# On Local Functions of Fields*

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**Abstract.** Properties of local functions of fields are discussed. A condition, called the Borchers condition, is introduced which is weaker than duality but allows the construction of a maximal local extension of a system of local algebras. This extension will satisfy duality. The local structure of the generalized free field is studied, and it is shown that duality does not hold for the local algebras associated with certain generalized free fields, whereas the Borchers condition is satisfied for all generalized free fields. The appendix contains an elementary proof of duality for the free field.

## 1. Introduction

In the construction of local dynamical theories one usually considers Hamiltonian densities which are local functions of the basic fields. Equations of motion involve polynomials or other local functions of the fields. It is therefore of interest to investigate general properties which all local functions of fields should possess. The fields will be assumed to satisfy the restriction that the vacuum vector is analytic for the field operators, so that according to Borchers and Zimmerman [1] a system of local algebras $B_\phi(R)$ can be associated with each field $\phi(x)$. This means that an algebra $B_\phi(R)$ of bounded operators is associated with each region $R$ and satisfies locality:

$$B_\phi(R_1) \subset B_\phi(R_2)' \quad \text{if} \quad R_2 \subset R_1{}^c. \tag{1}$$

Here $B_\phi(R)'$ denotes the commutant of $B_\phi(R)$ and $R^c$ denotes the causal complement of $R$: the interior of the set of points space-like to all points in $R$. The local algebra $B_\phi(R)$ is constructed as the von Neumann algebra generated by the spectral projections of fields averaged with test functions with support in the region $R$.

In addition to the locality condition (1) a variety of restrictions on the local algebraic structure may be imposed. One such restriction of this sort is that of duality:

*Definition 1.* Duality: The duality condition holds for the region $R$ if

$$B_\phi(R) = B_\phi(R^c)'$$

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Duality is known to be satisfied by the algebras associated with the free field [2] if the regions $R$ are restricted to be diamonds[1]. In what follows we shall consider only two sorts of regions: diamonds $R$ and their causal complements $R^c$. The algebra $B(R^c)$ is defined as the von Neumann algebra generated by diamonds whose closures are contained in $R^c$:

$$B(R^c) = \left\{ \bigcup_{\overline{R}_2 \subset R^c} B(R_2) \right\}''.$$

We shall show that duality does not hold for the algebras associated with certain generalized free fields, but a weaker condition, called the Borchers condition, and introduced in the next section, will hold for all generalized free fields. If we are given a system of local algebras which satisfy the Borchers condition but not necessarily duality, it will be shown that there exists a maximal local extension of the given algebras, and that this extension will satisfy duality.

Langerholc and Schroer [3] have given a definition for a field $\psi(x)$ to be a local function of the field $\phi(x)$: Any bounded operator $b$ which commutes with $\phi(x)$ for all $x$ in a region $R$ should also commute with $\psi(x)$ for $x$ in the same region $R$. To make this precise, let $\mathscr{D}$ be the dense set of vectors obtained by applying polynomials in the fields $\phi$ and $\psi$ to the vacuum vector.

*Definition 2.* Local functions of fields: The field $\psi(x)$ is said to be a local function of the field $\phi(x)$ if $(b^\dagger f, \phi(h) g) = (\phi(h) f, bg)$ for all $f, g \in \mathscr{D}$ and all real test functions $h$ with support in the region $R$ *implies* that $(b^\dagger f, \psi(h) g) = (\psi(h) f, bg)$ for $f, g \in \mathscr{D}$ and supp $h \in R$.

Since we have assumed that the vacuum is analytic for $\phi$ and $\psi$ it follows [1] that $\phi(h), \psi(h)$ are essentially self-adjoint on $\mathscr{D}$, and therefore this definition is *equivalent* to

$$B_\psi(R) \subset B_\phi(R). \tag{2}$$

The equivalence of Definition 2 with Eq. (2) is important since the verification of the algebraic property (2) may be reduced to the study of functions of the form

$$(b^\dagger f, \phi(x) g) - (\phi(x) f, bg).$$

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[1] The class of regions called diamonds is defined as follows: Let $x_1$ and $x_2$ be space-time points such that $x_2 - x_1$ is parallel to the positive time axis: $x_2 - x_1 = (a, 0)$. The intersection of the interior of the forward light cone from $x_1$ and the interior of the backward light cone from $x_2$ defines a diamond $D$ which is said to be generated by the line segment $x_2 - x_1$. The base of the diamond, denoted $\boldsymbol{D}$, is the sphere $\{x = (t, x): t = t_0, |\boldsymbol{x} - \boldsymbol{x}_0| < a/2\}$ where $(x_1 + x_2)/2 = (t_0, \boldsymbol{x}_0)$. Diamonds form a base for the usual topology of space-time and are equal to their double causal complement: $R = R^{cc}$. Diamonds also satisfy the independence property given by extended locality [4].

Let $\phi(x)$ be a local irreducible field and suppose $\psi(x)$ is relatively local to $\phi$. Then [1] the algebras associated with $\psi$ and $\phi$ are relatively local:

$$B_\psi(R) \subset B_\phi(R^c)' . \tag{3}$$

If furthermore $\phi$ satisfies duality then (3) becomes

$$B_\psi(R) \subset B_\phi(R) .$$

In other words, duality implies that the Borchers class of an irreducible local field $\phi$ consists of local functions of $\phi$, a fact which has been pointed out by Guenin and Misra [5]. On the other hand if there exists a field $\psi$ in the Borchers class of $\phi$ such that $\psi$ is *not* a local function of $\phi$, i.e. $B_\psi(R) \not\subset B_\phi(R)$, then duality cannot hold for the algebras $B_\phi(R)$. This situation will be seen to arise for certain generalized free fields.

## 2. The Borchers Condition and Duality

Borchers has shown the following important result, which is the basis for the definition of the Borchers class of relatively local fields.

**Theorem.** (Borchers) [6]. *Let $\phi(x)$ be an irreducible local field and let $\psi_1(x)$ and $\psi_2(x)$ be local relative to $\phi$: $[\psi_i(x), \phi(y)] = 0$ for $(x-y)^2 < 0$ $i = 1, 2$. Then it follows that $\psi_1$ and $\psi_2$ are relatively local: $[\psi_1(x), \psi_2(y)] = 0$ for $(x-y)^2 < 0$.* $\square$

An analogue of this result for a system of local algebras can be formulated as follows. Let $\{B(R)\}$ be an irreducible system of local algebras. Let $b_1$ be a bounded operator which commutes with the operators space-like to the diamond $R_1 : b_1 \in B(R_1{}^c)'$. Likewise let $b_2 \in B(R_2{}^c)'$. Then if $R_1$ and $R_2$ are space-like separated, the analogue of the above theorem would be that $b_1$ commutes with $b_2$. This statement may be reformulated if we define

*Definition 3.* If $R$ is a diamond, $\hat{B}(R) = B(R^c)'$.

If $R^c$ is the causal complement of a diamond, $\hat{B}(R^c) = \left\{ \bigcup_{R_2 \subset R^c} \hat{B}(R_2) \right\}''$.

We may then state the Borchers condition for local algebras as

*Definition 4.* The Borchers Condition: $\{\hat{B}(R)\}$ form a system of local algebras.

A system of local algebras for which the Borchers condition is valid satisfies additional restrictions on its local structure which are illustrated in the following propositions.

Suppose $\{B_2(R)\}$ is a local extension of $\{B(R)\}$; i.e. $\{B_2(R)\}$ is a system of local algebras such that $B_2(R) \supset B(R)$ for each diamond $R$. Then

$$B_2(R) \subset B(R^c)' = \hat{B}(R)$$

and we have shown the following proposition.

**Proposition 1.** *The Borchers condition implies that $\{\hat{B}(R)\}$ is the maximal local extension of $\{B(R)\}$.*      □

Note further that the Borchers condition implies that $\hat{B}(R)$ satisfies duality:

$$\hat{B}(R) \subset \hat{B}(R^c)' \subset B(R^c)' = \hat{B}(R).$$

Therefore

$$\hat{B}(R) = \hat{B}(R^c)'. \tag{4}$$

On the other hand it is clear that the duality condition for $\hat{B}(R)$ implies that the system $\{\hat{B}(R)\}$ is local. Thus,

**Proposition 2.** *The Borchers condition holds if and only if $\hat{B}(R)$ satisfies duality.*      □

We see therefore that although the original system need not satisfy duality, when the Borchers condition holds there exists a maximal local extension which does satisfy duality. The algebra $B(R)$ is maximal only if duality holds. However, $B(R^c)$ – the algebra associated with the region space-like to the diamond $R$ – is already maximal as the following proposition shows.

**Proposition 3.** *The Borchers condition implies that $B(R^c)$ is maximal.* □

*Proof.* Equation (4) implies that $\hat{B}(R^c) = \hat{B}(R)' = B(R^c)$. Since $\hat{B}$ is the maximal local extension it follows that all local extensions coincide on $R^c$.

This interesting result will be shown explicitly in the next section for the generalized free field.

## 3. The Generalized Free Field

The results of the preceding section can be illustrated with the local algebras associated with the generalized free field. These algebras are constructed as follows. Consider the Hilbert space

$$\mathscr{H} = L^2(P^3, d^3(p)) \otimes L^2(M, d\varrho(m))$$

where $P^3$ denotes the real three-dimensional "momentum" space with Lebesque measure $d^3(p)$, and $M$ denotes the real line with the measure $d\varrho(m)$ in the "mass variable" $m$. The measure $d\varrho$ is a regular Borel measure of slow increase [7] which implies that

$$\varrho[f] = \int d\varrho(m) f(m) = \int dm \varrho'(m) f(m)$$

defines a tempered distribution $\varrho'$. The support of $\varrho$ is contained in the positive real axis $\{m \geq 0\}$. The Fock space $\mathscr{F}$ over $\mathscr{H}$ is constructed from symmetrized tensor products of the "one particle space" $\mathscr{H}$. Creation and annihilation operators $a^\dagger(h)$, $a(h)$ are defined for all $h \in \mathscr{H}$

and satisfy the commutation relations

$$[a^\dagger(h_1), a^\dagger(h_2)] = 0 \qquad [a(h_1), a(h_2)] = 0$$

$$[a(h_1), a^\dagger(h_2)] = \int d^3(p) \, d\varrho(m) \, h_1(\boldsymbol{p}, m) \, h_2(\boldsymbol{p}, m) \,. \tag{5}$$

Equation (5) can also be written in the form

$$[a(\boldsymbol{p}, m), a^\dagger(\boldsymbol{p}', m')] = \varrho'(m) \, \delta(m - m') \, \delta^3(\boldsymbol{p} - \boldsymbol{p}') \,.$$

Define the field

$$\phi(x, m) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3(p)}{\sqrt{2w(\boldsymbol{p}, m)}}$$

$$\cdot \{a^\dagger(\boldsymbol{p}, m) \, e^{i[w(\boldsymbol{p}, m)t - \boldsymbol{p} \cdot \boldsymbol{x}]} + a(\boldsymbol{p}, m) \, e^{-i[w(\boldsymbol{p}, m)t - \boldsymbol{p} \cdot \boldsymbol{x}]}\}$$

$$w(\boldsymbol{p}, m) = \sqrt{\boldsymbol{p}^2 + m^2}$$

which is defined analogously to the free field of a single mass. The fields $\phi(\boldsymbol{x}, m) = \phi(t = 0, \boldsymbol{x}, m)$ and $\pi(\boldsymbol{x}, m) = \partial_t \phi(t = 0, \boldsymbol{x}, m)$ are tempered distributions satisfying the commutation relations

$$[\phi(\boldsymbol{x}, m), \phi(\boldsymbol{x}', m')] = 0 \qquad [\pi(\boldsymbol{x}, m), \pi(\boldsymbol{x}', m')] = 0$$

$$[\phi(\boldsymbol{x}, m), \pi(\boldsymbol{x}', m')] = i\delta^3(\boldsymbol{x} - \boldsymbol{x}') \varrho'(m) \, \delta(m - m') \,. \tag{6}$$

It is easily seen by their construction that the fields $\phi(\boldsymbol{x}, m)$, $\pi(\boldsymbol{x}, m)$ form an irreducible set of operators and the set $\mathscr{D}$ obtained by applying polynomials in the (averaged) fields $\phi(\boldsymbol{x}, m)$, $\pi(\boldsymbol{x}, m)$ to the vacuum is dense in the Fock space $\mathscr{F}$.

It is useful at this point to introduce an additional variable $s$ and to define

$$\phi(x, s) = \int dm \cos(ms) \, \phi(x, m) \,.$$

This is essentially the Fourier transform[2] of $\phi(x, m)$ with respect to the mass variable $m$. It is symmetrized in the variable $s$ so that

$$\partial_s \phi(x, s = 0) = 0 \,.$$

The field $\phi(x, s)$ satisfies the five-dimensional wave equation, with $s$ playing the role of an additional space-like variable:

$$(\partial_t^2 - \boldsymbol{V}_x^2 - \partial_s^2) \, \phi(x, s) = 0 \,. \tag{7}$$

The customary generalized free field is just the boundary value of $\phi(x, s)$ at the surface $s = 0$:

$$\phi(x) = \phi(x, s = 0) \tag{8}$$

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[2] We use the same symbol $\phi$ for the field as a function of $(x, m)$ and its fourier transform as a function of $(x, s)$. No confusion should arise since the variables will always be explicitly indicated.

and satisfies the commutation relations

$$[\phi(x), \phi(x')] = \int d\varrho(m)\, \Delta_0(x - x', m)$$

where

$$\Delta_0(x, m) = \frac{1}{(2\pi)^3} \int \frac{d^3(p)}{2w(p, m)} \{e^{-i[wt - p\cdot x]} - e^{i[wt - p\cdot x]}\}.$$

The above construction involving the five-dimensional wave equation is closely related to the Jost-Lehman-Dyson representation [8] as well as to the construction of Araki [9]. A similar technique was used by Haag and Schroer [10] in discussing the time-slice axiom for the generalized free field. The following uniqueness theorems for the wave equation can now be used to analyze the relation between $\phi(x)$, which is the boundary value of $\phi(x, s)$ on the surface $s = 0$, and $\phi(x, s)$, $\pi(x, s)$, which are the boundary values of $\phi(x, s)$ on the surface $t = 0$.

**Uniqueness Theorems** [8]. *Suppose the distribution $F(x, s)$ satisfies the wave Eq. (7) and let $R^5$ be the five-dimensional diamond centered about the origin $x = 0$, $s = 0$ with[3] base $R^5$. Then*

  i) *if $F(t = 0, x, s)$ vanishes together with its time derivative $\partial_t F(t = 0, x, s)$ in $R^5$, then it will vanish in the entire diamond $R^5$.*

  ii) *If $F(x, s = 0)$ vanishes together with its normal derivative $\partial_s F(x, s = 0)$ on the surface $s = 0$ in a neighborhood of the time-like line which generates the diamond $R^5$, then it will vanish in the entire diamond $R^5$.*    $\square$

The vacuum is analytic for all the fields which have been introduced in this section so that we may define several systems of local algebras.

  *Definition 5.* a) $B(R) = B_{\phi(x)}(R)$.

  b) $B_0(R) = B_{\phi(x,m)}(R \times M) = B_{\phi(x,s)}(R \times S)$.

  c) $B_0(R) = B_{\phi(x,m)}_{\pi(x,m)}(R \times M) = B_{\phi(x,s)}_{\pi(x,s)}(R \times S)$.

In the above definition

  a) $B(R)$ is the algebra we are primarily interested in, generated by the spectral projections of the generalized free field, Eq. (8), averaged with test functions in the variable $x$ with support in the four-dimensional diamond $R$.

  b) $B_0(R)$ is generated by the spectral projections of the fields $\phi(x, m)$ averaged with test functions whose support in the variable $x$ is contained in the diamond $R$ but whose support in the variable $m$ is arbitrary; i.e. with support in $R \times M$. In terms of the variables $x, s, B_0$ can clearly also be defined by $B_{\phi(x,s)}(R \times S)$ which is generated by the spectral projections of the fields $\phi(x, s)$ averaged with test functions with support in $R \times S$, where $S$ denotes the real line (variable $s$).

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  [3] See Footnote 1 for the definition of the base and generator of a diamond. (In the case of five-dimensional diamonds the base will naturally be four-dimensional.)

c) $B_0(\mathbf{R})$ is generated by the spectral projections of the fields $\phi(\mathbf{x}, m)$, $\pi(\mathbf{x}, m)$ averaged with test functions with support in $\mathbf{R} \times M$.

For any bounded operator $b$ we may define

$$F(x, s) = (b^\dagger f, \phi(x, s) g) - (\phi(x, s) f, bg) \tag{9}$$

where $f$ and $g$ are chosen from $\mathscr{D}$. As discussed in Section 1, the vanishing of $F(x, s)$ for $(x, s)$ in a region $R$ and all $f, g \in \mathscr{D}$ is equivalent to $b$ being in the commutant of the algebra generated by $\phi(x, s)$ for $(x, s)$ in $R$. This fact, combined with the uniqueness theorems for the wave equation, leads to relations among the algebras in Definition 5.

**Theorem 1.** *Let $R$ be a diamond. Then*

i) $B_0(\mathbf{R}) = B_0(R)$.

ii) *The algebras $\{B_0(R)\}$ form a local extension of the algebras $\{B(R)\}$.*

iii) *The algebras $\{B(R)\}$ associated with the generalized free field $\phi(x)$ form an irreducible set of operators.*

iv) $B(R^c) = B_0(R^c)$

v) $B(R) = B_{\substack{\phi(x,s)\\ \pi(x,s)}}(\mathbf{R}^5)$, *where $\mathbf{R}^5$ denotes the base of the five-dimensional diamond $R^5$ which has the same generator as $R$.* □

*Proof.* i) Let $b \in B_0(\mathbf{R})'$; then $F(x, s)$, Eq. (9), satisfies the wave Eq. (7) and vanishes together with its time derivative in the region $\mathbf{R} \times S$. It follows from uniqueness Theorem (i) that $F(x, s)$ vanishes in the larger region $R \times S$. Thus the operator $b$ is contained in $B_0(R)'$ and this shows that $B_0(R) \subset B_0(\mathbf{R})$. It is obvious that if $F(x, s) = 0$ in $R \times S$ then it is also zero, together with its time derivative, in $\mathbf{R} \times S$ which means $B_0(\mathbf{R}) \subset B_0(R)$ and thus i) is proved. Since $F(x, s)$ vanishes in $R \times S$ it vanishes in particular in the diamond $R$ on the surface $s = 0$, which shows that $B(R) \subset B_0(R)$ which proves ii).

iii) Suppose $b$ commutes with $\phi(x)$ for all $x$. Then $F(x, s)$ vanishes together with its normal derivative on the entire $s = 0$ surface. Uniqueness Theorem (ii) then implies that $F(x, s)$ is identically zero. Therefore $b$ commutes with the fields $\phi(\mathbf{x}, m)$, $\pi(\mathbf{x}, m)$ for all $\mathbf{x}$ and $m$. Since these fields are irreducible it follows that $b$ is a multiple of the identity. (This result can of course be obtained directly from the definition of $\phi(x)$ but it is useful to see how it arises from the point of view considered here.)

A sharpening of the preceding argument yields iv): Let $b$ commute with $\phi(x)$ for $x$ in the causal complement $R^c$ of a diamond $R$. Then $F(x, s)$, together with its normal derivative, is zero in the region $R^c$ on the surface $s = 0$. It is then easy to see, using uniqueness Theorem ii) that $F(t = 0, \mathbf{x}, s)$ and $\partial_t F(t = 0, \mathbf{x}, s)$ are zero for all $s$ when $\mathbf{x}$ is in the

interior of the complement of $R$. (See for example the discussion of Lemma 1 in Ref. [4].) This Proves iv), and by a similar argument, using both uniqueness Theorems i) and ii), the result v) also follows.

Consider the particular generalized free field given by $d\varrho(m) = \theta(m)dm$ where $dm$ is Lebesque measure and $\theta(m)$ is the characteristic function of the positive real axis $\{m \geq 0\}$. In this case the commutation relations (6) become, in terms of the variable $s$ rather than $m$,

$$[\phi(x, s), \pi(x', s')] = \frac{i\pi}{2}\, \delta^3(x - x')\, [\delta(s - s') + \delta(s + s')] \,. \tag{10}$$

Let $R$ be a diamond centered about the origin and let $R^5$ be the five-dimensional diamond with the same generator as $R$. Let $B$ be the algebra generated by the fields $\phi(x, s)$, $\pi(x, s)$ for $x$ in $R$ and $s$ sufficiently large so that $(x, s)$ is outside $R^5$. Then by Theorem 1 v) and Eq. (10) it follows that $B \in B(R)'$. On the other hand $B$ is noncommutative by (10) and is contained in $B_0(R) = B_0(R)$. Thus $B(R) \neq B_0(R)$. There is a wide class of generalized free fields for which $B(R) \neq B_0(R)$, but we are content with exhibiting this one example.

Nevertheless there certainly are many cases for which $B(R)$ does equal $B_0(R)$. If the measure $d\varrho(m)$ falls off exponentially in $m$, i.e. $\int d\varrho(m)\, e^{am} < \infty$ for some positive $a$, then $F(x, s)$, when averaged in $x$, is analytic in $s$ in a strip about the real axis, and so if $F$ vanishes in $R^5$ it will vanish in $R \times S$. This has the consequence that

$$B(R) = B_0(R) \quad [\varrho(m) \text{ exponentially decreasing}] \,. \tag{11}$$

In particular this equality holds for any generalized free field of compact support in the mass variable.

These methods are also useful in discussing causality for the generalized free field. (See Mollenhoff [11] and the discussion of Haag and Schroer [10].) Let $R$ be a diamond centered about the origin and define $R_\tau = R \cap \{|t| < \tau\}$ where $\tau$ may be taken arbitrarily small but nonzero. Assume the measure $d\varrho(m)$ is exponentially decreasing. If $b$ is a bounded operator which commutes with $B(R_\tau)$ then $F(x, s)$ will vanish in $R_\tau$ on the surface $s = 0$. From uniqueness Theorem ii) it is seen that for each $x$ in $R$ there is a neighborhood of $s = 0$ for which $F(t = 0, x, s)$ and $\partial_t F(t = 0, x, s)$ are equal to zero. Since these functions are analytic in $s$, it follows that they are equal to zero in $R \times S$. Thus the operator $b$ commutes with $B_0(R) = B(R)$ [Eq. (11)]. Therefore

$$B(R_\tau) = B(R) \quad [\varrho(m) \text{ exponentially decreasing}] \,.$$

Returning to the consideration of the Borchers condition and duality for the generalized free field, we note first that the algebras $B_0(R)$ satisfy duality:

$$B_0(R) = B_0(R^c)' . \tag{12}$$

This follows because the fields $\phi(x, m)$, $\pi(x, m)$ behave essentially like superpositions of free fields, and duality has been shown to hold for a free field by Araki [2]. A discussion of the duality relation (12) for $B_0$ is given in Section 4.

Equation (12) and Theorem 1 iv) imply

$$\hat{B}(R) = B(R^c)' = B_0(R^c)' = B_0(R) . \tag{13}$$

Since $\{B_0(R)\}$ is a system of local algebras we have

**Theorem 2.** *The Borchers condition is valid for the algebras $B(R)$.* $\quad\square$

We have previously seen that there exist generalized free fields for which $B(R) \neq B_0(R) = \hat{B}(R)$ [Eq. (13)] and therefore for these fields duality does not hold. Duality will however hold for all generalized free fields with exponentially decreasing measures $d\varrho(m)$ since Eqs. (11) and (13) imply

$$\hat{B}(R) = B(R) \qquad [\varrho(m) \text{ exponentially decreasing}]$$

which holds in particular for all generalized free fields with compact support in the mass variable.

The maximality of $B(R^c)$ follows from Theorem 1. iv) and Eq. (13):

$$\hat{B}(R^c) = B_0(R^c) = B(R^c) .$$

Summarizing we see that although duality does not hold for all generalized free fields, the Borchers condition is valid for all generalized free fields. Duality will hold for those generalized free fields with an exponentially decreasing measure in the mass variable $m$. From the general results of Section 2 we know that $\{\hat{B}(R)\}$ is the maximal local extension of $\{B(R)\}$. The fact that the algebra $B(R^c)$ associated with the region space-like to a diamond is maximal depends on Theorem 1 iv) which in turn results from uniqueness Theorem ii) for hyperbolic differential equations.

It is interesting to observe that for an arbitrary generalized free field $\phi$, there exists a local field $\phi_\alpha$ in the Borchers class of $\phi$ which is irreducible and satisfies duality: The Fourier transform is given by

$$\tilde{\phi}_\alpha(p) = e^{-\alpha p^2} \tilde{\phi}(p) .$$

## 4. Duality for $B_0(R)$

A basic result of Araki [12] (see also [13]) concerning duality for Fock-representations of creation-destruction operators can be used to prove duality for $B_0(R)$. Araki's basic duality result can be described as follows. Let $\mathscr{H}_R$ be a real Hilbert space and $\mathscr{H}$ the complexification of $\mathscr{H}_R$. Let $\mathscr{F}$ be the Fock space over $\mathscr{H}$. For each vector $h \in \mathscr{H}$, creation and destruction operators $a^\dagger(h)$, $a(h)$ are defined in the usual way, so that e.g. $a^\dagger(h)$ creates a one particle state with wave function $h$ from the no-particle state. *Auxiliary* fields are defined for $h \in \mathscr{H}_R$ by

$$\Phi(h) = \frac{1}{\sqrt{2}} \left[ a^\dagger(h) + a(h) \right]$$

$$\Pi(h) = \frac{i}{\sqrt{2}} \left[ a^\dagger(h) - a(h) \right].$$

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be subspaces of $\mathscr{H}_R$ and define $B(\mathscr{H}_1, \mathscr{H}_2)$ as the von Neumann algebra generated by the spectral projections of the fields $\Phi(h)$ with $h \in \mathscr{H}_1$ and $\Pi(h)$ with $h \in \mathscr{H}_2$. With $\mathscr{H}_i^\perp$ denoting the orthogonal complement in $\mathscr{H}_R$ of $\mathscr{H}_i$, Araki showed

**Theorem.** (Araki) [12]

$$B(\mathscr{H}_1, \mathscr{H}_2)' = B(\mathscr{H}_2^\perp, \mathscr{H}_1^\perp). \quad \square \tag{14}$$

This result may be applied to the study of the *free* field [2,14] by taking $\mathscr{H} = L^2(X^3, d^3(x))$ where $X^3$ denotes three dimensional coordinate space with Lebesque measure $d^3(x)$, and $\mathscr{H}_R$ is the set of real $L^2$-functions. The (time-zero) free fields $\phi(h)$, $\pi(h)$ of mass $m$ are defined for[4] $h \in \mathscr{S}$ by

$$\phi(h) = \Phi\left(\frac{1}{\sqrt{w_m}} h\right) \quad \pi(h) = \Pi(\sqrt{w_m} h) \tag{15}$$

where $w_m = \sqrt{-V^2 + m^2}$. Denoting by $B(R)$ the von Neumann algebra generated by the spectral projections of the fields $\phi(h)$, $\pi(h)$ with[4] $h \in \mathscr{D}(R)$, we have

$$B(R) = B\left(\frac{1}{\sqrt{w_m}} \mathscr{D}(R), \sqrt{w_m} \mathscr{D}(R)\right).$$

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[4] $\mathscr{S}$ denotes infinitely differentiable real functions of rapid decrease as $|x| \to \infty$. $\mathscr{D}(R)$ denotes the set of $h \in \mathscr{S}$ with compact support in the region $R$.

The duality relation $B(R^c)' = B(R)$ for the free field then follows from Eq. (14) and the result [14]

$$\left\{\frac{1}{\sqrt{w_m}}\mathscr{D}(\mathbf{R}')\right\}^{\perp} = \overline{\sqrt{w_m}\mathscr{D}(\mathbf{R})}, \qquad \{\overline{\sqrt{w_m}\mathscr{D}(\mathbf{R}')}\}^{\perp} = \overline{\frac{1}{\sqrt{w_m}}\mathscr{D}(\mathbf{R})} \quad (16)$$

where the bar —— denotes closure in $L^2$ and $\mathbf{R}'$ is the interior of the complement in $X^3$ of $\mathbf{R}$.

It is now a simple matter to prove the duality relation (12) for $B_0(R)$, which is the goal of this section. For the generalized free field we take[5] $\mathscr{H} = L^2(X^3, d^3(x)) \otimes L^2(M, d\varrho(m))$. The fields $\phi(x, m)$, $\pi(x, m)$ are given in terms of the auxiliary fields $\Phi, \Pi$ by Eq. (15) where $h \in \mathscr{S}$ is now a function of the variables $x$ and $m$. The duality relation for $B_0(R)$ then follows from the analogue of Eq. (16):

**Lemma.** $\quad \left\{\frac{1}{\sqrt{w}}\mathscr{D}(\mathbf{R}' \times M)\right\}^{\perp} = \overline{\sqrt{w}\mathscr{D}(\mathbf{R} \times M)},$

$\{\overline{\sqrt{w}\mathscr{D}(\mathbf{R}' \times M)}\}^{\perp} = \overline{\frac{1}{\sqrt{w}}\mathscr{D}(\mathbf{R} \times M)}.$ $\qquad \square$

Consider for example the first equation of the lemma. If $g$ is a vector in $\left\{\frac{1}{\sqrt{w}}\mathscr{D}(\mathbf{R}' \times M)\right\}^{\perp}$ then

$$\int d\varrho(m)\, F(m) \int d^3(x)\, g(x, m)\, \frac{1}{\sqrt{w_m}}\, f(x) = 0$$

for all $F \in \mathscr{D}(M)$ and $f \in \mathscr{D}(\mathbf{R}')$. It then follows that for almost every $m$,

$$\int d^3(x)\, g(x, m)\, \frac{1}{\sqrt{w_m}}\, f(x) = 0$$

which implies that $g(x, m) \in \overline{\sqrt{w_m}\mathscr{D}(\mathbf{R})}$ by Eq. (16). This then implies that

$$g \in \overline{\sqrt{w}\mathscr{D}(\mathbf{R} \times M)}.$$

The second equation of the lemma is proved similarly.

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[5] This is the same space as in Section 3, the relation between $X$ and $P$ being given by Fourier transformation.

## Appendix

Several proofs of duality for the *free* field have appeared in the literature [2, 14, 15]. We present an approach based on the expansion formula [16]:

$$Q = (\Omega, Q\Omega) + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^3(x_1)\ldots d^3(x_n) \, (\Omega, [\phi(x_1,0),\ldots[\phi(x_n,0),Q]\ldots]\Omega)$$
$$\overleftrightarrow{\partial}_{t_1}\ldots\overleftrightarrow{\partial}_{t_n} :\phi(x_1,0)\ldots\phi(x_n,0): \tag{A.1}$$

where $A\overleftrightarrow{\partial_t}B = A(\partial_t B) - (\partial_t A)\, B$, $\partial_t\phi(x,0) = \pi(x,0)$, and $\Omega$ is the vacuum state. The colons: : denote Wick ordering.

If $Q \in B(R^c)'$ then all the functions $(\Omega, [\phi(x_1,0),\ldots,Q]\ldots]\Omega)$ have have support in $R$ and thus expression (A.1) leads to the duality condition $Q \in B(R)$. Araki's original approach [2, 12] used an expansion of $Q$ in creation-destruction operators. The expansion (A.1) is based on a $\phi$, $\pi$ expansion as used by Osterwalder [15]. The advantage of the expression (A.1) is its manifest local structure. In fact we need only the weak convergence of (A.1) on the vacuum vector, as the following discussion shows.

Duality for the algebras $\{B(R)\}$ is clearly equivalent to the statement:

$$Q_1 \in B(R^c)', \qquad Q_2 \in B(R)' \Rightarrow [Q_1, Q_2] = 0.$$

Define
*Weak Duality: $Q_1 \in B(R^c)'$, $Q_2 \in B(R)' \Rightarrow (\Omega, [Q_1, Q_2]\Omega) = 0$.*

**Lemma A.1.** *Weak duality is equivalent to duality.* $\qquad\qquad\square$

*Proof.* Let $Q_1 \in B(R^c)'$, $Q_2 \in B(R)'$, $A, B \in B(R)$. Then $A^\dagger Q_1 B \in B(R^c)'$ and weak duality implies that

$$(A\Omega, [Q_1, Q_2]\, B\Omega) = (\Omega, [A^\dagger Q_1 B, Q_2]\,\Omega) = 0.$$

By the Reeh-Schlieder theorem $B(R)\,\Omega$ is dense in the Hilbert space and therefore $[Q_1, Q_2] = 0$.

**Lemma A.2.** *Let $Q \in B(R^c)'$ and*

$$Q_{n,\varepsilon} = \frac{(-i)^n}{n!} \int d^3(x_1)\ldots(\Omega, [\phi_\varepsilon(x_1,0),\ldots Q]\ldots]\Omega) \overleftrightarrow{\partial}_{t_1}\ldots :\phi_\varepsilon(x_1,0)\ldots:$$

*where $\phi_\varepsilon(x,0) = $ translation by $x$ of $\phi_\varepsilon$, $\phi_\varepsilon = \int \phi(x,0)\, h_\varepsilon(x)\, d^3(x)$, $h_\varepsilon(x) = \varepsilon^{-1} h(x/\varepsilon)$, and $h$ is a positive, infinitely differentiable function with support in $\{|x| < 1\}$, $\int h(x)\, d^3(x) = 1$.*

Then $Q_{n,\varepsilon}\Omega = H_\varepsilon E_n Q\Omega \xrightarrow[\varepsilon \to 0]{} E_n Q\Omega$ *where* $E_n$ *is the projection in Fock space onto n-particle vectors and* $H_\varepsilon$ *is multiplication by* $|\tilde{h}_\varepsilon(\boldsymbol{p})|^2$:

$$H_\varepsilon F(\boldsymbol{p}_1, \dots \boldsymbol{p}_n) = |\tilde{h}_\varepsilon(\boldsymbol{p}_1)|^2 \dots |h_\varepsilon(\boldsymbol{p}_n)|^2 \, F(\boldsymbol{p}_1, \dots \boldsymbol{p}_n).$$

*The operator norm* $\|H_\varepsilon\| \leqq 1$.    □

*Proof.* Writing $Q_{n,\varepsilon}$ in terms of creation-destruction operators gives

$$Q_{n,\varepsilon}\Omega = \frac{1}{n!} \int d^3(p_1) \dots d^3(p_n) \prod_i |\tilde{h}_\varepsilon(\boldsymbol{p}_i)|^2 \, (\Omega, a(\boldsymbol{p}_1) \dots a(\boldsymbol{p}_n) Q\Omega) \, a^\dagger(\boldsymbol{p}_1) \dots a^\dagger(\boldsymbol{p}_n) \, \Omega$$

$$= H_\varepsilon E_n Q\Omega.$$

Since $\tilde{h}_\varepsilon(\boldsymbol{p}) = \tilde{h}(\varepsilon \boldsymbol{p})$ we have $|\tilde{h}_\varepsilon(\boldsymbol{p})| \leqq 1 \Rightarrow \|H_\varepsilon\| \leqq 1$, and $\tilde{h}_\varepsilon(\boldsymbol{p}) \xrightarrow[\varepsilon \to 0]{} 1$ $\Rightarrow H_\varepsilon \xrightarrow[\varepsilon \to 0]{\text{strongly}} I$ and the lemma is proved.

Lemma A.2 leads to the conclusion that, with $Q^{(n)} \equiv (\Omega, Q\Omega) + \sum_{j=1}^{n} Q_{j,1/n}$,

$$Q^{(n)}\Omega \xrightarrow[n \to \infty]{\text{strongly}} Q\Omega.$$

Now suppose $Q \in B(R^c)'$ and $Q_2 \in B(R_\varepsilon)'$ where $R_\varepsilon = \{x: \text{dist}(x, R) < \varepsilon\}$. Then $[Q_2, Q^{(n)}] = 0$ for all $n$ sufficiently large which implies, according to Lemmas A.1 and A.2, that $[Q_2, Q_1] = 0$. We have thus shown $B(R^c)' \subset B(R_\varepsilon)$ for all $\varepsilon > 0$, and so

$$B(R^c)' \subset \bigcap_\varepsilon B(R_\varepsilon).$$

In addition, any operator $Q \in \bigcap_\varepsilon B(R_\varepsilon)$ will commute with $B(R_2)$ if the closure of $R_2$ is contained in $R^c$. Thus

$$B(R^c)' = \bigcap_\varepsilon B(R_\varepsilon). \tag{A.2}$$

Equation (A.2) is a statement of duality. We do not pursue questions concerning the boundary of $R : \bigcap_\varepsilon B(R_\varepsilon) = B(R)$ [2].

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