

# Was the Big Bang a Whimper?

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**Abstract.** In many cases the spatially homogeneous cosmological models of General Relativity begin or end at a “big bang” where the density and temperature of the matter in the universe diverge. However in certain cases the spatially homogeneous development of these universes terminates at a singularity where all physical quantities are well—behaved (a “whimper”) and an associated Cauchy horizon. We examine the existence and nature of these singularities, and the possible fate of matter which crosses the Cauchy horizon in such a universe. The nature of both kinds of singularity is illustrated by simple models based on two-dimensional Minkowski space-time; and the possibility of other types of singularity occurring is considered.

## 1. Introduction

In the standard spatially homogeneous and isotropic cosmological models of General Relativity (see e.g. Refs. [1, 2]) a singularity necessarily develops at the beginning of an expansion phase if the energy density  $\mu$  and pressure  $p$  of the matter obey the inequality

$$\mu + 3p \geq 0 \quad (1.1)$$

at all times. (We shall here assume the cosmological constant  $\Lambda$  is zero; if it is non-zero, the corresponding inequality is  $\mu + 3p \geq 2\Lambda$ , where notation is as in [2].) Further this singularity is necessarily a physical one if either of the sets of inequalities

$$p \geq 0, \quad \mu_0 > 0 \quad (1.2a)$$

or

$$1 \geq dp/d\mu \geq 0, \quad \mu_0 + p_0 > 0, \quad \mu_0 + 3p_0 > 0 \quad (1.2b)$$

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are satisfied, where  $\mu_0$  and  $p_0$  are respectively the density and pressure at some arbitrary time  $t_0$ , for then the curvature invariant

$$R^{ab}R_{ab} = \mu^2 + 3p^2 \quad (1.3)$$

diverges at the singularity. [Equation(1.1) is automatically satisfied in these cases.]

The high symmetry imposed on these universe models means that they cannot necessarily be regarded as good models of the actual universe at all times; in particular they could be very misleading about the nature, or even the existence, of a singularity. However a series of theorems by Penrose and Hawking (see e.g. Refs. [3, 4]) show that singularities do occur in realistic models of the universe in which the idealizations of homogeneity and isotropy have been dropped. These idealizations may be replaced by an assumption that causality is not violated, plus some geometrical restrictions which can be shown to be satisfied in a realistic universe model because of the existence of the cosmic microwave background radiation. The matter content is unrestricted except that it is supposed that the stress tensor  $T_{ab}$  obeys certain inequalities which are generalizations of condition (1.1). When these conditions are satisfied, the theorems indicate that singularities exist; but they do not give any idea of the nature of the singularity. What they in fact prove is that space-time must be geodesically incomplete.

In a separate attack on this problem, Lifschitz, Khalatnikov and Belinskii looked at power series solutions of the field equations near a singularity. Their earlier investigations [5] showed the existence of generic solutions in which an apparent singularity was not a physical singularity, and their later investigations [6] showed the existence of generic solutions which do have physical singularities; however one does not obtain from their work a definitive conclusion as to the nature of the singularity one would expect in a realistic model of the physical universe.

In the absence of a more general argument, one can try to examine this problem by looking at exact solutions of Einstein's equations. After the homogeneous and isotropic world models, the simplest cosmologies are the spatially homogeneous but anisotropic world models [7]. It is known [4, 8, 9] that singularities occur in these universe models also if inequalities like (1.1), (1.2) are obeyed (we shall for simplicity assume that the matter takes a perfect fluid form). In the case when the matter moves orthogonally to the surfaces of homogeneity [7, 10] it moves without acceleration or rotation and a physical singularity occurs. In the case of "tilted" models, when the matter moves relative

to these surfaces [11], it is not clear that this is the case<sup>1</sup>. While it has been shown that a physical singularity occurs in tilted Type IX universes if they can be extended far enough [12, 13], Shepley [14] has given an example (a L.R.S. Type V universe) which has a singularity where the matter density does not go infinite.

We wish to examine here in which homogeneous cosmologies such singularities can occur, and to examine further the nature of these singularities. It turns out that in Shepley's example, there is a second singularity where the matter density probably does go infinite; but nevertheless it is an important example, for there could be similar model universes with no initial singularity where the density, temperature or pressure diverges; and we might then say that there was no "big bang" in such a universe.

The existence of a singularity in an inextendible space-time is recognized by the existence of an inextendible geodesic which is incomplete, or more generally of an inextendible curve which has finite length as measured by a generalized affine parameter [4, 15]. Suppose such a geodesic or curve  $\lambda(v)$ , defined for affine parameter or generalized affine parameter values  $0 \leq v < v_+$ , cannot be extended to the parameter value  $v_+$ . To classify the nature of the singularity the curve runs into, [4, 16], consider the components  $R_{abcd}(v)$  of the Riemann tensor with respect to an orthonormal basis  $e_a(v)$  of vectors defined along  $\lambda(v)$ . If (i) all the components  $R_{abcd}$  tend to finite limits as  $v \rightarrow v_+$  for some orthonormal basis which is parallelly propagated along  $\lambda(v)$ , the singularity will be said to be a *locally extendible* singularity. If (ii) some component  $R_{abcd}(v)$  with respect to a parallelly propagated basis does not go to a limit, but there is some other orthonormal basis along  $\lambda(v)$  in which they do all go to finite limits as  $v \rightarrow v_+$ , we say the singularity is an *intermediate singularity*. If (iii) there is no orthonormal basis along  $\lambda(v)$  such that the components  $R_{abcd}(v)$  all go to finite limits as  $v \rightarrow v_+$ , we say that the singularity is a *curvature singularity*. Such a singularity can be further characterized as a *matter singularity* if some Ricci tensor components  $R_{ab}(v)$  do not go to finite limits for any orthonormal frame along  $\lambda(v)$ , or as a *conformal singularity* if the Weyl tensor components  $C_{abcd}(v)$  do not all go to finite limits<sup>2</sup>.

When there is a locally extendible singularity, there always exists an open neighbourhood of  $\lambda(v)$  which can be extended in such a way that  $\lambda(v)$  can be continued beyond  $\lambda(v_+)$  in this local extension [17]. Roughly

<sup>1</sup> References [8, 9] do *not* prove there is a *matter* singularity, despite statements claiming this.

<sup>2</sup> Obviously further classification could be given according to whether these components diverge, oscillate finitely or oscillate infinitely, in some particular family of frames; and according to the behaviour of the covariant derivatives up to the  $r^{\text{th}}$  order.

speaking, the curvature tensor is perfectly regular near the singularity but there are either too many or too few directions present in the limit as  $v \rightarrow v_+$  for  $\lambda(v_+)$  to be a regular space-time point. Cone singularities, covering space singularities and the singularity in Taub-NUT space are examples of such singularities [16].

In the case of an intermediate singularity, space-time is quite regular near the singular point if a suitable reference frame is used, but a non-convergent Lorentz transformation relates this frame to a parallelly propagated frame. Thus arbitrarily large or irregular tidal forces may tear apart an observer falling into such a singularity, but other observers can move arbitrarily close to the singularity without experiencing very large or irregular forces. As well as in the homogeneous cosmologies we consider here, such a singularity occurs in certain plane wave solutions [31], and in conformally transformed Taub-NUT universes ([4], p. 291).

Both intermediate and curvature singularities are *p. p. singularities* in the notation of Hawking and Ellis [4], that is, are singularities in which the curvature tensor components do not go to finite limits if a parallelly propagated frame is used as a reference frame; clearly if such a singularity occurs, this is sufficient to prove there cannot be a continuation of the curve  $\lambda(v)$  in any extension of the space-time. When a curvature singularity occurs, every observer moving arbitrarily near it experiences unboundedly large or irregular gravitational forces. Examples are the singularities in Robertson-Walker spaces where (1.2) hold (a matter singularity) and in the Schwarzschild solution (a conformal singularity). It is plausible, but has not been rigorously proved, that a curvature singularity occurs if and only if a *s.p. curvature singularity* [4] occurs, that is, if some scalar polynomial constructed from the curvature tensor, the metric tensor  $g_{ab}$  and the totally skew tensor  $\eta^{abcd}$  does not go to a limit along  $\lambda(v)$  as  $v \rightarrow v_+$ . It is obvious that an s.p. singularity implies a curvature singularity, but the converse is not obvious, particularly in view of the fact that some curvature tensors are non-zero but define no non-zero scalar polynomials. In the particular case we are interested in, when the Ricci tensor has a perfect fluid form with a uniquely defined timelike eigenvector, and we only consider the curvature tensor rather than its derivatives, these singularities are the same (Lemma 6.2).

For our present purposes the important point is that any singularity along a given curve must obey either (i), (ii), or (iii), and so can be classified as belonging to one of these three classes.

In Section 2 we show that spatially homogeneous universes cannot in general remain spatially homogeneous – a breakdown of prediction occurs. In Section 3 we discuss those cases in which we can show there is a matter singularity, and in Section 4 those cases in which there is an intermediate singularity, when the spatial homogeneity breaks down;

in the latter cases the density does not go infinite, and the space-time can be extended across a Cauchy horizon. In Section 5 we discuss the possible extensions which can be made across the Cauchy horizon into a stationary inhomogeneous region, and into further regions. In Section 6 we discuss the possibility that a conformal singularity might occur but no matter singularity, and in Section 7 we summarize our results and discuss their relation to more realistic universe models. We also give a simple model for the intermediate singularities, which explains their main features.

When detailed calculations are needed, we use the notation of [11]; however, one should be able to read the theorems and understand their meaning without having to consult that paper. Our discussion will not include the Kantowski-Sachs Type I [7] universes explicitly; we expect no surprises in this case.

## 2. The Breakdown of Prediction in Spatially Homogeneous Cosmologies

The space-times  $(\mathcal{M}, g)$  we consider will satisfy the following conditions (cf. [4], Section 3, and § 5.4):

(1)  $(\mathcal{M}, g)$  is a space-time (i.e. a connected 4-dimensional  $C^\infty$  Hausdorff manifold  $\mathcal{M}$  with  $C^3$  Lorentz metric  $g$ ) which is inextendible;

(2)  $g$  satisfies Einstein's field equations with a perfect fluid matter source, i.e. there is a vector field  $u$  on  $\mathcal{M}$  such that

$$R_{ab} = (\mu + p) u_a u_b + \frac{1}{2}(\mu - p) g_{ab}, \quad u^a u_a = -1; \quad (2.1)$$

(3) a  $C^1$  equation of state  $p = p(\mu)$  for the fluid is given, and is such that the inequalities

$$\mu > 0, \quad \mu \geq 3p \geq 0, \quad \frac{1}{3} \geq dp/d\mu \geq 0 \quad (2.2)$$

are satisfied everywhere on  $\mathcal{M}$ ;

(4) the Cauchy data for the field equations on some spacelike surface  $\mathcal{S}$  in  $\mathcal{M}$  is invariant under a continuous group  $G$  of diffeomorphisms of  $\mathcal{S}$  which is simply transitive on  $\mathcal{S}$ ; and the domain of development  $(D(\mathcal{S}), g)$  of  $\mathcal{S}$  in  $(\mathcal{M}, g)$  is isometric to the unique maximal Cauchy development of this data. Further the closure  $\overline{D(\mathcal{S})}$  of  $D(\mathcal{S})$  in  $\mathcal{M}$  is maximal.

The first part of condition (4) implies that the initial data on  $\mathcal{S}$  is analytic; as the intrinsic geometry of  $\mathcal{S}$  is invariant under the group of diffeomorphisms  $G$ , this is in fact a group of isometries.  $\mathcal{S}$  is therefore a complete surface (i.e. without edge). We may assume that no non-spacelike curve intersects it more than once, for if this were not so there would be a covering space  $(\tilde{\mathcal{M}}, \tilde{g})$  in which this was true for each image

of  $\mathcal{S}$  [4, 18], and we could then consider this space instead of  $(\mathcal{M}, \mathbf{g})$ . Because of the existence of the vector field  $\mathbf{u}$  at each point of  $\mathcal{S}$ ,  $\mathcal{S}$  is two-sided; we choose the unit normal  $\mathbf{n}$  to  $\mathcal{S}$  to point into the same half of the light cone as  $\mathbf{u}$  at each point.

By condition (3), the Cauchy problem for the field equations has a unique local solution when regular Cauchy data  $\{\mu, u^a, \tilde{h}_{ab}, \chi_{ab}\}^3$  are given on a spacelike surface and satisfy the constraint equations on that surface. Therefore the Cauchy data on  $\mathcal{S}$  determines a unique maximal Cauchy development (see e.g. [4], Chapter 7); and the second half of condition (4) ensures that the actual development of  $\mathcal{S}$  in  $\mathcal{M}$  is just this unique maximal development (without this condition, one could for example cut out points in  $\mathcal{M}$  to the future of  $\mathcal{S}$  and then go to a covering space in such a way as to obtain an inextendible domain of development  $D(\mathcal{S})$  of  $\mathcal{S}$  in  $(\mathcal{M}, \mathbf{g})$  which was different from the maximal development).

We define the map  $\Phi_s: \mathcal{S} \rightarrow \mathcal{M}$  to be the map sending a point  $q \in \mathcal{S}$  a proper distance  $s$  along the geodesic  $\gamma(s)$  normal to  $\mathcal{S}$  through  $q$  in the  $+n^a$  direction (negative values of  $s$  corresponding to a map a distance  $|s|$  in the  $-n^a$  direction); and similarly  $\Psi_\tau: \mathcal{S} \rightarrow \mathcal{M}$  maps a point  $q \in \mathcal{S}$  a proper distance  $\tau$  along the integral curve of  $+u^a$  through  $q$ . There will be positive numbers  $s_\pm, \tau_\pm$  which are the largest numbers (possibly infinite) such that the maps  $\Phi_s, \Psi_\tau$  are defined for  $-s_- < s < s_+$ ,  $-\tau_- < \tau < \tau_+$  respectively; we denote the images of  $\mathcal{S}$  under these maps by  $\mathcal{N}(s) \equiv \Phi_s(\mathcal{S})$ ,  $\mathcal{F}(\tau) \equiv \Psi_\tau(\mathcal{S})$ . Because of (2.2),  $u^a$  is uniquely defined at each point, up to a sign, by (2.1); therefore its integral curves (the fluid flow lines) cannot intersect. As no non-spacelike curve intersects  $\mathcal{S}$  more than once, different surfaces  $\mathcal{F}(\tau)$  cannot intersect. Thus as long as, for a given value  $\tau$ ,  $\Psi_\tau(q)$  is defined for each point  $q \in \mathcal{S}$ , the surface  $\mathcal{F}(\tau)$  is homeomorphic to  $\mathcal{S}$  and does not intersect any surface  $\mathcal{F}(\tau')$  with  $\tau' \neq \tau$ . The same is not obviously true for the surfaces  $\mathcal{N}(s)$ , for the normal geodesics to  $\mathcal{S}$  could possibly intersect each other.

Let  $S_\pm, T_\pm$  be the largest numbers such that the images  $\mathcal{N}(s), \mathcal{F}(\tau)$  lie in the domain of development  $D(\mathcal{S})$  of  $\mathcal{S}$  for  $-S_- < s < S_+$ ,  $-T_- < \tau < T_+$  respectively. Because of (4), this Cauchy development will be spatially homogeneous. More precisely,

**Lemma 2.1.** *The domain of development  $(D(\mathcal{S}), \mathbf{g})$  of  $\mathcal{S}$  admits the group  $G$  as a group of isometries, with the homogeneous spacelike surfaces  $\mathcal{N}(s)$ ,  $-S_- < s < S_+$ , as the surfaces of transitivity. These surfaces are the same as the surfaces  $\mathcal{F}(\tau)$ ,  $-T_- < \tau < T_+$ ; they are Cauchy surfaces for  $D(\mathcal{S})$ , and the map  $\Psi: (-T_-, T_+) \times \mathcal{S} \rightarrow D(\mathcal{S})$  which maps  $(\tau, \mathcal{S})$  to  $\mathcal{F}(\tau) \equiv \Psi_\tau(\mathcal{S})$  is a diffeomorphism.*

<sup>3</sup>  $h^{ab}$  is the first fundamental form of  $\mathcal{S}$  and  $\chi^{ab}$  its second fundamental form;  $u^a$  may be expressed in terms of a magnitude  $\beta$  and a vector  $\mathbf{c}$  lying in  $\mathcal{S}$  [see (2.3)].

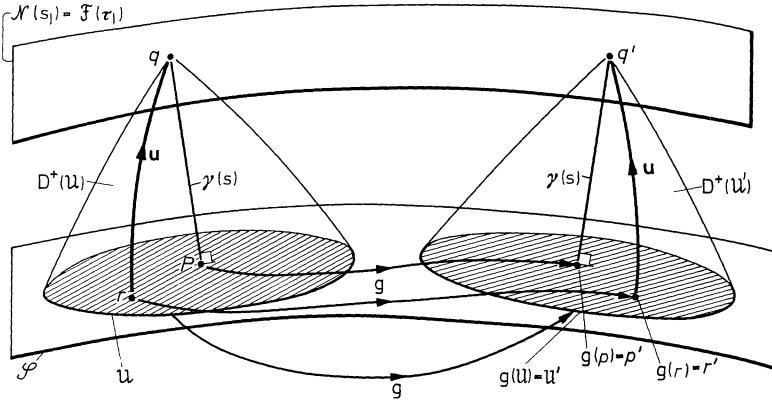


Fig. 1.  $\mathcal{U}, \mathcal{U}'$  are identical and have identical Cauchy data, so their Cauchy developments  $D^+(\mathcal{U}), D^+(\mathcal{U}')$  are identical. In particular the distance  $\tau_1$  along the fluid line through  $r$  till it intersects the normal geodesic through  $p$  at a distance  $s_1$  from  $p$ , is identical to that for the fluid flow line through  $r'$  which intersects the normal geodesic through  $p'$  at a distance  $s_1$  from  $p'$

Consider a point  $q \in D(\mathcal{S})$ . Then there is a geodesic  $\gamma(s)$  through  $q$  normal to  $\mathcal{S}$  ([4], §6.7) so there is an  $s_1, -S_- < s_1 < S_+$ , such that  $q \in \mathcal{N}(s_1)$ . By (2.2),  $\mu + p > 0$ , so (2.1) shows there is a unique integral curve of the timelike vector field  $\mathbf{u}$  through  $q$ ; as  $q \in D(\mathcal{S})$ , this curve intersects  $\mathcal{S}$ , so there is a unique  $\tau_1, -T_- < \tau_1 < T_+$ , such that  $q \in \mathcal{F}(\tau_1)$ . Let  $\mathcal{U} \equiv (J^-(q) \cup J^+(q)) \cap \mathcal{S}^4$ ; this is non-empty, and  $p \equiv \Phi_{s_1}^{-1}(q)$ ,  $r \equiv \Psi_{\tau_1}^{-1}(q)$  are in  $\mathcal{U}$ . Given an element  $g \in G$ , it maps  $\mathcal{U} \subset \mathcal{S}$  to  $\mathcal{U}' \subset \mathcal{S}$  where  $\mathcal{U}$  and the Cauchy data on  $\mathcal{U}$  are identical to  $\mathcal{U}'$  and the Cauchy data on  $\mathcal{U}'$ . The maximal Cauchy development of  $\mathcal{U}$  is unique, and is therefore also the maximal Cauchy development of  $\mathcal{U}'$ ; in particular, the geodesics normal to  $\mathcal{S}$  and the integral curves of  $\mathbf{u}$  in this maximal development are uniquely determined. By (4), this maximal Cauchy development is isometric to the domains of development  $(D(\mathcal{U}), \mathbf{g}), (D(\mathcal{U}'), \mathbf{g})$  of  $\mathcal{U}, \mathcal{U}'$  in  $(\mathcal{M}, \mathbf{g})$ ; so  $(D(\mathcal{U}), \mathbf{g}), (D(\mathcal{U}'), \mathbf{g})$  are isometric. In particular, for each  $g \in G$  the map  $\tilde{g} \equiv \Phi_{s_1}^{-1} \circ g \circ \Phi_{s_1}$  is an isometry mapping  $q \in \mathcal{N}(s_1)$  into  $\tilde{g}(q) \in \mathcal{N}(s_1)$ , the map  $\hat{g} \equiv \Psi_{\tau_1}^{-1} \circ g \circ \Psi_{\tau_1}$  is an isometry mapping  $q \in \mathcal{F}(\tau_1)$  into  $\hat{g}(q) \in \mathcal{F}(\tau_1)$ , and  $\tilde{g}(q) = \hat{g}(q)$  (see Fig. 1); thus these are in fact the same surfaces, and  $G$  acts as a group of isometries of  $(D(\mathcal{S}), \mathbf{g})$  transitive on them. It acts transitively because for each point  $q' \in \mathcal{N}(s_1)$  there is an isometry  $g \in G$  such that  $g(p) = \Phi_{s_1}^{-1}(q')$ ; then the associated map  $\tilde{g}$  is an isometry mapping  $q$  to  $q'$ . The rest of the statement now follows easily.  $\square$

<sup>4</sup>  $J^-(q)$  is the causal past of  $q$ , and  $J^+(q)$  the causal future of  $q$  (see e.g. [4]).

Note that in fact there is nothing very special about the surface  $\mathcal{S}$ ; given  $D(\mathcal{S})$ , we could have chosen *any* of the surfaces  $\mathcal{N}(s)$ ,  $-S_- < s < S_+$ , as the homogeneous initial surface  $\mathcal{S}$ . From the homogeneity, it follows that the following lemma holds:

**Lemma 2.2.** *The geodesics orthogonal to  $\mathcal{S}$  are orthogonal to the surfaces  $\mathcal{N}(s)$ ,  $-S_- < s < S_+$ , and have no conjugate points in  $D(\mathcal{S})$ . If  $\gamma(s)$  is a geodesic orthogonal to  $\mathcal{S}$  such that there are no points conjugate to  $\mathcal{S}$  in  $\gamma(s)$ ,  $-s_- \leq -s' < s < 0$ , then  $\gamma(s)$  lies in  $D(\mathcal{S})$  for  $-s' < s < 0$ . The map  $\Phi : (-S_-, S_+) \times \mathcal{S} \rightarrow D(\mathcal{S})$  which maps  $(s, \mathcal{S})$  to  $\mathcal{N}(s) = \Phi_s(\mathcal{S})$ , is a diffeomorphism.  $\square$*

Let  $n^a$  be the future-directed unit tangent vector to the congruence of geodesics orthogonal to the surfaces  $\mathcal{N}(s)$  in  $D(\mathcal{S})$ . The metric tensor  $\tilde{h}_{ab}$  of these surfaces is given by  $\tilde{h}_{ab} = g_{ab} + n_a n_b$ . The relation between  $\mathbf{u}$  and  $\mathbf{n}$  is determined by  $\beta, \tilde{c}_a$  where

$$u^a = \cosh \beta n^a + \sinh \beta \tilde{c}^a, \quad \tilde{c}^a \tilde{c}_a = 1, \quad \tilde{c}^a n_a = 0. \tag{2.3}$$

Further we define  $\tilde{\theta}_{ab} \equiv n_{a;b}$ ; then  $\tilde{\theta}_{ab} = \tilde{\theta}_{(ab)}$ ,  $\tilde{\theta}_{ab} n^b = 0$ , and  $\tilde{\theta}_{ab}|_{\mathcal{S}} = \chi_{ab}$  where  $\chi_{ab}$  is the second fundamental form of the surface  $\mathcal{S} \equiv \mathcal{N}(0)$ . The trace of  $\tilde{\theta}_{ab}$  is  $\tilde{\theta}$ , so  $\tilde{\theta} = \tilde{\theta}^a_{;a} = n^a_{;a}$ ; and the length  $\tilde{l}$  is defined by  $\tilde{l}^{-1} d\tilde{l}/ds = \frac{1}{3}\tilde{\theta}$ .

**Lemma 2.3.** *If  $(\beta, \tilde{\theta})$  are bounded for  $-s' < s \leq 0$ , where  $s' > 0$  is finite, then  $s_- \geq S_- > s'$ . Further  $S_- \geq T_-$ .*

The first integral relation ((2.17b) of [11]) can be written

$$\tilde{\theta}^2 = \tilde{\theta}^{ab} \tilde{\theta}_{ab} + 2\mu \cosh^2 \beta + 2p \sinh^2 \beta - {}^3R \tag{2.4}$$

where  ${}^3R$  is the Ricci scalar of the 3-surfaces  $\mathcal{N}(s)$ ; and  ${}^3R \leq 0$  if the universe is not Type IX [10], while if the universe is Type IX,  ${}^3R$  is bounded above if  $\tilde{l}$  is bounded below (we are indebted to Matzner for discussions on this point). Hence in both cases,  $\mu$  and  $\tilde{\theta}_{ab} \tilde{\theta}^{ab}$  are bounded for  $0 \geq s > -s'$  (or else  $\tilde{\theta}$  would be unbounded); so there are no points conjugate to  $\mathcal{S}$  in  $\gamma(s)$ ,  $0 \geq s > -s'$ , which shows that  $S_- \geq s'$ , by Lemma 2.2. The problem is to prove the strict inequality. For  $0 \geq s > -s'$ , local coordinates  $\{x^a\} = \{t, x^k\}$  may be used, such that the metric takes the form

$$ds^2 = -dt^2 + \tilde{h}_{\mu\nu}(x^a) dx^\mu dx^\nu$$

where  $n^a = \delta^a_0$ ,  $\tilde{\theta}_{\mu\nu} = \partial \tilde{h}_{\mu\nu} / \partial t$ , and  $\tilde{h}_{\mu\nu}$  are analytic functions on each surface  $\mathcal{N}(s)$ . As  $\tilde{\theta}_{\mu\nu}$  is bounded for  $0 \geq s > -s'$ , so is

$$\tilde{h}_{\mu\nu}(t, x^k) = \int_0^t \tilde{\theta}_{\mu\nu}(s, x^k) ds + \tilde{h}_{\mu\nu}|_{\mathcal{S}}.$$



The solution is analytic on each surface  $\mathcal{N}(s)$  in  $D(\mathcal{S})$ ; hence there is regular bounded Cauchy data  $\{\mu, \tilde{c}_a, \beta, \tilde{h}_{ab}, \tilde{\theta}_{ab}\}$  on each surface  $\mathcal{N}(s)$ ,  $0 \leq s > -s'$ . If  $s_- > s'$ , consider the surface  $\mathcal{N}(-s')$  in  $\mathcal{M}$ . By continuity there is regular bounded Cauchy data on  $\mathcal{N}(-s')$ ; and the constraint equations are satisfied, as the conservation equations guarantee that they are satisfied on each surface  $\mathcal{N}(s)$  in  $D(\mathcal{S})$ , because they are satisfied on  $\mathcal{S}$ . Hence by (4), there is a non-trivial Cauchy development  $D(\mathcal{N}(-s'))$  which contains  $\mathcal{N}(-s')$ ; so  $S_- > s'$ .

On the other hand, if  $s_-$  were equal to  $s'$ , one could attach a surface  $\mathcal{N}(-s')$  to  $\mathcal{M}$  by the condition that  $[0, -s'] \times \mathcal{S}$  is homeomorphic to  $D^-(\mathcal{S}) \cup \mathcal{N}(-s')$  under a mapping  $\Phi$  which is the diffeomorphism  $\Phi: (s, \mathcal{S}) \rightarrow \mathcal{N}(s)$  when restricted to  $[0, -s'] \times \mathcal{S}$ . Extending the Cauchy data to  $\mathcal{N}(-s')$  by continuity, the argument used previously shows there is a Cauchy development  $D(\mathcal{N}(-s'))$  whose past part is not contained in  $(\mathcal{M}, \mathbf{g})$ ; but this is a contradiction because of (4) and the fact that  $D(\mathcal{N}(-s'))$  is, by Lemma 2.1, the same as  $D(\mathcal{S})$ . Finally the inequalities follow because  $ds/d\tau = -u^a n_a = \cosh \beta \geq 1$ .  $\square$

There is an obvious dual to this result in which the future is replaced by the past.

Under the circumstances we are considering, the Cauchy development of  $\mathcal{S}$  is strictly limited.

**Theorem 2.4.** *One of  $S_+$ ,  $S_-$  is finite. If the universe is not Type IX, the expansion  $\tilde{\theta}$  of the normals diverges and  $\tilde{l} \rightarrow 0$  as  $s$  approaches this limit.*

The divergence  $\tilde{\theta}$  obeys Raychaudhuri's equation (cf. [2, 4, 11], Eq. (2.16c))

$$d\tilde{\theta}/ds + \tilde{\theta}^{ab} \tilde{\theta}_{ab} + \frac{1}{2}(\mu + 3p) + \sinh^2 \beta (\mu + p) = 0. \quad (2.5)$$

Combining this with (2.4), one obtains

$$d\tilde{\theta}/ds + \tilde{\theta}^2 = -{}^3R + \mu(\sinh^2 \beta + \frac{3}{2}) + p(\sinh^2 \beta - \frac{3}{2}). \quad (2.6)$$

Because of (2.2), (2.5) shows

$$d\tilde{\theta}/ds + \frac{1}{3}\tilde{\theta}^2 < 0; \quad (2.7)$$

and when the universe is not Type IX,  ${}^3R \leq 0$ , so (2.2) and (2.6) show

$$d\tilde{\theta}/ds + \tilde{\theta}^2 > 0. \quad (2.8)$$

One can integrate through the inequalities (2.7), (2.8) to show:

**Lemma 2.5.** *Suppose  $\tilde{\theta}_0 \equiv \tilde{\theta}|_{\mathcal{S}} > 0$  in a universe which is not Type IX. Then  $-S_- < s < 0 =$*

$$(\tilde{\theta}_0)^{-1} + s < (\tilde{\theta})^{-1} < (\tilde{\theta}_0)^{-1} + \frac{1}{3}s; \quad H'\tilde{l}^{-3} > \tilde{\theta} > H\tilde{l}^{-1},$$

where  $H', H$  are positive constants; for  $0 < s < S_+$ , these inequalities are reversed.  $\square$

Suppose now  $\tilde{\theta}_0$  is positive in a universe which is not Type IX. Then Lemma 2.5 shows  $\tilde{\theta}$  stays positive for  $-S_- < s < S_+$ , and will become infinite a finite distance  $s''$  to the past of  $\mathcal{S}$  if  $\gamma(s)$  can be extended that far. Then  $s'' \geq S_-$ , or else there would be a point conjugate to  $\mathcal{S}$  along  $\gamma(s)$  in  $D(\mathcal{S})$ , contradicting Lemma 2.2. Hence  $S_-$  is finite. Lemma 2.3 shows  $\gamma(s)$  can be extended in  $D(\mathcal{S})$  as long as  $\tilde{\theta}$ ,  $\mu$ , and  $\beta$  are all bounded. Equation (2.4) shows that if  $\mu$  is unbounded on some open interval,  $\tilde{\theta}$  is unbounded there also, and Lemmas 2.7 and 2.8 will show that  $\tilde{\theta}$  is unbounded if  $\beta$  is unbounded. Hence  $\gamma(s)$  can be extended as long as  $\tilde{\theta}$  is bounded, so  $s'' \leq S_-$ , which implies  $s'' = S_-$ . Lemma 2.5 then shows that  $\tilde{l} \rightarrow 0$  as  $s \rightarrow -S_-$ , and that  $\tilde{\theta}$  diverges there.

A similar argument applies when  $\tilde{\theta}_0$  is negative, and  $s'' = S_+$ , in a universe which is not Type IX. In the case of a Type IX universe, (2.7) still holds and shows that either  $S_-$  or  $S_+$  must be finite. The case  $\tilde{\theta}_0 = 0$  can only occur in a Type IX universe; and in this case both  $S_-$  and  $S_+$  are finite.  $\square$

This shows that either in the past or in the future, one cannot predict what happens for more than a finite time  $s''$  on the basis of the Cauchy data on  $\mathcal{S}$ . We shall from now on choose the direction of time such that this happens to the past. With this choice of sign,  $\tilde{\theta}_0 > 0$ ,  $S_-$  is finite,  $\tilde{\theta} > 0$  for  $-S_- < s < 0$ , and  $\tilde{\theta} \rightarrow \infty$  as  $s \rightarrow -S_-$  from above. We shall later show that in all universes which are not Type IX  $S_+$  is infinite, i.e. there is no singularity to the future of the surface  $\mathcal{S}$  (see Lemma 5.2).

Before proving Lemma 2.8, we prove two useful preliminary results. Define the functions  $w$ ,  $r$ , up to multiplicative constants, by  $dw/w = d\mu/(\mu + p)$ ,  $dr/r = dp/(\mu + p)$ . Then  $(rw)/(r_0 w_0) = (\mu + p)/(\mu_0 + p_0)$ , where  $\mu_0 \equiv \mu|_{\mathcal{S}}$ ,  $p_0 \equiv p(\mu_0)$ ,  $r_0 \equiv r(\mu_0)$ ,  $w_0 \equiv w(\mu_0)$ . We choose  $r_0, \omega_0$  so that  $r > 0$ ,  $\omega > 0$ . Now inequalities (2.2) readily show that:

**Lemma 2.6.**  $\mu \geq \mu_0 \Rightarrow p \geq p_0, r \geq r_0, w \geq w_0; \mu/\mu_0 \geq w/w_0 \geq (\mu/\mu_0)^{3/4}; (p/p_0)^{1/4} \geq r/r_0; \frac{1}{3}(\mu - \mu_0) \geq p - p_0; (w/w_0)^{1/3} \geq r/r_0; w/r_0 \geq w/r \geq w_0/r_0$ . For  $\mu \leq \mu_0$ , these inequalities are reversed.  $\square$

In terms of these quantities, the energy-momentum conservation equations are (cf. [11])

$$d \log(w \tilde{l}^3 \cosh \beta) / ds = 2 \tanh \beta \tilde{c}_a a^d, \tag{2.9}$$

$$d \log(r \sinh \beta) / ds = -\tilde{c}^a \tilde{\theta}_{ab} \tilde{c}^b, \tag{2.10}$$

where  $a_d$  is the trace of the commutation coefficients (or Lie derivative components)  $\gamma^k_{\mu\nu}$  of a basis of invariant vectors in the surfaces  $\mathcal{N}(s)$  (see [10, 11]); so  $a_\epsilon \equiv \frac{1}{2} \gamma^{\alpha}_{\epsilon\alpha}$ ,  $a_d n^d = 0$ .

**Lemma 2.7.** *In a universe which is not Type IX,  $-S_- < s < 0 \Rightarrow$*

$$B \tilde{l} \leq r \sinh \beta \leq B' \tilde{l}^{-3}, \quad M \tilde{l}^{-1} < w \cosh \beta < M' \tilde{l}^{-5}$$

where  $B, B'$  are constants  $\geq 0$ ,  $M, M'$  are constants  $> 0$ ;  $B, B'$  are  $> 0$  unless  $\beta_0 = 0$ , when  $B = B' = 0$ . When  $0 < s < S_+$ , these inequalities are reversed.

Given  $\tilde{\theta}$  and  $\tilde{\theta}_{ab}\tilde{\theta}^{ab}$ , the range of values  $\tilde{\theta}_{ab}e^ae^b$  can have (for any unit vector  $e^a$  orthogonal to  $n^a$ ) is bounded by  $\frac{1}{3}\tilde{\theta} \pm (\frac{2}{3}(\tilde{\theta}_{ab}\tilde{\theta}^{ab} - \frac{1}{3}\tilde{\theta}^2))^{1/2}$ . By (2.4),  $\tilde{\theta}^2 > \tilde{\theta}_{ab}\tilde{\theta}^{ab}$ , and  $\tilde{\theta} > 0$  for  $-S_- < s < S_+$ , so

$$-\frac{1}{3}\tilde{\theta} < \tilde{\theta}_{ab}e^ae^b < \tilde{\theta}. \quad (2.11)$$

Using this one can integrate through (2.10) to obtain the first inequality. When  $a_\alpha = 0$ , (2.9) easily implies the second inequality is satisfied; when  $a_b \neq 0$ ,  ${}^3R \leq -6a_d a^d$  (see equation (8b) of Appendix I in [10]; this is the case when  $n_1 = 0$ ) so Eq. (2.4) shows  $\frac{2}{3}\tilde{\theta}^2 > 6a_d a^d$ . As  $\tilde{\theta}$  is positive, it follows that for any unit vector  $e$  orthogonal to  $n$ ,

$$\frac{2}{3}\tilde{\theta} > 2a_d e^d \tanh \beta > -\frac{2}{3}\tilde{\theta}. \quad (2.12)$$

Using this one can integrate through (2.9) to obtain the second inequality.  $\square$

**Lemma 2.8.** *Consider a universe which is not Type IX. If  $\beta$  is unbounded on some open interval of  $s(s < 0)$ , then  $\tilde{\theta}$  is also unbounded there.*

From Lemma 2.5 and Eq. (2.4),  $(H')^2 \tilde{l}^{-6} > \tilde{\theta}^2 > 2\mu \cosh^2 \beta$ . When  $\mu \geq \mu_0$ , Lemmas 2.6 and 2.7 show  $\mu \cosh^2 \beta > (\mu_0 M/w_0) \tilde{l}^{-1} \cosh \beta$ ; and when  $\mu \leq \mu_0$ ,  $\mu \cosh^2 \beta > \mu_0 (M/w_0)^{4/3} \tilde{l}^{-4/3} (\cosh \beta)^{2/3}$ . Hence there are positive constants  $A, A'$  such that if  $\mu \geq \mu_0$ ,  $A \tilde{l}^{-5} > \cosh \beta$ , and if  $\mu \leq \mu_0$ ,  $A' \tilde{l}^{-7} > \cosh \beta$ ; therefore whenever (for  $s < 0$ )  $\beta$  is unbounded,  $\tilde{l}$  approaches arbitrarily close to zero, and  $\tilde{\theta}$  is unboundedly large then, by Lemma 2.5.  $\square$

### 3. Matter Singularities

In a great many cases one can show that the cause of the breakdown of prediction is a singularity where the fluid energy density and the space-time curvature go infinite.

The simplest situation is the case where  $u^a$  is parallel to  $n^a$ , i.e. the fluid flow is orthogonal to the homogeneous surfaces. (cf. [10]). The result is essentially well known:

**Theorem 3.1.** *If  $u^a$  is orthogonal to  $\mathcal{S}$ , then  $s_- = \tau_-$  is finite. Both the energy density  $\mu$  and the curvature invariant  $R_{ab}R^{ab}$  diverge on  $\mathcal{N}(s)$ ,  $-S_- < s < S_+$ , as  $s \rightarrow -S_- = -s_-$ .*

If  $u^a$  is parallel to  $n^a$  on  $\mathcal{S}$ , then  $u^a$  is parallel to  $n^a$  in  $D(\mathcal{S})$  (cf. [11]). Thus the fluid expansion  $\theta \equiv u^a{}_{;a}$  is the same as the normal geodesic expansion  $\tilde{\theta}$ , and diverges in the finite distance  $S_-$  if the universe is not Type IX, by Theorem 2.4; and  $\tilde{l} \rightarrow 0$  then. Now Lemmas 2.7 (with

$\beta = 0$ ) and 2.6 show  $\mu \rightarrow \infty$  as  $s \rightarrow -S_-$ . Then  $R^{ab}R_{ab}$  diverges by (1.3), which follows from (2.1).

The same result will be true for a Type IX universe, provided it can be extended far enough to imply  $\tilde{l} \rightarrow 0$ . However because  $\beta = 0$ , one can use the fact that  ${}^3R$  is bounded above if  $\tilde{l}$  is bounded below to generalize the second half of Theorem 2.4 to this case also; and the result holds also for Type IX universes.  $\square$

In the case of tilted cosmologies (when  $\mathbf{u}$  and  $\mathbf{n}$  are not parallel) the situation is more complicated. Matzner *et al.* [12] showed there was a matter singularity in all Type IX dust universes when  $\tilde{l} \rightarrow 0$ , and Matzner [13] extended this result to the case of perfect fluid Type IX universes. They did not however prove the universes could be extended until  $\tilde{l} \rightarrow 0$ . We here consider universes other than Type IX, and use essentially the same methods as Matzner *et al.* to prove that matter singularities occur in all Class A universes (except Type IX, which our method does not cover) and in certain Class B universes, where we use the notation introduced in Ref. [10] for the group types. The group structure constants  $C^\alpha_{\beta\gamma}$  obey the relation  $C^\alpha_{\beta\alpha} = 0$  if and only if the vector  $a_d$  (see Lemma 2.6) vanishes. If  $C^\alpha_{\beta\alpha} = 0 \Leftrightarrow a_d = 0$ , we say the group is in Class A; if  $C^\alpha_{\beta\alpha} \neq 0 \Leftrightarrow a_d \neq 0$ , we say the group is in Class B.

**Theorem 3.2.** *In a tilted homogeneous model with  $(a_b \tilde{c}^b)$  bounded above for  $s < 0$ ,  $s_- > \tau_-$  is finite; and if the universe is not Type IX, both the energy density  $\mu$  and the curvature invariant  $R^{ab}R_{ab}$  are unbounded on  $\mathcal{N}(s)$  as  $s \rightarrow -S_- = -s_-$  (or equivalently, on  $\mathcal{F}(\tau)$  as  $\tau \rightarrow -T_- = -\tau_-$ ).*

If the universe is Type IX, the first part follows from [13]<sup>5</sup>. Now suppose it is not Type IX. By Theorem 2.4,  $\tilde{l} \rightarrow 0$  as  $s \rightarrow -S_-$ ; so we can only hope to avoid a matter singularity if  $\beta \rightarrow \infty$  as  $s \rightarrow -S_-$  (by Lemmas 2.5–2.7).

Let there be some upper bound on  $(a_d \tilde{c}^d)$  for  $s < 0$ ; integrating through (2.9) shows

$$-S_- < s < 0 \Rightarrow w \cosh \beta > M'' \tilde{l}^{-3}, \quad M'' (> 0) \text{ constant}. \quad (3.1)$$

Also Eq. (2.4) shows

$$\tilde{\theta}^2 \geq \tilde{\theta}^{ab} \tilde{\theta}_{ab} + 2\mu \cosh^2 \beta. \quad (3.2)$$

Combining (3.1), (3.2), and Lemma 2.5 shows

$$-S_- < s < 0 \Rightarrow (\tilde{\theta}^{ab} \tilde{\theta}_{ab}) / (\tilde{\theta}^2) < 1 - (2\mu/w^2) (M''^2/H'^2). \quad (3.3)$$

Now suppose there were an upper bound  $\mu'$  to  $\mu$  for  $-S_- < s < 0$ . Then (3.3) and Lemma 2.6 show

$$(\tilde{\theta}^{ab} \tilde{\theta}_{ab}) / (\tilde{\theta}^2) < 1 - \eta$$

<sup>5</sup> And will also follow from Theorem 4.1.

where  $\eta \equiv 2\mu_0^2 M''^2 (\mu' w_0^2 H'^2)^{-1}$  if  $\mu \geq \mu_0$ , and  $\eta \equiv 2\mu_0^{3/2} M''^2 \cdot (\mu'^{1/2} w_0^2 H'^2)^{-1}$  if  $\mu \leq \mu_0$ . Using this and the relation  $(\tilde{\theta}_{ab} \tilde{\theta}^{ab})^{\frac{1}{2}} \geq |\tilde{\theta}_{ab} e^a e^b|$  which is valid for any unit vector  $e$  orthogonal to  $n$ , (2.10) shows

$$-S_- < s < 0 \Rightarrow |d \log(r \sinh \beta) / ds| \leq |\tilde{\theta}| (1 - \eta)^{\frac{1}{2}}.$$

Integrating through this inequality shows that

$$-S_- < s < 0 \Rightarrow r \sinh \beta < B \tilde{l}^{-3(1-\eta)^{\frac{1}{2}}}, \quad B(>0) \text{ constant.} \quad (3.4)$$

Finally (3.1) and (3.4) together show that, for  $-S_- < s < 0$ ,

$$w/r > (M'' \tanh \beta) / (B \tilde{l}^{\tilde{\epsilon}}), \quad \epsilon \equiv 3(1 - (1 - \eta)^{\frac{1}{2}}) > 0.$$

By Lemma 2.6, this would imply  $\mu \rightarrow \infty$  as  $s \rightarrow -S_-$  (when  $\tilde{l} \rightarrow 0$  and  $\beta \rightarrow \infty$ ); so the assumption that  $\mu$  is bounded above as  $s \rightarrow -S_-$  has led to a contradiction.  $\square$

When the conditions of Theorems 3.1 or 3.2 are satisfied in a universe which is not Type IX, we know that the density is unbounded within a finite proper distance along the normal geodesics and along the fluid flow lines. We also know that it is unbounded on every past inextendible non-spacelike curve, within a finite proper time, because the surfaces  $\mathcal{N}(s)$  are Cauchy surfaces; however we do not yet know if this happens within a finite affine distance along every past-directed non-spacelike geodesic.

**Lemma 3.3.** *When the conditions of Theorems 3.1 or 3.2 are satisfied in a universe which is not Type IX, the matter density is unbounded within a finite affine distance on every past-directed non-spacelike geodesic through  $\mathcal{S}$ .*

Let the geodesic  $x^a(v)$  have tangent vector  $k^a$ ; then the affine parameter  $v$  is related to  $s$  by

$$v = -\int (n_a k^a)^{-1} ds = -3 \int (\tilde{l} \tilde{\theta}(n_a k_a))^{-1} d\tilde{l}. \quad (3.5)$$

We write the geodesic tangent vector in the form  $k^a = (n_b k^b) \kappa^{-1} (-\kappa n^a + e^a)$  where  $e_a e^a = 1$ ,  $e^a n_a = 0$ ,  $\kappa > 0$  and  $\kappa^2 \geq 1$  as  $k^a$  is timelike or null. (As it is past-directed,  $n_b k^b > 0$ .) It then follows that

$$d((n_b k^b)^{-1}) / dv = -\tilde{\theta}_{ab} e^a e^b \kappa^{-2} < \frac{1}{3} \tilde{\theta}, \quad (3.6)$$

on using (2.11). Hence  $s < 0 \Rightarrow$

$$(n_b k^b)^{-1} < A \tilde{l}^{-1}, \quad A(>0) \text{ constant.} \quad (3.7)$$

With (3.5), this shows

$$-S_- < s < 0 \Rightarrow v < -3A \int_0^{\tilde{l}} (\tilde{l}^2 \tilde{\theta})^{-1} d\tilde{l}.$$

From the proof of Lemma 2.8, this is bounded as  $s \rightarrow -S_-$ , i.e. as  $\tilde{l} \rightarrow 0$ . Thus the geodesics, which pass through every surface  $\mathcal{N}(s)$ ,  $-S_- < s < 0$ , as these are Cauchy surfaces, do so in a finite affine parameter distance.  $\square$

#### 4. Intermediate Singularities

Theorem 2.4 shows that the Cauchy development of the spatially homogeneous initial data is limited. Theorems 3.1 and 3.2 show that in a wide variety of circumstances this is due to the occurrence of a matter singularity; in these cases, the causal past  $J^-(\mathcal{S})$  of the surface  $\mathcal{S}$  is precisely equal to its past Cauchy development  $D^-(\mathcal{S})$  (so  $T_- = \tau_-$ ). We now consider the cases where  $J^-(\mathcal{S}) - D^-(\mathcal{S}) \neq \emptyset$ ; these are the cases where the fluid crosses a Cauchy horizon  $H^-(\mathcal{S})$ , coming from a region of space-time which is not determined by the data on the surface  $\mathcal{S}$  (and in this case,  $T_- < \tau_-$ ).

An example of a universe in which this happens has been given by Shepley [14]. We shall extend this result to show there exist other such solutions.

**Theorem 4.1.** *There exist tilted homogeneous cosmologies in which  $J^-(\mathcal{S}) - D^-(\mathcal{S}) \neq \emptyset$  for every group type in Class B, and for both zero and non-zero vorticity. There exist no such universes in Class A.*

The past Cauchy horizon  $H^-(\mathcal{S})$  is defined to be the past boundary of  $D^-(\mathcal{S})$  in  $\mathcal{M}$ . First, we note a lemma which follows from (4) and the homogeneity of  $D(\mathcal{S})$ :

**Lemma 4.2.** *If  $J^-(\mathcal{S}) - D^-(\mathcal{S}) \neq \emptyset$ , the past Cauchy horizon  $H^-(\mathcal{S})$  is the homogeneous null surface  $\mathcal{F}(-T_-)$ , which is homeomorphic to  $\mathcal{S}$ .  $\square$*

Consider now the tilted homogeneous cosmologies in the ‘‘fluid basis’’ of [11]. If we can find a solution in which the tilt parameter  $\lambda \equiv r \tanh \beta$  is less than  $r$  in some region  $\mathcal{U} \subset \mathcal{M}$  and equal to  $r$  on the boundary  $\partial\mathcal{U}$  of  $\mathcal{U}$ , we have a solution of the desired type, as the surfaces of transitivity then make a transition from being spacelike to being null. As most of the field equations and Jacobi identities determine the propagation off a given homogeneous surface, all we have to do (cf. [11]) is find a set of initial data on a null surface  $\lambda = r$  such that the constraints ((A) – (D) of [11]) are satisfied, and the Jacobi identities  $n_{\alpha\beta} a^\beta = 0$  of the group are also satisfied (these are just the equations obtained on eliminating the derivatives between the  $(\begin{smallmatrix} 0 \\ 123 \end{smallmatrix})$ ,  $(\begin{smallmatrix} 0 \\ 012 \end{smallmatrix})$ ;  $(\begin{smallmatrix} 1 \\ 123 \end{smallmatrix})$ ,  $(\begin{smallmatrix} 1 \\ 012 \end{smallmatrix})$ ; and  $(\begin{smallmatrix} 2 \\ 123 \end{smallmatrix})$ ,  $(\begin{smallmatrix} 2 \\ 012 \end{smallmatrix})$  Jacobi identities of Appendix A in [11]). We can, for example, do so by setting all the  $\gamma^a{}_{bc}$  to zero except  $\theta_1 = \theta_2$ ,  $\theta_3$ ,  $\gamma^1{}_{23}$ ,  $\gamma^2{}_{13}$ , and  $\gamma^1{}_{13} = \gamma^2{}_{23}$ , ensuring that  $\gamma^1{}_{13} - \theta_1 \equiv k \neq 0$ . Then  $n_{\alpha\beta} a^\beta = 0$ , and the constraints (A) – (D) reduce to

$$\mu = -2k\theta_3 - 2k^2 - \frac{1}{2}(\gamma^1{}_{23} + \gamma^2{}_{13})^2 = 2\theta_1\theta_3 - 4k\theta_3 - 3k^2 - \frac{1}{4}(\gamma^1{}_{23} + \gamma^2{}_{13})^2$$

on the surface  $\lambda = r$ , in the case  $\Lambda = p = 0 = \omega$ ,  $r = 1$ . One can now easily see that these constraints can be satisfied for each group type in Class B. More complicated examples show that non-zero  $\Lambda$ ,  $p$  or  $\omega$  do not prevent the constraints being satisfied on the surface  $\lambda = r$ , in Class B universe. They cannot be satisfied in any Class A universe (in agreement with Theorem 3.2).  $\square$

The simplest such universes are tilted L.R.S. (Type V) models; the example given by Shepley [14] is a zero-pressure solution of this kind (whose local behaviour has been previously studied by Farnsworth [19] and Ellis [20]); it is the special case of Theorem 4.1 in which  $\gamma^1_{23} = \gamma^2_{13} = 0$ .

We now wish to examine the nature of the singularity in these models. We first prove two preliminary results.

**Lemma 4.3.** *If the density  $\mu$  is bounded in  $D^-(\mathcal{S})$  in a universe which is not Type IX, then  $\beta \neq 0$ ,  $a_a \neq 0$ ; and  $\tilde{\theta} \rightarrow \infty$ ,  $\beta \rightarrow \infty$ ,  $a^2 \equiv a_a a^d \rightarrow \infty$ , and  $a_a \tilde{c}^d$  is unbounded below on the surfaces  $\mathcal{N}(s)$  as  $s \rightarrow -S_-$ . Every past inextendible non-spacelike geodesic through  $\mathcal{S}$  passes through every surface  $\mathcal{N}(s)$ ,  $-S_- < s < 0$ , within a finite affine parameter distance.*

For  $-S_- < s < 0$ , there are constants  $A_i, B_i$  such that

$$A_1 \tilde{l}^3 < w < A_2 \tilde{l}^{-1}, \tag{4.1}$$

$$B_1 \tilde{l}^4 < \sinh \beta < B_2 \tilde{l}^{-11/2}. \tag{4.2}$$

If  $\mu \rightarrow 0$  as  $s \rightarrow -S_-$ , then there is a singularity as  $s \rightarrow -S_-$ , and  $J^-(\mathcal{S}) = D^-(\mathcal{S})$ .

The statements for  $\tilde{\theta}$  and  $a_a \tilde{c}^d$  follow from Theorems 2.4 and 3.2. The divergence of  $\beta$  if  $\mu$  is bounded follows from Lemmas 2.6 and 2.7. The equation for  $a$  ((2.16a) of [11]) and (2.11) show  $-\frac{1}{2}d(a^2)/ds = \tilde{\theta}_{ab} a^a a^b \leq \tilde{\theta} a^2$ , so  $a^2 \tilde{l}^6 > A$ ,  $A = \text{constant} > 0$ . The statement about the geodesics follows from the proof of Lemma 3.3.

To prove the second part, note that Eqs.(2.9)–(2.12) imply the inequalities (for  $\beta > 0$ )

$$\begin{aligned} \tilde{\theta}(\tanh^2 \beta + \frac{2}{3} \tanh \beta - 1) &> w^{-1} dw/ds (1 - \tanh^2 \beta dp/d\mu) \\ &> -\tilde{\theta}(\frac{1}{3} \tanh^2 \beta + \frac{2}{3} \beta + 1), \end{aligned} \tag{4.3}$$

$$\begin{aligned} \tilde{\theta}(\frac{1}{3} + dp/d\mu + \frac{2}{3} dp/d\mu \tanh \beta) &> (\tanh \beta)^{-1} d\beta/ds (1 - \tanh^2 \beta dp/d\mu) \\ &> -\tilde{\theta}(1 - dp/d\mu + \frac{2}{3} dp/d\mu \tanh \beta). \end{aligned} \tag{4.4}$$

Using (2.2) one can integrate through (4.3), (4.4) to obtain (4.1), (4.2). (In fact if  $dp/d\mu$  is constant, one can get much better bounds on  $w, \beta$  by integrating through (4.3), (4.4).) Finally, if  $\mu \rightarrow 0$  then  $w \rightarrow 0$ ; but the conservation equations show  $w = w_0 \exp[-\int_0^t \theta d\tau]$ , which can only tend to zero in the finite interval  $(-T_-, 0)$  if  $\theta$  is unboundedly large on

$\mathcal{N}(s)$  as  $s \rightarrow -S_-$ . Hence the fluid congruence is singular on any curve  $\gamma(s)$ ,  $-S_- < s < 0$ , in the limit  $s \rightarrow -S_-$ , and the space-time cannot be extended to  $\gamma(-S_-)$ .  $\square$

Lemma 4.3 applies to the homogeneous cosmologies with  $J^-(\mathcal{S}) - D^-(\mathcal{S}) \neq \emptyset$ , for then, given any point  $p \in \mathcal{S}$ ,  $\mu$  is bounded on the curve  $\Psi_\tau(p)$  for  $0 > \tau > -\tau_-$ , so  $\mu$  is bounded in  $D^-(\mathcal{S})$ . In fact in this case it is obvious that  $\beta \rightarrow \infty$  because this is precisely the statement that the spacelike surfaces of transitivity go null.

The last part of the statement shows that if  $J^-(\mathcal{S}) - D^-(\mathcal{S}) \neq \emptyset$ , then  $\mu$  (which must tend to *some* limit as  $s \rightarrow -S_-$ , as  $D^-(\mathcal{S})$  is extendible) tends to a *non-zero* limit as  $s \rightarrow -S_-$ .

When  $H^-(\mathcal{S})$  exists, its null generators are group orbits; they are geodesics which are complete in the future but incomplete in the past. In order to simplify this and the following proofs, we add to the Conditions (1)–(4) of § 2 the requirement: (5)  $\mathcal{S}$  is simply connected.

It then follows, as the group is not Type IX, that  $\mathcal{S}$  is homeomorphic to  $R^3$  (Schmidt, [21]). This requirement is not essential, as if it is not true in  $(\mathcal{M}, \mathbf{g})$  there is a covering space  $(\hat{\mathcal{M}}, \hat{\mathbf{g}})$  of  $(\mathcal{M}, \mathbf{g})$  in which it is true; and the properties of  $(\mathcal{M}, \mathbf{g})$  can then be found from those of  $(\hat{\mathcal{M}}, \hat{\mathbf{g}})$  by use of the canonical projection.

**Lemma 4.4.** *If  $J^-(\mathcal{S}) - D^-(\mathcal{S}) \neq \emptyset$ , the null geodesic generator  $\lambda(u)$  of  $H^-(\mathcal{S})$  through any point  $q \in H^-(\mathcal{S})$  is the orbit of a canonically parameterized group of isometries  $\{H_\lambda(v)\}$ ,  $-\infty < v < \infty$ . The range of  $u$  is  $+\infty > u > -u_-$ , where  $u_-$  is finite; the divergence  $\hat{\theta} \equiv k^a{}_{;a}$  of the (future directed) null tangent vectors  $\mathbf{k} = \partial/\partial u$ , or its derivative  $d\hat{\theta}/du$ , diverges to  $\infty$  as  $u \rightarrow -u_-$ . There is a spacelike 2-surface  $\mathcal{Q}$  (homeomorphic to  $R^2$ ) such that the mapping  $H : (-\infty, \infty) \times \mathcal{Q} \rightarrow H^-(\mathcal{S})$  which maps  $(v, \mathcal{Q})$  to  $\mathcal{Q}_v = \{H_\lambda(v)q, q \in \mathcal{Q}\}$  is a diffeomorphism.*

By Lemma 4.2 there is a unique null geodesic segment  $\lambda(u)$  through each point  $q \in H^-(\mathcal{S})$ , which is complete in the future direction. As  $H^-(\mathcal{S})$  is a homogeneous surface, there is some Killing vector field  $\xi$  in  $\overline{D^-(\mathcal{S})}$  such that  $\xi|_q$  is parallel to the null geodesic tangent vector  $\partial/\partial u|_q$ . Let the canonically parameterized group of isometries generated by  $\xi$  be  $H_q(v)$ ; this is a complete group because of the homogeneity of  $H^-(\mathcal{S})$ , which follows from that of  $\mathcal{S}$ . Because  $\xi$  is a Killing vector, its magnitude  $\xi^2 \equiv \xi_a \xi^a$  satisfies  $d(\xi^2)/dv = 0$ ; so the group orbit  $\mathcal{H}_q = \{H_q(v)q, -\infty < v < \infty\}$ , is a null curve. However this orbit lies in  $H^-(\mathcal{S})$ , and the only null curve through  $q$  lying in  $H^-(\mathcal{S})$  is the null geodesic  $\lambda(u)$ ; so this null geodesic is the orbit of  $q$  under the group of isometries  $H_q$ , (which we can write as  $\mathcal{H}_\lambda(v)$ ), and much of the elegant Boyer-Ehlers analysis [22] applies. The Killing vector  $\xi$  is spacelike in the surfaces  $\mathcal{F}(\tau)$ ,  $0 > \tau > -T_-$ , as these are spacelike surfaces, and



is null on  $H^-(\mathcal{S})$ . On  $\lambda(u)$  the Killing vector  $\xi = \partial/\partial v$  and geodesic tangent vector  $k = \partial/\partial u$  are parallel vectors (we take them both future directed) obeying the relations  $k^a_{;b}k^b = 0$ ,  $\xi^a_{;b}\xi^b = +c\xi^a$ ; and just as in Lemma 2 of [22],  $c$  is a (real) constant on  $\lambda(u)$  (which, if non-zero, can be chosen to be positive).

Two cases arise: (i)  $c \neq 0 \Leftrightarrow \xi^2_{,a|q} \neq 0$ . In this case the parameters  $u, v$  and the vectors  $\xi, k$  are related by

$$k = \exp(-cv) \xi, \quad u = -u_- + c^{-1} \exp(+cv); \quad u_-, c(>0) \text{ constant} \tag{4.5}$$

on  $\lambda(u)$ . Clearly  $u$  can be extended beyond every value greater than  $-u_-$ ; it cannot be extended to the value  $-u_-$ , for then the points  $\lambda(-u_-)$  would be fixed points of the group in  $H^-(\mathcal{S})$ , and the flow lines of the fluid passing through these points would be a preferred subset (at most 2-dimensional) of  $H^-(\mathcal{S})$ , contradicting the condition that  $H^-(\mathcal{S})$  be homogeneous. Thus in this case the geodesic generators of  $H^-(\mathcal{S})$  are incomplete. (ii)  $c = 0 \Leftrightarrow \xi^2_{,a|q} = 0$ . In this case the geodesic generators of  $H^-(\mathcal{S})$  are complete (see [22]). We shall now see that only the first case is possible.

Because of Condition (5) and Lemma 2.1,  $H^-(\mathcal{S})$  is diffeomorphic to  $R^3$ ; as there is a future-oriented null geodesic generator of  $H^-(\mathcal{S})$  (complete in the future, but possibly incomplete in the past) through each point of  $H^-(\mathcal{S})$ , one can find a smooth spacelike 2-surface  $\mathcal{Q}$  which intersects each geodesic generator just once. Defining  $\mathcal{Q}_v$  by mapping  $\mathcal{Q}$  a group parameter distance  $v$  along each geodesic generator, it is a spacelike surface in  $H^-(\mathcal{S})$ , as it is not tangent to  $\lambda(u)$ . This map is regular as long as  $k^{a;b}k_{a;b}$  is bounded; but this is true as long as  $d\hat{\theta}/du$  is bounded, because  $\hat{\theta}$  obeys the equation

$$d\hat{\theta}/du = k^{b;a}k_{a;b} + R_{ab}k^ak^b < \frac{1}{2}\hat{\theta}^2 \tag{4.6}$$

along  $\lambda(u)$  [which is just the null analogue of (2.5), (2.7), the inequality following because of (2.1) and (2.2)]. Hence  $u$  can be extended beyond every value  $u'$  for which  $d\hat{\theta}/du$  is finite. On the other hand (4.6) shows that  $\hat{\theta} \rightarrow \infty$  (and  $d\hat{\theta}/du \rightarrow \infty$ ) for some finite value  $u''$  of  $u$  if  $\lambda(u)$  can be extended that far; however it cannot be extended to the value  $u''$ , for if it could the points at which  $\hat{\theta}$  was infinite would form a preferred subset (at most two-dimensional)<sup>6</sup> of  $H^-(\mathcal{S})$ , again contradicting the homogeneity of  $H^-(\mathcal{S})$ . Thus there is a finite value  $u_-$  such that  $\frac{d\hat{\theta}}{du} \rightarrow \infty$  precisely as  $u \rightarrow -u_-$ ; the geodesic generators of  $H^-(\mathcal{S})$  are incomplete; and (4.5) holds.

<sup>6</sup> If an affine transformation of  $u$  is made, the points of  $H^-(\mathcal{S})$  on which  $\hat{\theta} \rightarrow \infty$  remain invariant (see [9]).

Using any remaining freedom in the normalization (on  $\mathcal{Q}$ ) of each Killing vector  $\xi|_q$  parallel to the null generator of  $H^-(\mathcal{S})$  at  $q \in \mathcal{Q}$  to ensure that  $\{\xi|_q, q \in \mathcal{Q}\}$  is a  $C^1$  vector field on  $\mathcal{Q}$ , the map  $H$  is a diffeomorphism.  $\square$

Note that while  $H^-(\mathcal{S})$  is a local isometry horizon, according to Carter's definition [23], it is not a Killing horizon (see remarks below); in fact the Killing vector field will in general be null, in  $H^-(\mathcal{S})$ , only on the null geodesic  $\lambda(u)$  through  $q$ , and will be spacelike elsewhere in  $H^-(\mathcal{S})$ . The essential difference between our situation and that studied by Boyer-Ehlers is that in their case the fixed point of the group is in the horizon  $H^-(\mathcal{S})$ , and so the horizon bifurcates there; in our case the fixed point is excluded from the space-time, and so the horizon  $H^-(\mathcal{S})$  is a single smooth null surface which is incomplete in the past.

The fluid is invariant under the group  $H_\lambda(v)$ , so an infinite number of flow lines intersect the geodesic  $\lambda(u)$  to the past of  $\mathcal{Q}$  (for this corresponds to  $-\infty < v < 0$ ). However this happens in the finite affine distance  $-u_- < u < 0$ , so in some sense an infinite amount of matter passes through a finite region of space-time. The matter itself encounters only bounded densities and curvatures on each world line near  $H^-(\mathcal{S})$ ; so it seems likely that the singularities which occur are intermediate singularities, in the classification introduced in Section 1. This expectation is fulfilled, as we now see.

**Theorem 4.5.** *If  $J^-(\mathcal{S}) - D^-(\mathcal{S}) \neq \emptyset$ , each past directed null geodesic generator of  $H^-(\mathcal{S})$  ends at an intermediate singularity. Further each past directed geodesic orthogonal to  $\mathcal{S}$  remains in  $D^-(\mathcal{S})$  and has a past endpoint at an intermediate singularity; so  $s_- = S_-$ .*

Given any curve  $\lambda(r)$  with tangent vector  $\mathbf{k} = \partial/\partial r$ , (2.1) shows

$$R_{ab}k^ak^b = (\mu + p)(u_ak^a)^2 + \frac{1}{2}(\mu - p)(k_ak^a). \quad (4.7)$$

First, let  $\lambda(r)$  be a future-directed null geodesic generator  $\lambda(u)$  of  $H^-(\mathcal{S})$ . Then (4.5) relates  $\mathbf{k}$  to the future-directed Killing vector  $\xi$  which is parallel to  $\mathbf{k}$  on  $\lambda(u)$ ; so

$$R_{ab}k^ak^b = (\mu + p) \exp(-2cv) (\xi_au^a)^2 \quad \text{on} \quad \lambda(u). \quad (4.8)$$

Now  $(\mu + p)$  is constant and positive on  $\lambda(u)$ , as this is an isometry orbit and (2.2) holds on  $\mathcal{M}$ ; and  $(\xi_au^a)$  is constant on  $\lambda(u)$ , because  $\xi$  is a Killing vector, and so must commute with  $\mathbf{u}$ . Thus the first and third factors on the right hand side of (4.8) are constant and positive on  $\lambda(u)$ ; hence as  $v \rightarrow -\infty$ , i.e. as  $u \rightarrow -u_-$ ,  $R_{ab}k^ak^b \rightarrow \infty$ . As  $R_{ab}k^ak^b$  is just a Ricci tensor component in a parallelly transported null tetrad along  $\mathbf{u}$  in which  $\mathbf{k}$  is one of the null vectors, this suggests that a Ricci tensor component must diverge in a parallelly transported orthonormal tetrad along  $\lambda(u)$ . To prove this

rigorously, let  $(\mathbf{k}, \mathbf{l}, \mathbf{n}, \bar{\mathbf{n}})$  be a parallelly transported null tetrad along  $\lambda(u)$ , with  $\mathbf{k} \cdot \mathbf{l} = -1$ ,  $\mathbf{n} \cdot \bar{\mathbf{n}} = 1$  and the other scalar products zero, and let  $(\xi, \eta, \mathbf{v}, \bar{\mathbf{v}})$  be a null tetrad dragged along  $\lambda(u)$  by  $\xi$ , with  $\xi \cdot \eta = -1$ ,  $\mathbf{v} \cdot \bar{\mathbf{v}} = 1$  and the other scalar products zero. As  $\mathbf{k}$  and  $\xi$  are parallel null vectors on  $\lambda(u)$  obeying (4.5), the most general transformation relating these bases on  $\lambda(u)$  can be written

$$\mathbf{k} = \exp(-cv) \xi, \quad \mathbf{l} = \exp(cv) (\eta + B\mathbf{v} + \bar{B}\bar{\mathbf{v}} + B\bar{B}\xi), \quad \mathbf{n} = e^{i\theta} (\mathbf{v} + B\xi)$$

where  $\theta$  is a real function of  $v$  and  $B$  a complex function of  $v$ . The components of  $R_{ab}$  in the basis  $(\xi, \eta, \mathbf{v}, \bar{\mathbf{v}})$  are constant along  $\lambda(u)$  because the Lie derivatives with respect to  $\xi$  of  $R_{ab}$  and of the basis both vanish on  $\lambda(u)$ ; so if  $|B|$  does not diverge as  $v \rightarrow -\infty$ , the only divergent component of  $R_{ab}$  in the basis  $(\mathbf{k}, \mathbf{l}, \mathbf{n}, \bar{\mathbf{n}})$  is  $R_{ab} k^a k^b$ , and hence in the canonically associated parallelly transported orthonormal basis the Ricci component  $\frac{1}{2} R_{ab} (k^a + l^a) (k^b + l^b)$  diverges. On the other hand if  $|B|$  diverges, at least one of the components  $\frac{1}{2} R_{ab} (n^a + \bar{n}^a) (n^b + \bar{n}^b)$ ,  $\frac{1}{2} R_{ab} (n^a + \bar{n}^a) (n^b - \bar{n}^b)$  is unbounded as  $v \rightarrow -\infty$ . Hence in any parallelly transported orthonormal basis along  $\lambda(u)$ , there is always some Ricci tensor component which does not go to a limit as  $u \rightarrow -u_-$ . On the other hand this is not a curvature singularity, as all Ricci (and Weyl) components are constant along  $\lambda(u)$  in the orthonormal frame which is canonically associated with the frame  $(\xi, \eta, \mathbf{v}, \bar{\mathbf{v}})$ .

Second, let  $\lambda(r)$  be a past directed normal geodesic  $\gamma(s)$ . Then  $k^a = -n^a$ ; by (2.3), Eq. (4.7) becomes

$$R_{ab} n^a n^b = \cosh^2 \beta (\mu + p) - \frac{1}{2} (\mu - p).$$

As  $\mu$  and  $p$  are bounded in  $D^-(\mathcal{S})$  but  $\beta \rightarrow \infty$  as  $s \rightarrow -S_-$ , this shows that  $R_{ab} n^a n^b \rightarrow \infty$  as  $s \rightarrow -S_-$ <sup>7</sup>. In this case,  $n^a$  is a unit timelike vector parallelly propagated along the geodesic  $\gamma(s)$ ; so adding three unit, orthogonal spacelike vectors which are parallelly propagated along  $\gamma(s)$ , we have a parallelly propagated orthonormal tetrad along  $\gamma(s)$  in which one Ricci tensor component diverges as  $s \rightarrow -S_-$ . This is not a curvature singularity, because on some fluid flow line we can choose an orthonormal tetrad in which the timelike vector is the fluid flow vector  $u^a$  and in which [as it crosses  $H^-(\mathcal{S})$ ] all the Ricci and Weyl components are well behaved as  $s \rightarrow -S_-$ . Spreading this tetrad over the surfaces  $\mathcal{N}(s)$ ,  $-S_- < s < 0$ , by the simply transitive group of isometries, it provides an orthonormal tetrad at each point of  $D^-(\mathcal{S})$  in which the curvature tensor components are well behaved; so in particular, we have an orthonormal tetrad along  $\gamma(s)$  in which the curvature tensor components are well-behaved.  $\square$

<sup>7</sup>  $\mu + p$  cannot go to zero as  $s \rightarrow -S_-$ , because it takes the same values as on the fluid lines which cross  $H^-(\mathcal{S})$ ; and  $\mu + p > 0$  on  $H^-(\mathcal{S})$ .

It is instructive to consider explicitly what parallel propagation of vectors along  $\lambda(u)$  looks like in the fluid basis of [11], with  $\beta$  replaced by the parameter  $\lambda \equiv r \tanh \beta$ . To do this consider a surface on which  $\lambda = r$ , which is the condition for a null surface of transitivity. The condition  $\xi^2_{,a} \neq 0$  is just the condition  $(\lambda/r)_{,0} \neq 0$ , so (by the argument just given) we must have

$$\xi^2_{,a}|_{H^+(\mathcal{S})} \neq 0 \Leftrightarrow \theta_3 + r_{,0} \neq 0 \quad \text{on } H^+(\mathcal{S}). \quad (4.9)$$

In fact a check shows that (4.9) must be true because of constraint equation (C) of [11]; for this shows that if  $\theta_3 + r_{,0} = 0$  when  $\lambda = r$ , the condition  $\mu + p > 0$  cannot be satisfied. Now the vector  $\mathbf{C} \equiv \mathbf{u} - \mathbf{c}$  is a null vector in  $H^-(\mathcal{S})$  and so is parallel to the null geodesic generators of  $H^-(\mathcal{S})$ ; because it is invariant [on  $\lambda(u)$ ] under the group of isometries generated by  $\xi$ , it must differ from  $\xi$  (which satisfies the same condition) by at most a constant multiple on  $\lambda(u)$ . Thus we can normalize  $\xi$  so that on  $\lambda(u)$ ,  $\mathbf{C} = \partial/\partial v$  where  $v$  is the group parameter. Now parallel propagation of a vector  $\mathbf{X}$  along the integral curves of  $\mathbf{C}$  is given by  $dX^b/dv = +(\Gamma^b_{3f} - \Gamma^b_{0f})X^f$ , where the components are taken with respect to the fluid basis. The quantities  $\Gamma^a_{bc}$  are constant on  $\lambda(u)$ , so unless all the combinations  $(\Gamma^b_{3f} - \Gamma^b_{0f})$  are zero, a Lorentz transformation with unbounded parameter values will relate a parallelly propagated basis to a fluid basis, as  $v \rightarrow -\infty$ ; and (4.9) shows they are not all zero. Thus an intermediate singularity will occur as  $v \rightarrow -\infty$  if this corresponds to a finite value of the affine parameter  $u$ . However an explicit check shows that the vector  $\mathbf{K} = \exp(-(r_{,0} + \theta_3)v)\mathbf{C}$  obeys the above equation for  $\mathbf{X}$ , and so is parallelly propagated along  $\lambda(u)$ ; this shows that the constant  $c$  in (4.5) is just  $+(r_{,0} + \theta_3)$ , which is non-zero, and so  $v \rightarrow \infty$  corresponds to a finite value of  $u$ .

One can also check whether  $\mathbf{C}$  is parallel to a Killing vector field on  $H^-(\mathcal{S})$  [and not just on  $\lambda(u)$ ]. In fact if  $\xi^a$  were a Killing vector field on  $H^-(\mathcal{S})$ , then  $\xi^a$  would be a null geodesic congruence generating  $H^-(\mathcal{S})$  and Eq. (4.6) would show  $R_{ab}\xi^a\xi^b = 0$  on  $H^-(\mathcal{S})$ ; but this implies  $(\mu + p)(u_a\xi^a)^2 = 0$ , contradicting  $\mu + p \neq 0$  on  $H^-(\mathcal{S})$ . Hence  $H^-(\mathcal{S})$  is not a Killing Horizon.

While we now know what happens on the normal geodesics to  $\mathcal{S}$ , and on the generators of  $H^-(\mathcal{S})$ , we do not yet have information about what happens on a general non-spacelike geodesic through  $\mathcal{S}$ . The geometry of the spaces we are considering is complicated by the fact that  $\mathbf{u}$  and  $\mathbf{n}$  do not, in general, span 2-surfaces in space-time.

By (4.7), (2.3), and Lemma 4.3, as  $s \rightarrow -S_-$

$$R_{ab}k^ak^b \rightarrow \frac{1}{4}(\mu + p) \exp(2\beta) ((n_a + \tilde{c}_a)k^a)^2 + \frac{1}{2}(\mu - p)k^ak_a$$

on any past-directed non-spacelike curve  $\lambda(r)$  with tangent vector  $\mathbf{k}$ . Here  $r$  must be a monotonic function of  $-s$ ;  $(n_a + \tilde{c}_a)k^a \geq 0$  and  $\mu, p$  go to

finite limits  $\mu_0, p_0$  such that  $\mu_0 + p_0 > 0$ , so

$$\lim_{s \rightarrow -S_-} (n_a + \tilde{c}_a) k^a \neq 0 \Rightarrow R_{ab} k^a k^b \rightarrow \infty \quad \text{as } s \rightarrow -S_-;$$

and if  $\lim_{s \rightarrow -S_-} (n_a + \tilde{c}_a) k^a$  does not exist,  $R_{ab} k^a k^b$  is unbounded as  $s \rightarrow -S_-$ .

Now suppose that  $\lim_{s \rightarrow -S_-} (n_a + \tilde{c}_a) k^a = 0$ . We write out the components in a normalized tilted basis, i.e.

$$\mathbf{k} = -k^0 \mathbf{u} + k^1 \tilde{\mathbf{c}} + k^2 \mathbf{e}_2 + k^3 \mathbf{e}_3$$

where  $(\tilde{\mathbf{c}}, \mathbf{e}_2, \mathbf{e}_3)$  are an orthonormal triad in the surfaces  $\mathcal{N}(s)$ . By (2.3), the condition  $k_a k^a \leq 0$  shows that  $-k^0 e^\beta \leq k^1 \leq k^0 e^{-\beta}$ ; and the limit condition shows that, given any positive number  $\varepsilon$ ,  $|k^1 - e^{-\beta} k^0| < \varepsilon$  for  $s$  sufficiently close to  $-S_-$ . Hence there is an  $s'' > 0$  such that  $s < -s''$  implies  $e^{-\beta} - \varepsilon/k^0 < k^1/k^0 \leq e^{-\beta}$ . Now a detailed geometric argument shows that any non-spacelike geodesic satisfying this condition crosses  $H^-(\mathcal{S})$ . Thus any past-directed non-spacelike geodesic in  $D^-(\mathcal{S})$  either crosses  $H^-(\mathcal{S})$  or runs into an intermediate singularity. The same conclusion can be shown to be true for any timelike curve of bounded acceleration.

Finally we remark that if Condition (5) had not been put on, our conclusion would have been qualitatively similar, except that the singularities occurring would have been intermediate singularities which in addition could have had some of the nature of a cone singularity; that is, there would have been too few directions, in the limit  $s \rightarrow -S_-$ , for one to be able to construct a regular tangent space for any of the singular points (cf. [4], p. 274, where certain identifications result in a cone-type singularity in Minkowski space-time, and [16]).

## 5. Continuation across the Horizon

When  $J^-(\mathcal{S}) - D^-(\mathcal{S}) \neq \emptyset$ , one cannot in general say much about this region of space-time from the data available in  $D(\mathcal{S})$ ; for new information is available in this region, and affects the development of the fluid across the horizon  $H^-(\mathcal{S})$ . In particular cases one can deduce something about the further behaviour; for example, in the Farnsworth (Type V L.R.S.) solution [14, 19] a matter singularity will probably eventually occur, because the matter moves without acceleration or rotation; so Raychaudhuri's equation applied to the fluid congruence shows that a matter singularity must eventually occur on each fluid flow line, if they can be extended far enough. However we cannot deduce this result in more complex models with non-zero pressure or rotation.

To deduce the behaviour beyond the horizon we have to make some further assumption, which in effect determines what new information has come into this region. We can only hope for some sort of unique prediction by assuming that no new information has come in; and there seem to be effectively two ways of doing this.

The first way is to assume that the symmetry properties of the space-time in  $D(\mathcal{S})$  are continued across the horizon. More precisely, we can replace assumption (4) by assumption (4'):

(4') A continuous group  $G$  acts as a  $C^2$  group of isometries of  $(\mathcal{M}, \mathbf{g})$ , simply transitive on 3-dimensional surfaces of transitivity; and at least one of these surfaces is spacelike<sup>8</sup>.

The second possibility is to assume that the solution in  $J^-(\mathcal{S}) - D^-(\mathcal{S})$  is obtained by analytic continuation of the solution in  $D(\mathcal{S})$ , which is analytic if the equation of state is analytic. More precisely, we can replace assumption (1) by (1'):

(1')  $(\mathcal{M}, \mathbf{g})$  is an analytic space-time which is inextendible and locally inextendible<sup>9</sup>.

It can be shown that when  $p(\mu)$  is analytic, these two ways of extending beyond the horizon are equivalent, providing one does not run into a locally extendible singularity. As the symmetry extension [i.e. assuming (4')] is less restrictive (for one cannot obtain an analytic extension, in general, if  $p(\mu)$  is not analytic) and unique even if there are locally extendible singularities (we cannot guarantee that analytic extension will give a unique answer then), we shall proceed by using (4') to determine the solution beyond the horizon; in general, our results will also hold for the obvious analytic extension.

Thus we proceed by assuming (1), (2), (3), (4'), and (5). Now (4.7) and (4.9) show

**Lemma 5.1.** *The map  $\Psi : \mathcal{S} \times (0, -\tau) \rightarrow J^-(\mathcal{S})$  defined by  $\Psi(p, \tau) = \Psi_\tau(p)$  if  $p \in \mathcal{S}$  and  $\tau \in (0, -\tau_-)$  is a diffeomorphism of  $\mathcal{S} \times (0, -\tau_-)$  onto  $J^-(\mathcal{S})$ . The group  $G$  has the surfaces  $\mathcal{F}(\tau) \equiv \Psi_\tau(\mathcal{S})$ ,  $0 > \tau > -\tau_-$ , as surfaces of transitivity. If  $\tau_- > T_-$ , then for values of  $\tau$  just less than  $-\tau_-$  these homogeneous surfaces are timelike; and each surface  $\mathcal{F}(\tau)$  which is null divides regions of  $J^-(\mathcal{S})$  where they are spacelike and timelike, and is a Cauchy horizon for the spacelike surfaces.  $\square$*

After proving two preliminary results, we shall show there can be at most two such null surfaces  $\mathcal{F}(T_i)$ . We first return to consider the Cauchy development  $D(\mathcal{S})$  when Conditions (1)–(5) hold. In § 2, we

<sup>8</sup> One could use the slightly weaker concept of a 3-transitive group [24] here but would obtain the same set of solutions.  $\mathcal{S}$  can be chosen to be one of the spacelike surfaces of transitivity; then (1)–(4) hold in  $D(\mathcal{S})$ .

<sup>9</sup> If the solution runs into a locally extendible singularity, one cannot make this stipulation; but without it, analytic extension is not unique.

considered this development to the past of  $\mathcal{S}$ , i.e. for negative values of  $s$  and  $\tau$ . Now we consider the region of  $D(\mathcal{S})$  to the future of  $\mathcal{S}$ .

**Lemma 5.2.** *Let (1)–(5) hold with  $G$  not a group of Bianchi Type IX. Then  $S_+ = s_+$  is infinite. As  $s \rightarrow \infty$ ,  $\tilde{\theta} \rightarrow 0$ ,  $\tilde{l} \rightarrow \infty$ ;  $\tilde{\sigma}^{ab}\tilde{\sigma}_{ab} \rightarrow 0$ ,  ${}^3R \rightarrow 0$ ;  $\mu \rightarrow 0$ ,  $p \rightarrow 0$ ,  $l \rightarrow \infty$ . Every future inextendible timelike and null geodesic through  $\mathcal{S}$  is complete.*

The argument leading to (4.2) shows that for  $s > 0$ ,  $B'\tilde{l}^4 > \sinh \beta$ ,  $B' > 0$ ; hence  $\beta \rightarrow \infty \Rightarrow \tilde{l} \rightarrow \infty$ , which by Lemma 2.5 can only happen when  $\tilde{\theta} \rightarrow 0$  and  $s \rightarrow \infty$ . Thus an argument similar to that of Lemma 2.3 shows that the solution can be extended for infinite values of  $s$  in  $D^+(\mathcal{S})$ . Now Eq. (2.4) shows  $\tilde{\sigma}^{ab}\tilde{\sigma}_{ab} \rightarrow 0$ ,  ${}^3R \rightarrow 0$ ,  $\mu \rightarrow 0$ ; by (2.2),  $p \rightarrow 0$  also. Lemma 2.7 shows  $w \rightarrow 0$ , and so (cf. [11], p. 213)  $l \rightarrow \infty$ .

The statement for the geodesics follows much as in Lemma 3.3. We have  $\mathbf{k}$  as the future directed tangent vector; Lemma 2.5 shows  $(-k^a n_a)^{-1} > C\tilde{l}^{-1}$ , where  $C$  is some positive constant, and then  $v > 3C \int (\tilde{l}^2 \tilde{\theta})^{-1} d\tilde{l}$ , which shows  $v \rightarrow \infty$  as  $\tilde{l} \rightarrow \infty$ , i.e. the geodesics are complete in the future direction.  $\square$

One might hope that the following conditions would hold:

$$\beta \rightarrow 0 \quad \text{as} \quad s \rightarrow \infty, \quad \text{and} \quad \tau_+ = T_+ \quad \text{is infinite.} \quad (5.1)$$

However in general we are unable to state this; in fact B. Collins has found some Type V L.R.S. solutions in which  $\beta \rightarrow \infty$  as  $s \rightarrow \infty$ , and in which  $\tau_+ = T_+$  is finite. When this occurs, the fluid runs into a conformal singularity (cf. the proof of Lemma 4.3). However we expect this disagreeable behaviour to be the exception rather than the rule. We shall refer to a region  $J^+(\mathcal{S})$  for which (5.1) holds as a *regular future infinity*. Lemma 5.2 assures us that in the case  $p = 0$ , we can at least say that  $\tau_+$  is infinite.

Next we consider when we can continue the solution along the fluid flow lines. The assumption (4') of a global group of isometries in effect changes the equations from a hyperbolic system of partial differential equations, to a system of ordinary differential equations determining the development along the fluid flow lines. Their solution exists and is unique as long as the fluid flow is non-singular. More precisely,

**Lemma 5.3.** *Let  $\tau' \leq \tau_-$ . If the fluid shear and expansion go to well-defined finite limits as  $\tau \rightarrow -\tau'$ , then the solution can be extended beyond  $-\tau'$ , i.e.  $\tau' < \tau_-$ .*

The essential point is that as long as the fluid shear and expansion are regular, the surface  $\mathcal{F}(\tau)$  is regular and the solution can be extended beyond it. One way to see this is to use Taub's coordinates ([25]; see also [26]), obtained by choosing usual group coordinates in the surface  $\mathcal{S}$  and then dragging them along by the fluid to all the surfaces  $\mathcal{F}(\tau)$ ,

$-\tau_- < \tau < \tau_+$ . This provides a global set of comoving coordinates for  $(\mathcal{M}, \mathbf{g})$ , in which the metric takes the form

$$ds^2 = h_{\alpha\beta}(x^d) dx^\alpha dx^\beta - r(x^c)^{-2} (dx^0 + V_\alpha(x^\gamma) dx^\alpha)^2 \quad (5.2)$$

where  $V_\alpha$  are the coordinate components of the vector  $\mathbf{v} = \lambda c$  ( $V_0 = 0$ , as  $c_a u^a = 0$ ) and  $h_{\alpha\beta}$  are the coordinate components of  $h_{ab}$  (again,  $h_{a0} = 0$ ); Greek letters run from 1 to 3. Because of Lemma 2.6, if  $r \rightarrow \infty$ , a matter singularity occurs. In fact the conservation equations along the fluid flow lines show that the matter variables will be well-behaved as long as  $\theta$  is. Apart from the fluid index  $r$ , the variation along the flow lines is in the functions  $h_{\alpha\beta}(V_\alpha$  are constants along the fluid flow lines because of the conservation equations); and  $\theta_{a0} = 0$ ,  $\theta_{\alpha\beta} = \frac{1}{2} r h_{\alpha\beta,0}$ ,  $\theta = \frac{1}{2} r g^{\alpha\beta} h_{\alpha\beta,0}$ .

Thus as long as the coordinate components of  $\sigma_{\alpha\beta}$ ,  $\theta$  are bounded and have regular limits, which will be true when the same is true for the components of  $\theta_{ab}$  in a parallelly propagated orthonormal frame along the fluid flow lines, the metric is regular; the field equations (which involve essentially only  $h_{\alpha\beta}$ ,  $\theta_{\alpha\beta}$  and matter variables) are regular, and their solution is regular beyond  $-\tau'$ . (One could alternatively use a tetrad basis, such as the fluid basis of [11], and consider the problem in this basis.) Thus the solution can be extended beyond  $-\tau'$  unless either the shear or expansion of the fluid [measured in coordinates (5.2), or in a parallelly propagated orthonormal tetrad along the fluid] does not go to a bounded limit as  $\tau \rightarrow -\tau'$ .  $\square$

We can now characterize the only possible solutions in which the fluid crosses a horizon into a region which has the same symmetry group as  $D(\mathcal{S})$ .

**Theorem 5.4.** *If (1), (2), (3), (4'), (5) hold with  $J^-(\mathcal{S}) - D^-(\mathcal{S}) \neq \emptyset$ , then  $D^+(\mathcal{S})$  obeys the conditions of Lemma 5.2. The spacetime  $(\mathcal{M}, \mathbf{g})$  consists of the spatially homogeneous region  $\mathcal{D}_1 = D(\mathcal{S})$ , plus a stationary, spatially inhomogeneous region  $\mathcal{D}_2$ , which either (a) terminates at a finite value  $-\tau_-$  where the fluid runs into a singularity where either  $\theta$  or  $\sigma_{ab}$  does not go to a bounded limit; or (b) contains fluid flow lines which all extend to infinite proper times ( $\tau_-$  is infinite); or (c) can be continued across a horizon to a spatially homogeneous region  $\mathcal{D}_3$ , where  $\mathcal{D}_3$  again obeys (in a time reversed sense) the conditions of Lemma 5.2.*

This follows from the information we have available; having crossed the horizon, the fluid either ends at a singularity in  $\mathcal{D}_2$  (where either the fluid shear or expansion does not have a good limit, by Lemma 5.3); or can be continued indefinitely in  $\mathcal{D}_2$ ; or else crosses a horizon into  $\mathcal{D}_3$ , where now all the conditions of  $\mathcal{D}_1$  hold but in a time-reversed sense.  $\mathcal{D}_1$  (geodesically complete to the future) and  $\mathcal{D}_3$  (geodesically complete to the past) cannot be extended further (if  $\tau_+$  or  $\tau_-$  is finite, then by the proof of Lemma 4.3, at least a conformal singularity occurs there).  $\square$



Note that every past-directed timelike curve of bounded acceleration, or null geodesic, in  $\mathcal{D}_1$  either runs into an intermediate singularity, or crosses the horizon. In Case (a) such curves in  $\mathcal{D}_2$  either run into the intermediate singularity, or into the singularity which ends the fluid's history; Case (b), they either run into an intermediate singularity, or continue for ever in  $\mathcal{D}_2$ ; and in Case (c), they can cross the horizon into  $\mathcal{D}_3$  instead of running into the intermediate singularity. The timelike surfaces of transitivity in  $\mathcal{D}_2$  have many properties similar to those of the spacelike surfaces in  $\mathcal{D}_1$  and  $\mathcal{D}_3$ ; for the geodesics normal to one of these surfaces are normal to all of them, and just as in § 2 these geodesics can have no conjugate points in  $\mathcal{D}_2$ , but their divergence is unbounded as the horizons are approached. It is probable that there will be causal violations in  $\mathcal{D}_2$  in Cases (b) and (c); since otherwise the existence of closed trapped surfaces in  $\mathcal{D}_2$ , which implies that the future null geodesic generators of  $J^+(p)$  (for any point  $p$ ) form a compact set, makes it unlikely that any null geodesics could reach infinity. (Hawking, private communication). In Cases (b) and (c) there is no curvature singularity, i.e. no "big bang", in  $(\mathcal{M}, g)$ ; rather there are one or two intermediate singularities which we shall colloquially refer to as "whimpers". In Case (a) ("whimper-bang") we have both an initial singularity and an intermediate singularity (but note that the singularity where the fluid begins might not be a matter singularity, cf. the next section). One can more readily understand the nature of these solutions by examining their structure as shown in Fig. 2; these are similar to the usual Penrose diagrams (cf. [4]) indicating conformal structure, but here each point represents a 2-surface diffeomorphic to  $R^2$  rather than a 2-sphere. They represent diagrammatically the splitting of space-time into blocks  $\mathcal{D}_i$ , with infinity brought to finite values by suitable transformations<sup>10</sup>; because of the group of isometries, the structure is the same through  $(\mathcal{M}, g)$ .

For completeness we include the diagrams for the cases where the solution cannot be extended across a horizon (discussed further in § 3 and § 6). Note that while the normal geodesics run into the intermediate singularities, the matter sidesteps these singularities and crosses the horizon in those cases where  $J^-(\mathcal{S}) - D^-(\mathcal{S}) \neq \emptyset$ .

We have shown that these are the only possible behaviours when a Cauchy horizon exists. The Farnsworth solution is one of kind (a), where the fluid begins at a singularity (probably a matter singularity). We do not have explicit examples of solutions of kinds (b) ("whimper") or (c) ("whimper-whimper"); and indeed it is possible that a detailed examination of the field equations might show that no such solutions actually

<sup>10</sup> These diagrams do not represent actual cross-sections of the space-time, in general, because  $u$  and  $n$  do not in general span 2-surfaces. Each diagram shows one of the various conformal structures, and assumes the infinities are regular.

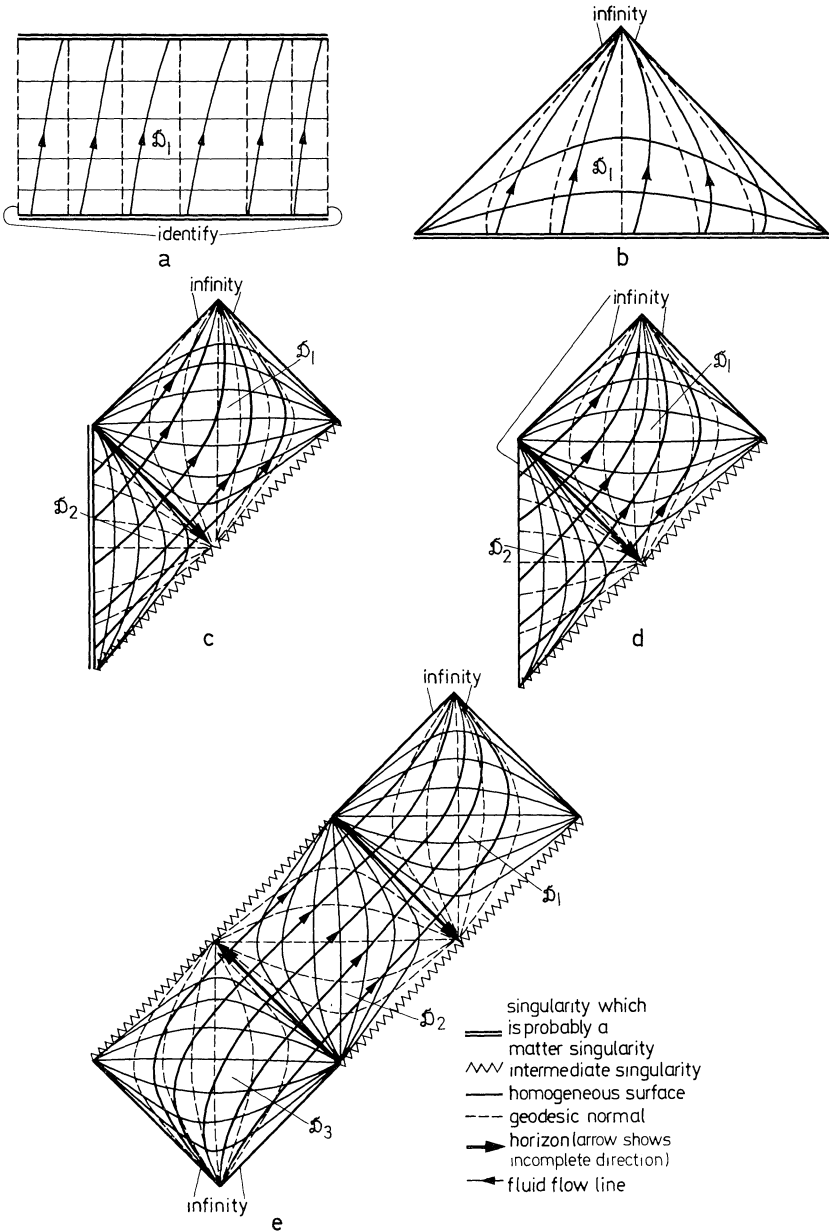


Fig. 2a—e. The structure of spatially homogeneous universes (showing one of the possible conformal structures at infinity, which has been assumed to be regular). Solutions where  $\mathcal{M} = D(\mathcal{S})$ : (a) Type IX, (b) all other Bianchi types. Solutions where  $\mathcal{M} - D(\mathcal{S}) \neq \emptyset$ : (c) “whimperbang”, (d) “whimper”, (e) “whimper-whimper”

exist (our examination has effectively been based on four of Einstein's equations; the other six equations might imply non-existence). In order to find examples of these solutions, or to prove their non-existence, one will have to undertake a more detailed study of the field equations than has been made here. A Lagrangian or Hamiltonian approach will be difficult (see [25]) and it may be that only machine integrations will settle this question. There are particular cases where one can obtain further information; for example examination of the conservation equations shows that if  $\mathbf{a}$  is always parallel to  $\mathbf{c}$ , or is always timelike in  $\mathcal{D}_2$ , the fluid cannot cross two horizons (here  $a_\epsilon \equiv \frac{1}{2}\gamma^\alpha_{\epsilon\alpha}$  in a group invariant tilted basis, and  $c$  is the unit projection of  $\mathbf{u}$  into the homogeneous surfaces). An extension of results in [11] then enables one to state if  $\omega = 0$  and the group is not Type III, the fluid cannot cross two horizons.

## 6. Conformal Singularities and Singular Infinities

We have now (in § 3) discussed some cases where the breakdown of prediction in  $D^+(\mathcal{S})$  is known to occur because of a matter singularity, and (in § 4 and § 5) investigated those cases where the matter crosses a Cauchy horizon. The question that remains is whether these are the only possible behaviours, or not.

More precisely, we wish to pose the following question: suppose that (1)–(5) are satisfied, and that  $J^-(\mathcal{S})$  is the same as  $D^-(\mathcal{S})$  (so  $\tau_- = T_- \leq s_- = S_-$ ); does it then follow that the matter world lines in  $D^-(\mathcal{S})$  end at a matter singularity, or not?

We have been unable to obtain a definitive answer to this question. Suppose that  $J^-(\mathcal{S}) = D^-(\mathcal{S})$ , and that no matter singularity occurs; then we know some conditions that will have to be fulfilled (Lemma 4.3), and that coordinates (5.2) cover  $J^-(\mathcal{S})$ , which ends where either  $\theta$  or  $\sigma_{ab}$  is not well behaved (Lemma 5.3). Thus either the derivative of  $\theta$  or of  $\sigma_{ab}$  cannot go to a bounded limit as  $\tau \rightarrow -\tau_-$ ; so one might hope to be able to show that there was either a matter singularity or a conformal curvature singularity along the fluid flow lines, by using the propagation equations for  $\theta$  and  $\sigma_{ab}$ . The vorticity enters these equations, its evolution being determined by the vorticity propagation equations; but although the acceleration (given by (1.25) of [11]) is well behaved if  $\theta$  is, the same is not true of the derivatives of  $\dot{u}^a$  which enter these equations. So all that one can find from the Raychaudhuri equation is that if the solution cannot be extended beyond  $D^-(\mathcal{S})$  and no matter singularity occurs, either  $\sigma^2$  is well behaved but one of  $\sigma^a_b \sigma^b_a$ ,  $\sigma^a_b \sigma^b_c \sigma^c_a$  is not, or at least two of  $\theta$ ,  $\sigma^2$ ,  $\omega^2$  and  $\dot{u}^a_{;a}$  are not well-behaved. This information does not seem sufficient to enable one to use the shear propagation equations

(see [2]) to show that  $E_{ab}$  is not wellbehaved in a parallel propagated frame.

To examine this further, we use (1.25) and (1.16) of [11] to write Raychaudhuri's equation for the fluid in the form

$$\begin{aligned} \theta'(1 - \tanh^2 \beta dp/d\mu) + \theta^2 \left( \frac{1}{3} - (2 - \tanh^2 \beta) \frac{dp}{d\mu} \right) \\ + \cosh \beta \frac{dp}{d\mu} \theta (2\tilde{\theta} + \tilde{c}^a{}_{;a} \tanh \beta) (1 - \tanh^2 \beta) \\ + 2\sigma^2 + \frac{1}{2}(\mu + 3p) = 2\omega^2 + \tanh^2 \beta (dp/d\mu) \theta, \end{aligned} \tag{6.1}$$

which is valid in  $D(\mathcal{S})$ . As  $\tau \rightarrow -\tau_-$ ,  $\beta \rightarrow \infty$ ,  $\tilde{\theta} \rightarrow \infty$  and  $\tilde{c}^a{}_{;a} = -2a_e \tilde{c}^e$  is unbounded above. The methods of Lemma 2.7 show that  $\frac{4}{3}\tilde{\theta} < 2\tilde{\theta} + \tanh \beta \tilde{c}^a{}_{;a} < \frac{8}{3}\tilde{\theta}$ , so the third term in (6.1) acts as an anti-damping term causing instability. The sign of the coefficient of  $\theta'$  is always positive; the term in  $(dp/d\mu)'$  will in almost all conceivable circumstances be negligible. The problem is the second term; if  $0 \leq dp/d\mu \leq 1/6$ , this term is positive and assists collapse, but if  $1/3 \leq dp/d\mu \leq 1$ , this term is negative and resists collapse. It easily follows from (6.1) that:

**Lemma 6.1.** *Let  $dp/d\mu$  be constant and  $\omega = 0$ . If  $\theta$  is positive at any time, it remains positive in  $D^-(\mathcal{S})$ . If further  $0 \leq dp/d\mu \leq 1/6$ , a matter singularity occurs within a finite distance, if  $s$  can be extended far enough in  $D(\mathcal{S})$ . □*

Thus under these restricted circumstances, acceleration cannot prevent a matter singularity occurring in  $D^-(\mathcal{S})$ ; but it seems quite feasible that, for example, if  $p = \frac{1}{3}\mu$  the acceleration could prevent a matter singularity occurring even if the vorticity were zero. It is also conceivable that a conformal singularity might be avoided by suitable balancing of the other terms in the shear propagation equation.

Hence if  $J^-(\mathcal{S}) = D^-(\mathcal{S})$ , all we really know is that a singularity occurs along each matter world line; if this is not a matter singularity, it could conceivably be a locally extendible singularity, an intermediate singularity or a conformal curvature singularity. On the other hand the acceleration, shear and vorticity would have to be very finely balanced for any of these singularities to occur without the expansion diverging and a matter singularity occurring. Thus even though we have no rigorous proof, we may make

**Conjecture 1:** *the matter flow lines cannot end at a locally extendible or an intermediate singularity.*

It seems very likely that this is true, the essential point being that in the cases considered in § 3 and § 4, the matter lines did not run into the intermediate singularity, but bypasses it; if they had run into this

singularity (or into a locally extensible singularity), it seems likely that this would have turned it into a matter singularity. Similarly, we may make

**Conjecture 2:** *the matter flow lines do not end at a conformal curvature singularity.*

While it is conceivable that the shear, vorticity and acceleration terms could balance each other in Raychaudhuri's equation without  $\theta'$  becoming disastrously large<sup>11</sup>, this requires a very fine balance to be kept, and we can see no reason why such a fine balance should be maintained. The other possibility is that the shear (and other terms) can diverge with  $\theta'$  diverging but  $\theta$  remaining finite. For example, consider pressure-free matter moving without rotation, so  $\omega = 0 = \dot{u}^a$ ,  $\mu = M/l^3$  ( $M$  constant  $> 0$ ). Now suppose the Weyl tensor is such that for  $l$  less than some value  $l_0$ ,  $\sigma^2 = \Sigma^2 l^{-1} (l - l_1)^{-\alpha}$  where  $\Sigma, \alpha$  and  $l_1$  are constants,  $0 < \Sigma$ ,  $0 < \alpha < 1$ ,  $0 < l_1 < l_0$ . Then one can integrate Raychaudhuri's equation to show that along the fluid flow lines,  $0 < l_1 < l \Rightarrow$

$$3(l')^2 + 2\Sigma^2(1 - \alpha)^{-1} (l - l_1)^{1 - \alpha} - Ml^{-1} = \text{constant};$$

so as  $l \rightarrow l_1$  from above,  $\sigma^2 \rightarrow \infty$  but  $l'$  goes to a finite value; and so  $l$  and  $\mu$  go to finite values also. In this case, the evolution of the shear is such that a conformal curvature singularity occurs before a matter singularity can occur. However we know of no exact solution with  $p \leq 1/3\mu$  in which this behaviour occurs in  $D(\mathcal{S})$ <sup>12</sup> (although Collins has found an example in which this does occur, when  $p = \mu$ ).

The situation, then, is that we can give no proof of either conjecture; but it seems very likely that Conjecture 1 is true, and fairly probable that Conjecture 2 is also true. If they are true, then the end of the Cauchy development  $D^-(\mathcal{S})$  occurs either at a matter singularity as in § 3, or at a Cauchy horizon as in § 4 and § 5.

What we are able to say for certain is that when a curvature singularity occurs it is also a scalar polynomial (s.p.) singularity [4]:

**Lemma 6.2.** *When (1)–(3) hold, a curvature singularity occurs along a given curve if and only if a s.p. singularity occurs along that curve.*

This follows when one remembers that the vector  $\mathbf{u}$  is uniquely determined algebraically by the curvature tensor because of (2.1) and (2.2), so the “electrical” and “magnetic” parts of  $E_{ab}, H_{ab}$  of the Weyl tensor (see e.g. [2]) are uniquely determined, and their eigenvalues are scalar invariants. □

<sup>11</sup> It is even possible that  $\theta \rightarrow -\infty$  and  $\mu \rightarrow 0$  as  $\tau \rightarrow -\tau_-$ .

<sup>12</sup> However in the L.R.S. Type V solutions, if we do *not* assume the solution across the horizon is obtained by analytic continuation, then it seems perfectly possible one could have solutions with this behaviour in  $J^-(\mathcal{S}) - D^-(\mathcal{S})$ .

A similar question arises in the “whimper-bang” case of § 5; while we then know the flow lines come to an end at a singularity, we do not know which type of singularity occurs. This is similar to the problem we have just discussed, but more difficult because of the timelike nature of the surfaces of transitivity. Collins has found examples in which the singularity that occurs is a conformal singularity; however one might still hope to prove that Conjectures (1) and (2) are true in general.

Finally, a similar question arises in the case of the behaviour at infinity. In the case (§ 5) where only one whimper occurs, although we know that the matter flow lines are complete, and so that they do not run into a singularity, the nature of the final state could be singular rather than regular. More precisely, consider a fluid flow line  $\gamma(\tau)$ ; as  $\tau \rightarrow -\infty$ , it is quite conceivable that  $\mu$  and  $p$  diverge, in which case we would say there was a *singular infinity* in the universe model. Obviously one could classify the possible singular behaviours of infinity as in § 1, according to the behaviours of the curvature tensor components in an orthonormal frame as  $\tau \rightarrow -\infty$ , into that of a curvature singularity and that of an intermediate singularity. Clearly the future infinity in  $D^+(\mathcal{S})$ , and the past infinity in the “whimper-whimper” case, are in some sense singular when  $\beta \rightarrow \infty$  as  $s \rightarrow \pm \infty$ ; we have not investigated these cases (in which the fluid can run into some kind of singularity as  $s \rightarrow \pm \infty$ ) in any detail. When  $\beta \rightarrow 0$  as  $s \rightarrow \pm \infty$ , Lemma 5.2 makes it seem unlikely that any singular behaviour will occur at infinity if (2.2) holds; but the vorticity conservation theorem (see [2]) makes it quite likely that the shear (and hence Weyl) terms diverge if  $dp/d\mu = 1$ . Thus it is certainly quite possible that even when the matter flow lines do not run into singularities (as  $\tau \rightarrow \pm \infty$ ), in some cases the behaviour of the universe model is singular in a well-defined sense.

## 7. Conclusion

We have been considering perfect fluid spatially homogeneous cosmologically models, and have found that in certain cases (including all non-tilted models and all Class A models except perhaps Type IX) the Cauchy development of the fluid from a spacelike surface of homogeneity is terminated by a matter singularity where the matter density blows up. In some cases the matter bypasses an intermediate singularity and crosses a Cauchy horizon into further regions of space-time; and then either ends its history at a singularity, or possibly continues forever without running into a singularity. There remains a possibility, which seems rather unlikely, that the matter might be unable to escape beyond its Cauchy development not because of matter singularity but because of either a milder (locally extendible or intermediate) singularity, or

because of a singularity in the conformal structure. We shall not consider this third rather remote possibility further here.

To illustrate the essential features of the other two possibilities we shall briefly explain how they can easily be understood by examining a simple *model for their behaviour* based on two-dimensional Minkowski space. First remember that the action of the Lorentz transformation (“boost”) about a given point  $p$  in two-dimensional Minkowski space is such that it acts in surfaces at constant distance from  $p$  (see Fig. 3). The transformation through a hyperbolic angle  $\beta$  moves the straight lines through  $p$  into each other, leaving invariant the null rays through  $p$ , and leaving  $p$  itself fixed.

Now consider “painting on” a single matter flow line onto this space. First, paint on the matter flow line in some arbitrary manner through  $p$ . Then move this timelike line around by repeated action of the Lorentz group boosts about  $p$  through the angle  $\beta$ ; the resultant fluid flow field on the space is invariant under the action of this group of isometries. We really wish to consider the limit as  $\beta \rightarrow 0$ , when the group action is continuous and each surface at constant distance from  $p$  becomes a surface of constant density; but for didactic purposes it is easier to continue thinking of  $\beta$  as a finite angle. We now have a model of a spatially homogeneous universe, with all observable properties uniform on the surfaces at constant distances from  $p$ . However all the fluid flow lines “pile up” at the null cone through  $p$  (see Fig. 3b) and they intersect at  $p$  because this is a fixed point of the group. Hence the fluid divergence is unbounded at  $p$ , and the fluid conservation equations imply that  $\mu \rightarrow \infty$  there: a “big bang” develops, the density of the fluid increasing indefinitely on the surfaces of constant density as they approach  $p$ . Thus in fact one now has to cut out of the resulting space the singular point  $p$  and the null lines through  $p$ , where the matter density would otherwise be infinite. This behaviour of the matter is an exact model of the “big-bang” matter singularity in the Robertson-Walker universe, the non-tilted universes, the Class A universes, and so on; the energy density diverges on every past-directed non-spacelike curve through any homogeneous spacelike initial surface  $\mathcal{S}$ , and this remains true no matter what orthonormal basis might be used for measuring the stress tensor  $T_{ab}$  along such a curve.

Now suppose that the initial timelike line that was painted on did not pass through  $p$ , but went from I to II, successively crossing each surface at constant distance from  $p$  in II. One could again spread this line over the space, by repeated use of the Lorentz group about  $p$ , to obtain a world model which was initially spatially homogeneous. Again the normals to the homogeneous surfaces intersect at  $p$ , but now the matter flow lines, though they “pile up” at the null line at  $45^\circ$  through  $p$  (see

Fig. 3c), do not pass through this intersection; they cross  $H^-(\mathcal{S})$  into a region where the surfaces of homogeneity are timelike. As they continue across each of the surfaces of homogeneity the density, pressure and so forth are perfectly regular on each of these surfaces; so there is certainly no matter singularity bounding  $D^-(\mathcal{S})$ . However consider a null geodesic generating  $H^-(\mathcal{S})$ ; this geodesic crosses an infinite number of matter world lines in a finite affine distance. This suggests that a singularity occurs on the line at  $45^\circ$  where the matter world line pile up; and this

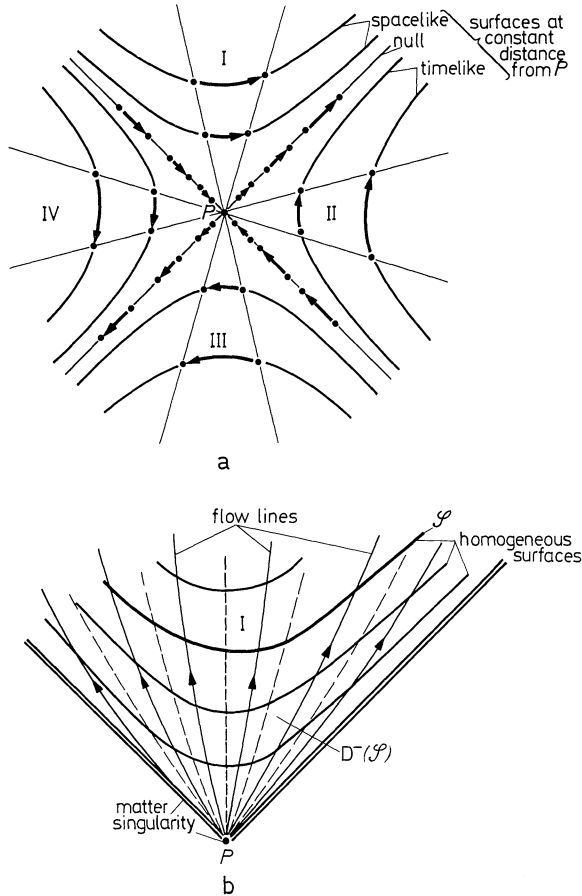
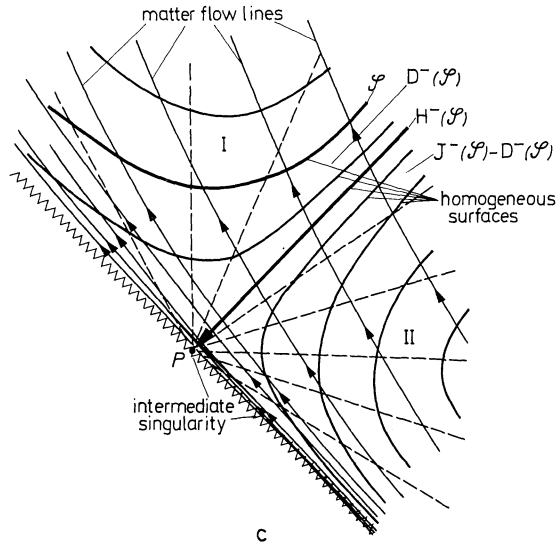


Fig. 3a—c. Two-dimensional Minkowski space as a model for singularities. (a) The action of the “boosts” about  $p$ . (b) A matter line through  $p$ , moved over the space by the boosts about  $p$ , leads to a flow diverging from  $p$ . A matter singularity occurs. (c) A matter line not through  $p$ , moved over the space by the boosts about  $p$ , leads to a flow which “piles up” towards  $p$  but does not converge there. An intermediate singularity and a Cauchy horizon for the homogeneous spacelike surface  $\mathcal{S}$  occur





is indeed so. For example, use of Eq. (4.7) with  $k^a$  chosen to be the unit normal vector  $n^a$  shows that the stress tensor component  $T_{ab}n^an^b$  diverges along the normals, because  $u_a n^a$  diverges on the surfaces of homogeneity as one approaches the horizon. Thus at least one matter tensor component diverges in an orthonormal frame (with  $n^a$  as the timelike unit vector) which is parallelly transported along a normal geodesic; so these normals cannot be continued to or beyond  $p$ . In fact the whole line at  $45^\circ$  has to be removed from this space. On the other hand one can obtain an orthonormal basis along a normal geodesic by moving along an orthonormal fluid basis (i.e. one with  $u$  as the timelike unit vector) under the action of the group, from some chosen fluid flow line to the normal geodesic; then all the matter tensor components are perfectly regular in this basis (both the matter tensor and the basis are invariant under the group) all the way up to  $p$ .

This behaviour of the matter<sup>13</sup> gives an exact model of an intermediate singularity, such as that which occurs in the Farnsworth solution.

While we know that an intermediate singularity is possible in each group type in Class B, we do not yet have any indication as to how general the occurrence of these singularities is. However one can investigate this by counting up the degrees of freedom in setting initial data on the null surface  $H^-(\mathcal{S})$ , for each group type in Class B; and it then becomes

<sup>13</sup> The *geometry* of the space-time is regular in all frames, for we have not used Einstein's equations in these models.

clear that there is an *open set of models in each group type in Class B* in which an intermediate singularity occurs. This might still be a small proportion of the possible models in each group type, but it is not negligible.

The next question is whether the occurrence of the intermediate singularities in these exact universe models is compatible with the present limits on the isotropy of the *microwave background radiation*. To check this we have considered L.R.S. Type V tilted universe models which are almost isotropic at late times; to simplify the calculations, we followed [28] by taking this model as a perturbation of the unique simplest “matter plus radiation” Robertson-Walker universe with  $k = -1$ . We assume decoupling takes place in the spatially homogeneous region  $\mathcal{D}_1$ . Just as in [28] (and cf. [29]) one can then find an expression for the observed temperature  $T_0$  of black-body radiation emitted at temperature  $T_e$  and universe radius  $l_e$ ; one finds

$$T_e/T_0 = (l_e/l_0) \exp \left\{ -A_1 \int_{l_e}^{l_0} \left( \frac{c^4 - 4c^2q^2 + q^4}{q(q-M)^2(c^2+q^2)^2} dq \right. \right. \\ \left. \left. + A_2 \frac{c^2q}{(3q-M)(c^2+q^2)^2} dq \right) \right\} \quad (7.1)$$

where  $A_1, A_2, M$  are constants describing the universe model;  $l_0$  is the radius of the model at the time of observation, and  $c = \cot(\phi/2)$  where  $\phi$  is the angle between the direction of observation and the axis of symmetry, measured from the  $+a$  direction. The anisotropy in this expression can be arbitrarily small, and so is certainly consistent with present day observations. A plot of this temperature as a function of  $\phi$ <sup>14</sup> is given in Fig. 4a and b for cases in which the matter runs into a singularity before it can cross a Cauchy horizon, and in Figs. 4c and d for cases where the matter crosses a Cauchy horizon and then later runs into a singularity. Note the somewhat different modes of anisotropy in these two cases. (We are indebted to Miss M. Shapcott for plotting these curves.)

One further point of interest is that of the *helium production* in these universes. If the production of helium were to take place in  $\mathcal{D}_1$  (or even  $\mathcal{D}_3$ ) the calculation would be as usual, allowing for a possibly different timescale of evolution through the helium production phase. However if it took place in  $\mathcal{D}_2$ , it would happen in a highly inhomogeneous situation; so very appreciable diffusion of particles could take place between regions at different temperatures and stages of nucleosynthesis, possibly producing markedly different results from the standard (spatially homogeneous) situations.

<sup>14</sup> The integrals in (7.1) can be worked out analytically.

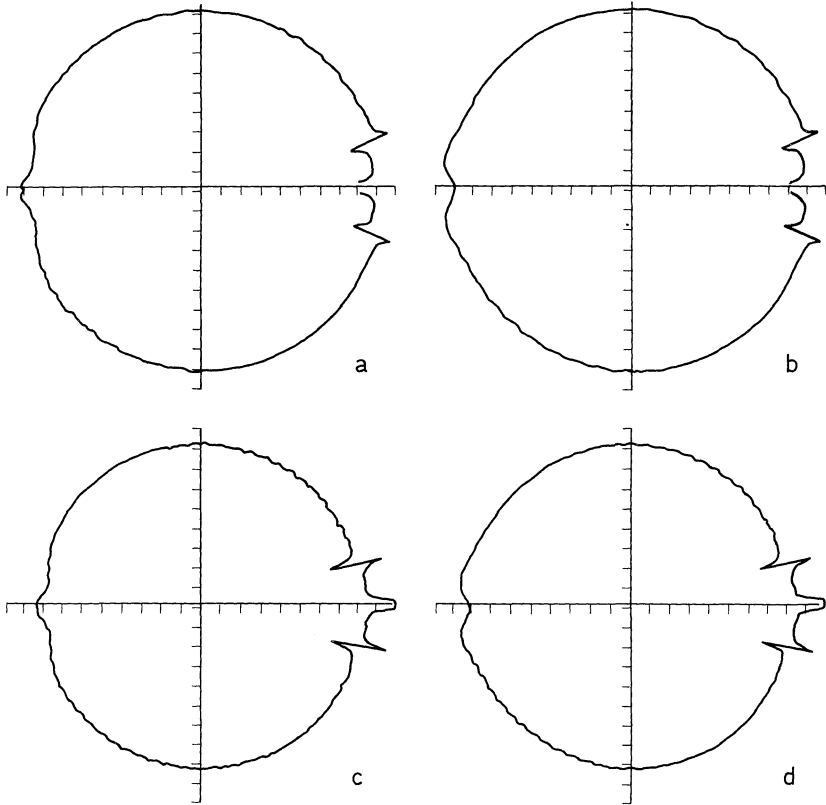


Fig. 4a—d. The temperature of the microwave background radiation in rotationally symmetric Type V tilted universes, as a function of the angle from the axis of symmetry. Cases (a) and (b) are ones where a matter singularity occurs before the matter can cross a Cauchy horizon; cases (c) and (d) are cases where the matter crosses a Cauchy horizon, and then a matter singularity occurs

Our examination of these homogeneous models has been on the basis of General Relativity with  $\Lambda = 0$ . If  $\Lambda \neq 0$  one can of course avoid the occurrence of any singularities by taking  $\Lambda$  positive and large enough (it then acts as a negative energy density term); but this is only possible if the deceleration parameter  $q_0$  is negative at the present time, and one may also raise philosophic objections (cf. [2]). Nevertheless it is a possibility that should be borne in mind. If the  $\Lambda$  term is non-zero but does not prevent the occurrence of a singularity, it does not seem likely that it would change the nature of the singularities occurring (but its effect on infinity could be considerable).

One can of course examine the types of singularity occurring in many other theories of space-time in a way similar to that we have used in General Relativity. It is interesting to note that in the steady-state universe, there is an incompleteness in the space-time (cf. [4]) which (just as in our Minkowski model) is associated with an intermediate singularity of the matter field but not of the space-time geometry (Carter and Hawking, private communication).

One further line of investigation which suggests itself is to consider what behaviour a quantum theory of gravity would predict near an intermediate singularity; it may be relatively simple to extend the previous examinations (see e.g. [30]) of homogeneous cosmological models to investigate this.

Ultimately, of course, one would like to drop the restriction to spatial homogeneity and examine more realistic universe models in which inhomogeneities occur. We know that intermediate singularities occur in a fairly wide class of homogeneous cosmological models; will they be likely to occur in more realistic universe models? We cannot provide a clear answer to this at present; as mentioned in the introduction, all that the Penrose-Hawking theorems predict is the existence of some singularity, without restricting the nature of the singularity. However, one can obtain some clues by considering perturbations of the homogeneous universes, i.e. considering an initial data surface  $\mathcal{S}$  on which the initial data is almost spatially homogeneous. The essential point here is that if one sent a photon down the horizon  $H^-(\mathcal{S})$  in the models we have considered, an infinite blue shift would occur (in the time-reversed situation) as it approached the intermediate singularity [cf. Eq. (4.5);  $\mathbf{u} \cdot \boldsymbol{\xi}$  is constant along the null geodesic generators of  $H^-(\mathcal{S})$ , so  $\mathbf{u} \cdot \mathbf{k}$  diverges]. Thus the photon would arrive with infinite energy the effect of the perturbation (firing the photon down the null geodesic) would be to turn the intermediate singularity into a curvature singularity in this time-reversed situation (we are indebted to Misner for discussions on this point). Hence we may surmise that *intermediate singularities are unstable*, and can only occur if exceptional, high symmetry conditions occur. One might be able to investigate this generally by an extension of the approximation techniques of Lifschitz and Khalatnikov. In the case of our homogeneous universes, it suggests that the perturbed universe would be an inhomogeneous universe in which instead of each intermediate singularity there would a curvature singularity. However there is no reason to think that much of the matter in the universe would fall into such a singularity; rather than being a big bang, it would be a "little bang" into which a small amount of matter and radiation fell (or from which a small amount of matter and radiation emerged), most of the matter in the universe missing it. Thus we may surmise that

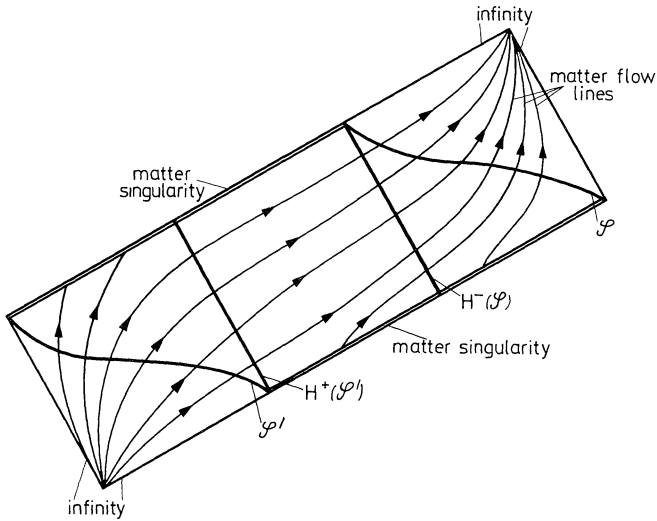


Fig. 5. A perturbed “whimper-whimper” universe probably leads to an inhomogeneous space-time with two “little bangs”; but most of the matter avoids these singularities

if the “whimper-whimper” universe of § 5 were perturbed, the resulting universe would be one in which matter collapsed from a state rather like a time-reversed Robertson-Walker universe; two little bangs occurred, most of the matter sidestepping them and then re-expanding to a state again like the presently observed universe (Fig. 5). Unfortunately this line of argument does *not* lead us nearer to an “oscillating” universe model; for there is no indication that curvature singularities are unstable, and in fact we surmise the contrary, that *curvature singularities are stable*. Thus perturbation of the Type IX universe (Fig. 2a) would probably lead to the same diagram, and not to a universe with an infinite number of expansion and contraction phases.

Finally we will need investigation of highly inhomogeneous spaces, determining in more general situations when intermediate and curvature singularities can occur (cf. [27] for an investigation of some matter singularities in a highly inhomogeneous situation).

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