

New Types of Singularity in General Relativity: The General Cylindrically Symmetric Stationary Dust Solution

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Received December 3, 1973

Abstract. The general solution of Einstein's equations for a stationary cylindrically symmetric distribution of pressure-free matter is obtained. It contains a function which may be freely prescribed. Using this freedom examples are given of new types of singularity in General Relativity.

1. Introduction

Recent work [1, 2] on the problem of singularities in relativistic cosmological models has made it clear that a variety of types of singularity may occur, such as Ricci curvature ("big bang") singularities and intermediate singularities ("whimpers"). In [1] further types were discussed, including *oscillating Ricci singularities* and *Weyl (conformal) singularities*, but no exact examples could be given. We aim here to provide such examples, so we begin by making precise the notions of "oscillating Ricci" and "Weyl" singularities.

In an inextendible spacetime the existence of a singularity is deduced from that of an inextendible geodesic (or more generally an inextendible curve) having finite length as measured by a generalised affine parameter. Let the curve be $\chi(v)$, defined for $0 \leq v \leq v_+$, and suppose it cannot be extended beyond v_+ . Consider the components $R_{abcd}(v)$ of the Riemann tensor in orthonormal bases $\{e_a\}$ defined along $\chi(v)$. Both oscillating Ricci singularities and Weyl singularities are types of *curvature singularities*: these occur when there is *no* orthonormal basis defined along $\chi(v)$ such that all $R_{abcd}(v)$ tend to finite limits as $v \rightarrow v_+$. The division into Ricci and Weyl singularities comes from considering which part of the Riemann tensor is failing to tend to a limit. If the Ricci tensor components do not tend to finite limits in any orthonormal frame we have a *Ricci singularity*, and this is the familiar "big bang" singularity if some components actually diverge in every such frame. If a Ricci singularity is such that in some orthonormal frames some components $R_{ab}(v)$ oscillate finitely (but do not tend to a limit) and the rest do tend

to limits, while in all other orthonormal frames at least some $R_{ab}(v)$ diverge, we call it an *oscillating Ricci singularity*.

Similarly, if the components $C_{abcd}(v)$ of the Weyl tensor do not tend to finite limits in any orthonormal frame, but the Ricci tensor components do tend to finite limits in at least one such frame, we say we have a *Weyl singularity*.

(In fact the two examples we shall give will be cases where scalar polynomials of the relevant tensor (such as $R_{ab}R^{ab}$) do not tend to limits; this is clearly *sufficient* to show that some components of the tensor do not have limits in any frame. It may moreover be shown [1] to be also *necessary* in the case where the source of the gravitational field is a perfect fluid.)

Two further types of singularity which are not curvature singularities should be mentioned: first, if the $R_{abcd}(v)$ go to finite limits in *some* orthonormal frame along $\chi(v)$ but not in any such frame *parallelly propagated* along $\chi(v)$ we have an *intermediate singularity* (“whimper”). (See [1].) If all the $R_{abcd}(v)$ have finite limits in some parallelly propagated orthonormal frame as $v \rightarrow v_+$ this is a *locally extensible singularity*: by a theorem of Clarke [3] there exists an open neighbourhood of $\chi(v)$ which can be extended in such a way that $\chi(v)$ can be continued beyond $\chi(v_+)$ in this local extension. Cone singularities, covering-space singularities and the Taub-NUT singularity are examples of this type [4].

The question naturally arises as to whether there exist solutions of the Einstein field equations for a fluid-filled spacetime which exhibit such singularities. We shall prove this for *oscillating Ricci* and *Weyl* singularities by giving explicit examples.

We consider in fact stationary dust-filled spacetimes possessing cylindrical symmetry. Several members of this family are known exactly [5, 6] and our reason for considering it is that, as was first remarked by Maitra [6], the field equations reduce to an underdetermined system of ordinary differential equations. This can easily be understood using the result [7] that the vacuum field inside an infinitely long rotating cylinder of matter is simply flat space. This is the cylindrical analogue of Birkhoff’s theorem, and just as Birkhoff’s theorem shows that concentric spherical shells of matter do not interact, therefore allowing one to specify a “density law” arbitrarily, so here coaxial cylindrical shells do not interact, and we may specify arbitrarily either the density law, or a vorticity or shear law. The fluid lines (integral curves of the fluid 4-velocity u^a) are inextensible and incomplete if the vorticity $\omega(r)$ and shear $\sigma(r)$ do not go to finite limits for some finite value r_+ of the proper radial coordinate r , so this will give us a singularity. This will not be a Ricci singularity if the density $\rho(r)$ has a finite limit at r_+ . Using the underdetermined nature of the field equations we are able to arrange

that we get Weyl singularities in some cases. Similarly we can arrange that $\varrho(r)$ oscillates but everything else goes to finite limits, implying an oscillating Ricci singularity.

2. The Field Equations

Following Ellis (e.g. [8]) we choose an orthonormal basis $\{e_a\}$ of vectors with $e_0 = u$, normalised so that $e_a \cdot e_b = g_{ab} = \text{diag}(-1, +1, +1, +1)$ and define the γ_{bc}^a by the commutation relations $[e_a, e_b] = \gamma_{ab}^c e_c$. A stationary, cylindrically symmetric spacetime admits a three-parameter Abelian group of motions acting on timelike hypersurfaces of transitivity. The vector u is tangent to these surfaces (since otherwise we would have a “tilted” Bianchi I dust universe, which is forbidden [9]) so without loss of generality we may take the normal n to the surfaces to be e_3 . As shown in [10] the vectors e_A (capital letters run from 0 to 2) generate the group reciprocal to the group of motions, and since the latter is Abelian so is its reciprocal group, giving

$$\gamma_{AB}^a = 0. \tag{2.1}$$

Since the surfaces orthogonal to $n = e_3$ are homogeneous n is geodesic. Using the relations $n_{a;b} = \partial_b n_a - n_f \Gamma_{ba}^f$, $\Gamma_{abc} = \frac{1}{2}(\gamma_{abc} + \gamma_{cab} - \gamma_{bca})$ (where $\partial_a(f) = e_a^i f_{;i}$ etc.) which follow [8] easily from the definitions, we find that

$$\gamma_{A3}^3 = -n_{A;b} n^b = 0. \tag{2.2}$$

The only non-zero γ_{bc}^a are now those of the form $\gamma_{B3}^A (= -\gamma_{3B}^A)$. Since the pressure in the fluid is zero, the Bianchi identities imply $\dot{u}_a \equiv u_{a;b} u^b = 0$, and from a similar computation to that giving (2.2) we find

$$\gamma_{03}^0 \equiv \dot{u}_3 = 0, \quad (\dot{u}_1 \equiv \gamma_{01}^0 = \dot{u}_2 \equiv \gamma_{02}^0 = 0). \tag{2.3}$$

We find easily that the vorticity vector $\omega^a = \frac{1}{2} \eta^{abcd} u_b u_{c;d}$ has components

$$\omega^1 = -\frac{1}{2} \gamma_{23}^0, \quad \omega^2 = \frac{1}{2} \gamma_{13}^0, \quad \omega^3 = \omega^0 = 0.$$

We have still available the freedom to rotate e_1, e_2 in their 2-space¹, and we choose to align e_1 along ω at all points. Thus

$$\omega^2 = \gamma_{13}^0 = 0. \tag{2.4}$$

¹ Which is actually a 2-surface, since they commute.

Defining the shear tensor $\sigma_{ab} = u_{(a;b)} - \frac{1}{3}u^c{}_{;c}(g_{ab} + u_a u_b)$ and collecting our results we find the only non-zero $\gamma^a{}_{bc}$ and σ_{ab} are

$$\left. \begin{aligned} \gamma_{23}^0 &= -2\omega & (\omega \equiv \omega^1) \\ \gamma_{03}^1 &= -2\sigma_{13}, & \gamma_{03}^2 &= -2\sigma_{23}. \\ \gamma_{13}^1, & \gamma_{23}^2, & \gamma_{23}^1, & \gamma_{13}^2 \end{aligned} \right\} \quad (2.5)$$

(the fluid expansion $u^a{}_{;a}$ vanishes identically).

Our specialisations of the tetrad now make it possible to use the forms of the field equations and Jacobi identities given in the Appendix of [8]. The homogeneity gives $\partial_0 \equiv \partial_1 \equiv \partial_2 = 0$, and we can choose coordinates (x^A, r) such that $e_3 = n = \partial/\partial r$, the surfaces of transitivity being $\{r = \text{constant}\}$. All variables are then functions of r only. [r is the proper distance along the spacelike integral curves of n , and is clearly not the intrinsically defined radial coordinate R , which in general is some function $R(r)$.] The Jacobi identities are all identically satisfied and the (00), (01) field equations are

$$\left. \begin{aligned} \frac{1}{2}\varrho &= 2\omega^2 - 2\sigma_{13}^2 - 2\sigma_{23}^2 \\ 0 &= \frac{d\sigma_{13}}{dr} + \omega\gamma_{23}^1 + \sigma_{23}\gamma_{13}^2 + \sigma_{13}(\gamma_{23}^2 + 2\gamma_{13}^1) \end{aligned} \right\} \quad (2.6)$$

(we take the cosmological constant $\Lambda = 0$).

To characterise our solutions as representing infinite rotating cylinders we must require also that the reflections

$$\mathcal{R}_1: (e_0, e_1, e_2, e_3) \rightarrow (e_0, -e_1, e_2, e_3)$$

$$\mathcal{S}_1: (e_0, e_1, e_2, e_3) \rightarrow (-e_0, e_1, -e_2, e_3)$$

should be isometries. (These correspond to the reflections $z \rightarrow -z$, and $(\varphi, t) \rightarrow (-\varphi, -t)$, of the usual cylindrical polar coordinates (r, φ, z, t) . $\mathcal{R}_1, \mathcal{S}_1$ will be isometries if [11]

(a) They are automorphisms of the Lie algebra of the e_A .

(b) The second fundamental form $\tilde{\theta}_{ab} = \tilde{h}_a{}^c \tilde{h}_b{}^d n_{c;d}$ of the surfaces $\{r = \text{constant}\}$ spanned by the e_A is invariant under them ($\tilde{h}_{ab} = g_{ab} + n_a n_b$). Condition (a) is automatically satisfied, and applying (b) in the case of \mathcal{R}_1 implies $\tilde{\theta}_{01} = \tilde{\theta}_{12} = 0$. We get no new conditions from \mathcal{S}_1 .

Now

$$2\tilde{\theta}_{01} = -\gamma_{31}^0 - \gamma_{30}^1 = -2\sigma_{13}$$

and

$$2\tilde{\theta}_{12} = \gamma_{23}^1 + \gamma_{13}^2.$$

Thus $\sigma_{13} = \gamma_{23}^1 + \gamma_{13}^2 = 0$. From the second of (2.6),

$$0 = (\omega - \sigma_{23})\gamma_{23}^1.$$

Hence either $\omega = \sigma_{23}, \sigma_{13} = 0$ or $\gamma_{23}^1 = \gamma_{13}^2 = \sigma_{13} = 0$.

Now from the first of (2.6) $\omega = \sigma_{23}$, $\sigma_{13} = 0$ implies $\varrho = 0$ so for non-vacuum solutions we take

$$\gamma_{23}^1 = \gamma_{13}^2 = \sigma_{13} = 0. \tag{2.7}$$

Writing now $\sigma_{23} = \sigma$, $\gamma_{13}^1 = \gamma_1$, $\gamma_{23}^2 = \gamma_2$ the field equations reduce to

$$(00) \quad \varrho = 4(\omega^2 - \sigma^2)$$

$$(02) \quad 0 = \frac{d}{dr}(\sigma - \omega) + (\sigma - \omega)\gamma_1 + 2\gamma_2\sigma$$

$$(11) \quad -\frac{1}{2}\varrho = \frac{d\gamma_1}{dr} + \gamma_1(\gamma_1 + \gamma_2)$$

$$(22) \quad 0 = \frac{d\gamma_2}{dr} + \gamma_2(\gamma_1 + \gamma_2)$$

and

$$0 = \gamma_1\gamma_2 + (\omega - \sigma)^2,$$

where the last equation is a linear combination of (00), (11), (22), (33). It is easy to show that it is a first integral of the other equations, so we drop now (02) from our system. We write

$$\omega = A \cosh \lambda, \quad \sigma = A \sinh \lambda. \tag{2.8}$$

where A, λ are (possibly complex) functions of r , and write also

$$\gamma_1 + \gamma_2 = \Gamma. \tag{2.9}$$

Then the equations become simply

$$\left. \begin{aligned} \varrho &= 4A^2 \\ \frac{d\Gamma}{dr} + \Gamma^2 &= -2A^2 \\ \frac{d\gamma_2}{dr} + \Gamma\gamma_2 &= 0 \\ \gamma_1\gamma_2 &= -A^2 e^{-2\lambda} \end{aligned} \right\} \tag{2.10}$$

where the second equation is (11)+(22). Writing $\alpha = (\gamma_2)^{-1}$ we can express all quantities in terms of α and its derivatives:

$$\begin{aligned} \gamma_2 &= \alpha^{-1}, \quad \Gamma = \alpha^{-1}\alpha', \quad \gamma_1 = \alpha^{-1}(\alpha' - 1) \\ \varrho &= 4A^2 = -\frac{2\alpha''}{\alpha}, \quad e^{2\lambda} = \frac{\alpha\alpha''}{2(\alpha' - 1)} = \frac{A^2\alpha^2}{1 - \alpha'} \end{aligned} \tag{2.11}$$

where

$$\alpha' = d\alpha/dr, \quad \alpha'' = d^2\alpha/dr^2.$$

This makes explicit the underdetermined nature of the solutions: we may choose α arbitrarily, subject only to the constraints that A^2 and $e^{2\lambda}$ are positive for some range of r . The resulting solutions are invariantly distinguished from one another by the differing values of $\varrho = 4A^2$.

3. Global Properties

We are chiefly interested in the singularities of solutions given by various choices of α in (2.11). In particular we shall require the components of the Weyl tensor C_{abcd} , so we compute its “electric” and “magnetic” parts, $E_{ac} = C_{abcd}u^b u^c$, $H_{ac} = \frac{1}{2}\eta_{ab}{}^{gh}C_{ghcd}u^c u^d$. Their non-vanishing components are

$$\left. \begin{aligned} E_{11} &= \frac{2}{3}(\sigma^2 - \omega^2), & E_{22} &= \frac{5}{3}\sigma^2 + \frac{\omega^2}{3} - 2\omega\sigma \\ E_{33} &= -\frac{7}{3}\sigma^2 + \frac{\omega^2}{3} + 2\omega\sigma, & H_{13} &= \gamma_1(\sigma - \omega) \end{aligned} \right\} \quad (3.1)$$

(Using (02) above).

These determine the Weyl tensor completely, since

$$C_{abcd} = (\eta_{abpq}\eta_{cdrs} + g_{abpi}g_{cdrs})u^p u^r E^{qs} - (\eta_{abpq}g_{cdrs} + g_{abpq}\eta_{cdrs})u^p u^r H^{qs}$$

where

$$g_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}.$$

It follows from (2.11), (3.1) that if $\alpha \rightarrow 0$ and σ, ω are well behaved, $H_{13} \rightarrow \infty$ in general. However, from the fact that

$$\tilde{\theta}_{11} = \gamma_1 = \alpha^{-1}(\alpha' - 1), \quad \tilde{\theta}_{22} = \gamma_2 = \alpha^{-1}, \quad \tilde{\theta}_{33} = 0$$

it is clear that what is causing H_{13} to be badly behaved is the focussing of the radial geodesics tangent to \mathbf{n} for some value of the proper distance r along them, which will occur at any axes of symmetry of our solutions. In general any such “axis” will be singular, for in view of the fact that \mathbf{u} is the *unique* (since $\varrho > 0$) timelike eigenvector of the Ricci tensor the divergence of H_{13} and hence $H_{ab}H^{ab}$ implies the divergence of a Riemann tensor scalar polynomial. Our tetrad basis is always singular where $\alpha = 0$, but in some cases we may interpret this as the usual removable singularity $r = 0$ of cylindrical polar coordinates. From (2.11), (3.1) we have

$$H_{13} = \alpha^{-2}(1 - \alpha')^{3/2}$$

and it follows easily that for H_{13} to be finite at $r = 0$ we must have $\alpha = r + (O)(r^3)$ near $r \dots 0$. This will fit in with the condition that γ_2 must be an odd function of r derived below.)

To be able to interpret the singularity $\alpha = 0$ as an axis we require certain extra conditions: first, we want to be able to identify all points $(t, -r, \varphi, z)$ with $(t, r, \varphi + \pi, z)$, so ϱ, ω, σ must be even functions of r and γ_1, γ_2 must be odd functions of r . (The radial geodesics converge ($\hat{\theta}_a^a \rightarrow -\infty$) as $r \rightarrow 0$ from one side and expand out symmetrically ($\hat{\theta}_a^a \rightarrow +\infty$) as r passes through zero to the other side.) Secondly, to be able to identify points (t, r, φ, z) with $(t, r, \varphi + 2n\pi, z)$, $n = \text{integer}$, we must have that the circumference of a small circle about the axis is equal to $2\pi \times$ the radius (elementary flatness). Finally we would like to have no more than two such “axes”, for this is the largest number of axes a spacetime can have if we demand that moving through an angle of 2π about one axis gives a rotation of 2π about the other axes (Hawking and Ellis, private communication). (Of course we can have any number of “local” axes, which do not obey this condition, as for example in LRS spaces [8].) To see if any intermediate singularities are occurring, we need to consider parallel propagated tetrads along various curves in the spacetime. (Although all Riemann tensor components may be finite in our frame, this might be infinitely Lorentz-rotated relative to a parallelly propagated frame; Riemann tensor components in this frame would diverge, giving an intermediate singularity – see [1].) An obvious curve to consider is an integral curve of \mathbf{n} ; v^b is parallel propagated along n^a if $v^b{}_{;a}n^a = 0$ i.e.

$$\frac{dv^b}{dr} = -\Gamma^b{}_{3f}v^f.$$

Using (2.5), (2.7) this gives $v^1 = \text{constant}$, $v^3 = \text{constant}$,

$$\frac{dv^0}{dr} = -(\sigma + \omega)v^2, \quad \frac{dv^2}{dr} = -(\sigma + \omega)v^0.$$

Thus if e.g. $v^a = v\delta_2^a$ at $r = r_0$ we find

$$\left. \begin{aligned} v^a(r) &= (v \sinh \beta, 0, 0, v \cosh \beta(r)) \\ \text{where} \quad \beta &= -\int_{r_0}^r (\sigma + \omega) dr = -\int_{r_0}^r A e^\lambda dr. \end{aligned} \right\} \quad (3.2)$$

Thus a parallel propagated frame along an integral curve of \mathbf{n} suffers a finite Lorentz rotation (boost) if the integral in (3.2) converges. This will certainly be so (for finite r, r_0) if σ, ω are bounded, and may even be so if they are not. Parallel propagation along the integral curves of the various e_A produces rotations at a constant rate of the parallel propagated basis relative to our basis. These therefore will always give finite Lorentz transformations for propagation through finite affine distances. Thus for the purpose of discovering whether the parallel propagated basis

along a given curve suffers a finite or infinite Lorentz rotation we need only consider the component of the curve parallel to e_3 , and the corresponding $\beta(r)$ in (3.2).

Given a solution (2.11) we may find coordinates x^i (although we shall not actually do this for the solutions considered here) by letting $e_a = e_a^i \partial/\partial x^i$ and choosing the e_a^i so that the commutation relations

$$[e_A, e_B] = 0, [e_0, e_3] = -2\sigma e_2, [e_1, e_3] = \gamma_1 e_1, [e_2, e_3] = -2\omega e_0 + \gamma_2 e_2$$

hold identically. We have already set $e_3 = \partial/\partial r$, and letting $e_1 = a(r) \partial/\partial z$ we find $a = \exp\left(-\int^r \gamma_1 dr\right)$ for suitable choice of z .

We set

$$\begin{aligned} u = e_0 &= b \partial/\partial t + c \partial/\partial \varphi \\ e_2 &= e \partial/\partial t + f \partial/\partial \varphi, \end{aligned}$$

where b, c, e, f are functions of r only. With this choice of coordinates the vectors $\partial/\partial t, \partial/\partial \varphi, \partial/\partial z$ are Killing vectors since they commute with all the e_a [10]. The commutation relations are satisfied if

$$\left. \begin{aligned} db/dr &= 2\sigma e, & dc/dr &= 2\sigma f \\ de/dr &= 2\omega b - \gamma_2 e, & df/dr &= 2\omega c - \gamma_2 f \end{aligned} \right\} \quad (3.3)$$

which imply

$$bf - ce = \exp\left(-\int^r \gamma_2 dr\right)$$

for suitable t, φ . The matrix e_a^i has determinant

$$(bf - ce) \exp\left(-\int^r \gamma_1 dr\right) = \exp\left(-\int^r (\gamma_1 + \gamma_2) dr\right) = \alpha^{-1}$$

so the transformation between the e_a and the x^i is non-singular for $\alpha \neq 0$ and singular when $\alpha = 0$. Thus it is possible for (t, r, φ, z) to be good coordinates even where the tetrad e_a is singular, i.e. on the "axis" $\alpha = 0$. The coordinate metric g_{ij} is given by

$$g_{ij} = e_i^a e_j^b g_{ab} = -e_i^0 e_j^0 + e_i^1 e_j^1 + e_i^2 e_j^2 + e_i^3 e_j^3.$$

where e_i^a is the inverse matrix to e_a^i . We find

$$\begin{aligned} ds^2 &= \exp\left(2 \int^r \gamma_2 dr\right) \\ &\cdot [-(f^2 - c^2) dt^2 + (b^2 - e^2) d\varphi^2 + 2(fe - bc) d\varphi dt] \quad (3.4) \\ &+ dr^2 + \exp\left(2 \int^r \gamma_1 dr\right) dz^2. \end{aligned}$$

Coordinates are often used [6] in which the invariantly defined (r, z) plane is given in conformally flat coordinates; to achieve this, let

$$\left. \begin{aligned} \bar{r} &= \int^r e^{-\Psi} dr \\ \Psi &= \int^r \gamma_1 dr. \end{aligned} \right\} \quad (3.5)$$

where

Then the last two terms in (3.4) become $e^{2\Psi}(d\bar{r}^2 + dz^2)$ as required. We may re-express the transformation (3.5) as

$$\gamma_1 = d\Psi/dr = e^{-\Psi} d\Psi/d\bar{r}.$$

Using (2.11) we find

$$e^{-\Psi} d\alpha/d\bar{r} - \alpha e^{-\Psi} d\Psi/d\bar{r} = 1$$

so

$$\alpha = \bar{r} e^{\Psi} \quad (3.6)$$

with suitable choice of \bar{r} . (3.6) and the relation $\gamma_2 = \alpha^{-1}$ imply now

$$\exp\left(\int^r \gamma_2 dr\right) = \exp\left(\int^r \frac{e^{-\Psi}}{\bar{r}} e^{\Psi} d\bar{r}\right) = \bar{r}. \quad (3.7)$$

From (3.4) we may also restate the “elementary flatness” conditions for $\alpha = 0$ to be an axis; if φ is our angular coordinate at $\alpha = r = 0$ the circumference of a small circle is

$$2\pi \exp\left(\int^r \gamma_2 dr\right) (b^2 - e^2)^{\frac{1}{2}}.$$

This must equal $2\pi \times$ the radius, which is $dr \sim r$, so the condition is

$$\lim_{r \rightarrow 0} (b^2 - e^2)^{\frac{1}{2}} = \lim_{r \rightarrow 0} r \exp\left(-\int^r \gamma_2 dr\right). \quad (3.8)$$

4. Solutions

We turn now to the discussion of exact solutions (2.11), discussing first the known solutions. The Ehlers-van Stockum solution [5] (see also [8]) are shear-free, so $\lambda = 0$. From (2.11) this gives

$$\alpha d^2\alpha/dr^2 - 2d\alpha/dr + 2 = 0,$$

which can easily be solved using the coordinates (3.5), (3.6), leading to

$$\Psi = k\bar{r}^2 + l$$

where k, l are constants. Letting $k = -m^2/8$ we quickly find $\varrho = 4\omega^2 = m^2 \exp\left[\frac{m^2\bar{r}^2}{4} - 2l\right]$. Thus ϱ, ω become infinite as $\bar{r} \rightarrow \infty$. Since

$r = \int e^{\Psi} d\bar{r} = \int \exp\left(\frac{-m^2}{8}\bar{r}^2 + l\right) d\bar{r}$ we see that $r \rightarrow$ finite value as $\bar{r} \rightarrow \infty$. Thus, as remarked by Shepley [12] in connection with van Stockum's solution there is a Ricci curvature singularity at a finite proper distance from the axis of symmetry. The axis is certainly non-singular since Ψ is even in \bar{r} (and hence r), and we find $b = \text{constant}$, $e = bm\bar{r}/2$, so the elementary flatness condition (3.8) is satisfied. ($\bar{r} \sim r$ as $r \rightarrow 0$ since $\Psi \rightarrow \text{constant}$.)

We turn next to the geodesically complete solution discovered by Maitra [6]. This solution is everywhere analytic and contains no closed timelike lines. It is characterised by the fact that there is a timelike Killing vector of unit length. In Maitra's coordinates (3.5), (3.6) this solution is easily obtained by writing

$$d\Psi/d\bar{r} = -y^2/4\bar{r}.$$

Then using (2.11), (3.5), (3.6) it follows that

$$\omega = \frac{e^{-\Psi}}{4} \left(\frac{dy}{d\bar{r}} + \frac{y}{\bar{r}} \right), \quad \sigma = \frac{e^{-\Psi}}{4} \left(\frac{dy}{d\bar{r}} - \frac{y}{\bar{r}} \right).$$

From (3.4) the timelike Killing vector $\partial/\partial t$ has unit length if

$$(f^2 - c^2) \exp\left(2 \int \gamma_2 dr\right) = 1.$$

Using (3.7) we therefore set

$$f = \bar{r}^{-1} \cosh \delta(\bar{r}), \quad c = \bar{r}^{-1} \sinh \delta(\bar{r}).$$

Equations (3.3) [using (3.5) and the above expressions for ω and σ] now imply on eliminating $\delta(\bar{r})$

$$y = \frac{1}{2} \left[\left(\frac{a^2}{\bar{r}^2} + 4 \right) - \frac{a}{\bar{r}} \right] \quad a = \text{constant}$$

leading to

$$\Psi = -\frac{1}{4x^2} [(1+x^2)^{\frac{1}{2}} - 1] + \frac{1}{8} - \frac{1}{4} \log \frac{1}{2} [(1+x^2)^{\frac{1}{2}} + 1]$$

where $x = 2\bar{r}/a$, the value given by Maitra. The expressions for other quantities ρ , b , e etc. now follow easily and the conditions for a good axis are all satisfied. We give now some solutions which as far as the author is aware, are new.

We consider first solutions with

$$\rho = 4A^2 = 2B^2 = \text{constant}.$$

From (2.11) we find that a solution is

$$\alpha = B^{-1} \sin Br, \quad e^{2\lambda} = \frac{2 \sin^2 Br}{1 - \cos Br}.$$

These solutions are clearly highly pathological, as we have an infinite number of candidates for “axes” $\alpha = 0$. In fact this choice of α satisfies the regularity conditions for an axis at $r = 0$, but the fact that $e^{2\lambda} \rightarrow 0$ means that there is a Weyl singularity there. We see from (2.11) that $\omega \rightarrow \infty$, $\sigma \rightarrow -\infty$ with $\sigma \sim -\omega$. (3.1) shows that $E_{11} \rightarrow 0$, $E_{22} \sim 4\omega^2$, $E_{33} \sim -4\omega^2$ so

$$E_{ab}E^{ab} \sim 32\omega^2 \rightarrow \infty.$$

Because all Ricci tensor components are either zero or proportional to ϱ , which is finite, this must be a Weyl singularity.

To give an example of an “oscillating Ricci” singularity, we consider solutions (2.11) with α of the form

$$\alpha = e^v$$

then

$$\left. \begin{aligned} A^2 &= -\frac{1}{2} \left[\frac{d^2 v}{dr^2} + \left(\frac{dv}{dr} \right)^2 \right] \\ \text{and} \\ e^{2\lambda} &= A^2 e^{2\lambda} \left[1 - e^v \frac{dv}{dr} \right]^{-1}. \end{aligned} \right\} \quad (4.1)$$

We try to find a suitable v with

$$\frac{d^2 v}{dr^2} = -1 - \sin^2 \frac{1}{r}.$$

We can choose our integration constant so that

$$\frac{dv}{dr} = -r - \int_0^r \sin^2 \frac{1}{s} ds.$$

For $r > 0$, dv/dr is monotonically decreasing in r , and so A^2 (and hence ϱ) has the oscillatory character of the function $\sin^2 \frac{1}{r}$. We have also to check that $A^2, e^{2\lambda} > 0$ for $0 \leq r \leq$ some $r_0 > 0$. The above expression implies

$$-r \geq \frac{dv}{dr} \geq -2r.$$

Thus we can choose a v such that

$$\frac{-r^2}{2} \geq v \geq -r^2$$

and

$$\left(\frac{dv}{dr} \right)^2 \leq 4r^2.$$

From (4.1) this implies

$$A^2 \geq \frac{1}{2} \left(1 + \sin^2 \frac{1}{r} \right) - 2r^2$$

$$e^{2\lambda} = A^2 e^{2\nu} (1+k)^{-1}$$

where

$$k \geq r e^\nu \geq 0.$$

Thus for $0 \leq r \leq \frac{1}{2}$ both A^2 and $e^{2\lambda}$ are strictly positive and finite, ϱ, ω, σ (and γ_1, γ_2) are finite and defined. However as $r \rightarrow 0, \varrho = 4A^2$ (which is a scalar polynomial: $\varrho^2 = R_{ab}R^{ab}$) oscillates finitely and does not tend to a limit (it has the form $2 \sin^2 \frac{1}{r} + f(r)$ where f is monotonic), so this is an ‘‘oscillating Ricci’’ singularity. This solution is again highly pathological, the density becoming negative for some r between $\frac{1}{2}$ and 2, and there being two ‘‘axes’’ $\alpha = 0$ at $r = \pm \infty$.

We give finally a family of solutions which are non-pathological in the sense that they have a proper axis and their Riemann tensor components are always finite in a paralley-propagated frame.

We let

$$\alpha = \int_0^r \frac{ds}{1+s^{2n}} \quad n = \text{integer} \geq 1$$

then

$$\left. \begin{aligned} A^2 &= nr^{2n-1} (1+r^{2n})^{-2} \left[\int_0^r \frac{ds}{1+s^{2n}} \right]^{-1} \\ e^{2\lambda} &= \frac{n}{r(1+r^{2n})} \int_0^s \frac{ds}{1+s^{2n}}. \end{aligned} \right\} \quad (4.2)$$

Thus $e^{2\lambda}, A^2 > 0$ for $r > 0$ or $r < 0$. At $r = 0$ we have to investigate $\lim_{r \rightarrow 0} r^{-1} \int_0^s \frac{ds}{1+s^{2n}}$. By l'Hôpital's Rule this is equal to $\lim_{r \rightarrow 0} \frac{1}{1+r^{2n}} = 1$. Hence $e^{2\lambda} \rightarrow n$ and $A^2 \rightarrow 1$ (if $n = 1$) or 0 (if $n > 1$) as $r \rightarrow 0$, and $A^2, e^{2\lambda}$ are positive and bounded for all r , since

$$\int_0^r \frac{ds}{1+s^{2n}} < 1 + \int_1^r \frac{ds}{1+s^2} < 1 + \frac{\pi}{4}.$$

As $r \rightarrow \infty, A^2 \rightarrow 0$. We find

$$\omega \sigma = \frac{1}{2} \left[\frac{nr^{n-1}}{(1+r^{2n})^{\frac{1}{2}}} \pm \frac{r^n}{(1+r^{2n})^{\frac{1}{2}} \int_0^r ds(1+s^{2n})^{-1}} \right] \quad (4.3)$$

so as $r \rightarrow \infty, \omega$ and σ tend to the same finite limit. The solutions are invariantly distinguished from Maitra's by the fact that $\varrho = O(r^{-2n-1})$

as $r \rightarrow \infty$ whereas Maitra's $g = O(r^{-2})$, and from the Ehlers-van Stockum solutions by the presence of shear. We must check that the "axis" $r = 0$ is a good axis. First $\alpha = 0$ only for $r = 0$. Secondly, since α is an odd function, g, σ, ω are even and γ_1, γ_2 are odd as required. Finally, to check the "elementary flatness" condition (3.8) we need the asymptotic form of b, e and γ_2 as $r \rightarrow 0$.

Now

$$\begin{aligned} \alpha &= \int_0^r ds(1 + s^{2n})^{-1} = \int_0^r [1 - s^{2n} + s^{4n} - \dots] ds \\ &= r \left[1 - \frac{r^{2n+1}}{2n+1} + \frac{r^{4n+1}}{4n+1} - \dots \right]. \end{aligned}$$

Thus

$$\gamma_2 = \alpha^{-1} \sim r^{-1} \left[1 + \frac{r^{2n+1}}{2n+1} + O(r^{4n+1}) \right] \tag{4.4}$$

and hence

$$\exp\left(-\int_0^r \gamma_2 dr\right) \sim \frac{1}{r} \exp\left[\frac{-r^{2n+2}}{(2n+1)(2n+2)} + O(r^{4n+2})\right]$$

so

$$\lim_{r \rightarrow 0} r \exp\left(-\int_0^r \gamma_2 dr\right) = 1.$$

From (3.8) the elementary flatness condition is then

$$\lim_{r \rightarrow 0} (b^2 - e^2) = 1. \tag{4.5}$$

From (4.3) we have $\omega, \sigma \sim \frac{1}{2}nr^{n-1}$ for $n > 1$, and $\omega \sim 1 - \frac{3}{4}r^2, \sigma \sim -\frac{3}{4}r^2$ for $n = 1$. Using (4.4) the Eq. (3.3) become

$$\left. \begin{aligned} db/dr &= nr^{n-2}(er) \\ d(er)/dr &= nr^n b \end{aligned} \right\} n > 1$$

and

$$\left. \begin{aligned} db/dr &= -\frac{3}{2}r(er) \\ d(er)/dr &= \left(2r - \frac{3r^3}{2}\right)b \end{aligned} \right\} n = 1.$$

These equations admit the solutions

$$\left. \begin{aligned} b &= 1 + nr^{2n}/2(n+1) + O(r^{3n}) \\ e &= nr^n/(n+1) + O(r^{2n}) \end{aligned} \right\} n > 1$$

$$\left. \begin{aligned} b &= 1 - \frac{3}{8}r^4 + O(r^6) \\ e &= r + O(r^3) \end{aligned} \right\} n = 1.$$

Thus in all cases (4.5) is satisfied, and all solutions have a good axis. To see what happens to R_{abcd} in a parallelly-propagated basis, we examine the integral $\beta(r)$ in (3.2); here this gives

$$-\beta = \int_{r_0}^r n s^{n-1} / (1 + s^{2n})^{\frac{3}{2}} ds$$

which is bounded for all r_0, r for consider the “worst” case $r_0 = 0, r = \infty$. Then $\beta < 0$ and

$$\begin{aligned} -\beta &= \int_0^1 \frac{n s^{n-1}}{(1 + s^{2n})^{\frac{3}{2}}} ds + \int_1^\infty \frac{n s^{n-1}}{(1 + s^{2n})^{\frac{3}{2}}} ds \\ &\leq \int_0^1 n dr + \int_1^\infty \frac{n s^{2n-1}}{(1 + s^{2n})^{\frac{3}{2}}} ds \\ &= n + [- (1 + r^{2n})^{-\frac{1}{2}}]_1^\infty = n + \sqrt{2}. \end{aligned}$$

Hence *all Riemann tensor components are finite (analytic!) in any parallel propagated orthonormal frame*. This follows since all such frames, by the above result, suffer a finite Lorentz rotation relative to our frame, and all R_{abcd} are finite in our frame as ϱ, σ, ω are everywhere finite and γ_1, γ_2 are finite except on the axis $r = 0$. Thus the only singularities that the family of solutions $\alpha = \int_0^r ds (1 + s^{2n})^{-1}$ can have are locally extensible, and it is highly probable that in fact there are *no* singularities in these spacetimes. Unfortunately we are unable to confirm this by explicit examination of the geodesics because of the difficulty of solving exactly the system (3.3): this would give us the metric, or equivalently the Killing vector components in the tetrad basis, leading to constants of the motion. Similarly we are unable to check whether there are any closed timelike lines; a sufficient condition for their absence is $b^2 - e^2 \geq 0$ for all r .

5. Conclusion

We have given a method of constructing all solutions of the field equations (with $\Lambda = 0$) for stationary cylindrical symmetric distributions of pressure-free matter. Since the field equations give an underdetermined system in this case, there are arbitrarily many such solutions, and after showing how known solutions fit into our construction we have given some solutions which so far as the author is aware are new. Some of these were designed to exhibit new kinds of singularity not previously given in the literature. In all cases the matter is flowing past these singularities (\mathbf{u} lies in the surfaces $\{r = \text{constant}\}$), and the singularities appear at particular values of r . This makes it seem highly likely that at least

the Weyl singularities are unstable against degenerating into "big bang" singularities: should any matter be projected into them it seems certain that we would get $\rho \rightarrow \infty$ here. Investigation of wave propagation in these spaces on the lines of [13] should help to resolve this point.

Acknowledgements. This work was carried out during the author's tenure of a U.K. Science Research Council Studentship at D.A.M.T.P., Cambridge, and he wishes to thank the members of the Relativity group at D.A.M.T.P. for many helpful comments.

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Communicated by J. Ehlers

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