} \tag{3.8}$$

Summing over j yields as before

$$h(t) - h(l) = O(e^{-\tilde{m}l})\,. \tag{3.9}$$

We see then that

$$\lim_{t \geqq l \to \infty} [h(t) - h(l)] = 0\,.$$

Thus h has a limit at infinity, say γ_∞; moreover, passing to the limit $t\to\infty$ in (3.9) yields

$$h(l)=\gamma_\infty+O(e^{-\tilde{m}l}).$$

Since $\eta_l=-\beta_\infty-h(l)/l+g(l)$, the Conclusion (ii) of Theorem 3 follows.

Finally, we combine Part (ii) of Theorem 2 with the asymptotics of E_l and η_l to see that as $t\geqq Cl\to\infty$,

$$G_{t,l}=-tE_l-lE_t-\alpha_\infty tl+\gamma_\infty+O(te^{-\tilde{m}t}+le^{-\tilde{m}l}). \tag{3.10}$$

We now use Nelson's symmetry, together with the fact that $t\exp(-\tilde{m}'t)=O(\exp(-\tilde{m}t))$ for $\tilde{m}<\tilde{m}'<\underline{m}$. Thus (3.10) is replaced by the uniform estimate:

$$G_{t,l}=-tE_l-lE_t-\alpha_\infty tl+\gamma_\infty+O(e^{-\tilde{m}t}+e^{-\tilde{m}l}) \tag{3.11}$$

as t, $l\to\infty$. Part (iii) of Theorem 3 is now obtained by substituting the asymptotic expansion for E_l and E_t into (3.11). Q.E.D.

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