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The Absence of Even Bound States for $\lambda(\varphi^4)_2$ *

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Abstract. We use correlation inequalities for $\lambda(\varphi^4)_2$ to exclude even bound states of energy less than $2m$.

Let H denote the Hamiltonian for the $\lambda\varphi^4$ quantum field model. We define m to be mass of H. The purpose of this note is to show that for $\lambda < \lambda$ (critical) H restricted to the even subspace has no spectrum in the interval $(0, 2m)$. Thus we exclude even bound states of energy less than $2m$. In [1] this result was established for λ sufficiently small. Feldman [2] obtained our results assuming $G_{4n} < 0$, where G_{4n} is the $4n^{th}$ truncated Green's function. These inequalities on G_{4n} are only known to hold for $n = 1$.

Our proof is an immediate consequence of correlation inequalities due to Lebowitz [3] and the spin 1/2 approximation of Simon and Griffiths [4]. Our methods are similar to those of Feldman.

Following [3] consider a spin 1/2 pair ferromagnetic interaction and two duplicate spin variables denoted by s_i and σ_i. Let

$$q_i = \sigma_i + s_i \qquad t_i = \sigma_i - s_i$$

and let

$$q_A = \prod_{i \in A} q_i \qquad t_A = \prod_{i \in A} q_i \,.$$

Then from [3] we have

$$\langle q_A t_B \rangle \leqq \langle q_A \rangle \langle t_B \rangle$$

where $\langle . \rangle$ denotes the product expectation.

Similarly let Φ and Φ' denote duplicate Euclidean fields for $\lambda\varphi^4$ and for $X_i = (\boldsymbol{X}_i, X_i^0) \in R^2$ set

$$Q_{X_i} = \Phi(X_i) + \Phi'(X_i) \qquad T_{X_i} = \Phi(X_i) - \Phi'(X_i)$$

and

$$Q_A = \prod_{X_i \in A} Q_{X_i} \qquad T_B = \prod_{X_i \in B} T_{X_i} \,.$$

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From the spin 1/2 approximation and multilinearity it follows that

$$\langle Q_A T_B \rangle \leqq \langle Q_A \rangle \langle T_B \rangle . \tag{1}$$

When $X_1^0 \leqq X_2^0 \ldots \leqq X_{2n}^0$ and $A = \{X_i\}$, let

$$\Psi_A = \varphi(X_1)\, e^{-X_1^0 H}\, \varphi(X_2)\, e^{-(X_2^0 - X_1^0)H} \ldots \varphi(X_{2n})\, \Omega .$$

To establish the absence of even bound states it suffices to show that for each n

$$0 \leqq \langle \Psi_A e^{-tH} \Psi_A \rangle - \langle \Psi_A \rangle^2 \leqq C_A e^{-2mt} . \tag{2}$$

Here Ω is the vacuum of H and φ denotes the time zero field. The above inequality implies that the bound state (of energy less than $2m$) is orthogonal to the even subspace.

We now turn to the proof of the estimate. Let

$$A = \{X_1 \ldots X_{2n}\} , \qquad B = \{X_1' + \tilde{t}, \ldots X_{2n} + \tilde{t}\}$$

where $\tilde{t} = (0, t)$.

Observe that since there is a gap in the product theory the Feynman-Kac formula shows that

$$\langle Q_A T_B \rangle - \langle Q_A \rangle \langle T_B \rangle$$

decays exponentially as $t \to \infty$. We determine the rate of decay by induction on n. For $n = 1$ (2) follows as in [1]. Now suppose (2) holds for $j \leqq n-1$. Let $R(t)$ be defined by

$$\begin{aligned} &\langle Q_A T_B \rangle - \langle Q_A \rangle \langle T_B \rangle \\ &\qquad = 2\{\langle \Psi_A e^{-tH} \Psi_A \rangle - \langle \Psi_A \rangle^2\} - R(t) . \end{aligned}$$

Note that $R(t)$ also vanishes exponentially. From (1) we have

$$0 \leqq 2\{\langle \Psi_A e^{-tH} \Psi_A \rangle - \langle \Psi_A \rangle^2\} \leqq R_t .$$

$R(t)$ is a sum of products of expectations obtained from expanding $\langle Q_A T_B \rangle - \langle Q_A \rangle \langle T_B \rangle$ into fields. The terms corresponding to $\langle Q_A \rangle \langle T_B \rangle$ are constant in time and do not require further analysis. The terms corresponding to $\langle Q_A T_B \rangle$ have the form

$$\langle \Phi_{A_1} \Phi_{B_1} \rangle \langle \Phi_{\sim A_1} \Phi_{\sim B_1} \rangle$$

where

$$A_1 \subset A , \qquad \sim A_1 = A \backslash A_1$$

$$B_1 \subset B , \qquad \sim B_1 = B \backslash B_1$$

and either A_1 or B_1 is a proper subset of A or B. Now if A_1 has an odd number of elements then so has B_1 because otherwise the term vanishes. Hence both factors contain fields localized in both A and $B = A + \tilde{t}$. It follows from the eveness of the theory and the definition of m that each factor decays like $\mathcal{O}(e^{-tm})$. If A_1 has an even number of elements then so has B_1 or again the expectation vanishes. By the induction hypothesis and the Schwartz inequality we can write

$$\langle \Phi_{A_1} \Phi_{B_1} \rangle = \langle \Phi_{A_1} \rangle \langle \Phi_{B_1} \rangle + \mathcal{O}(e^{-2mt}) .$$

Note that the products $\langle \Phi_{A_1} \rangle \langle \Phi_{B_1} \rangle$ are constant in t. Consequently we have

$$R(t) = \text{Const.} + \mathcal{O}(e^{-2mt}) .$$

Since $R(t)$ goes to zero for large t the constant vanishes and the estimate is complete.

References

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