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Self-similar Spacetimes: Geometry and Dynamics*

Douglas M. Eardley
California Institute of Technology, Pasadena, California, USA

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Abstract. The nature and uses of self-similarity in general relativity are discussed. A spacetime may be self-similar (homothetic) along surfaces of any dimensionality, from 1 to 4. A geometric construction is given for all self-similar spacetimes. As an important special case, the "spatially self-similar cosmological models" are introduced, and their dynamical properties are studied in some detail: The initial-value problem is posed, the ADM formulation is established (when applicable), and it is shown that the evolution equations preserve a self-similarity of initial data. The existence of a conserved quantity is deduced from self-similarity. Possible applications to cosmology and singularities are mentioned.

1. Introduction

Similarity solutions in classical hydrodynamics have been a fruitful source of models for physical systems having no intrinsic scale of length, or mass, or time. In (classical) general relativity, the fundamental constants G and c reduce the number of independent physical units to one; take it to be the unit of length. Therefore, the physical notion of self-similarity for spacetime amounts precisely to the geometric notion of *invariance under scale transformations*, as was first pointed out by Cahill and Taub [1].

If a strongly self-gravitating system evolves in size through many orders of magnitude, either expanding or contracting, one might reasonably expect it to "forget" its initial conditions and eventually become scale-invariant. For example, the expansion of the universe from the big bang and the collapse of a star to a singularity might both exhibit self-similarity in some form. This expectation is borne out in most of the popular models for these processes, as will be discussed briefly below. Conversely, one may hope to discover new facts about cosmology and singularities by building new models that presume self-similarity from the out-set.

With these applications in mind, this paper systematically defines and analyzes the notion of self-similarity in the context of general relativity. The immediate goal is a set of tools; physical applications will not be attempted here. The main conclusion is that self-similarity

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is very closely related to isometry, and that for most calculational purposes, *a similarity is just as good as an isometry*. For example: A similarity simplifies the Einstein equations. A similarity in initial data generally is preserved by the evolution equations (insofar as matter fields permit). A similarity entails the existence of a conserved quantity (for matter of zero rest mass). Similarities seem to be unique among generalized geometrical symmetries of spacetime, in enjoying these desirable properties.

Others have obtained many results about what we are calling self-similar spacetimes. Cahill and Taub [1] analyzed self-similar, spherically symmetric spacetimes, and Taub [2], plane-symmetric ones, with strong emphasis on the physical significance of self-similarity in general relativity. Godfrey [3] constructed all homothetic (≡ self-similar) Weyl spacetimes, and obtained many of the results of Section 3 below in this special case. The place of homothetic invariance (≡ similarity) in the scheme of more general geometric symmetries has been studied by Collinson and French [4], Katzin, Levine, and Davis [5], and Collinson [6]. Much of the material of Section 2 below is well known in differential geometry (see e.g., Yano [7]), but the main result, Proposition 1, is new.

Section 2 deals with the geometry of self-similar spacetimes, concentrating on the similarity group H_n and its Lie algebra $\boldsymbol{H}_n$ of infinitesimal generators $\boldsymbol{v}$, vector fields on spacetime. A self-similar spacetime usually is conformally related to another, unphysical spacetime that admits H_n as an isometry group; this settles the problem of existence and construction. The few exceptions are treated in Appendix A.

Section 3 constructs an important class of self-similar spacetimes, characterized by a similarity group H_3 acting on space slices. These "self-similar cosmological models" form a natural, inhomogeneous generalization of the familiar homogeneous cosmological models, and they share many of the formal properties of the homogeneous models.

Section 4 studies the dynamics of the self-similar cosmological models, with respect to the initial-value problem and the ADM formalism.

Section 5 illustrates the foregoing with some simple examples, and derives a conservation law.

The spacetime signature is $(-+++)$. Greek indices run over 0, 1, 2, 3, lower-case Latin indices over 1, 2, 3. Further conventions will be defined as needed.

2. Geometry of Self-similar Spacetimes

Consider a spacetime $(M, \boldsymbol{g})$; M is a 4-dimensional differentiable (say C^3) manifold, and $\boldsymbol{g}$ is the physical metric field. $(M, \boldsymbol{g})$ is further endowed with certain matter fields ϕ_A. The *isometry group* G_m of $(M, \boldsymbol{g})$ is the Lie group of smooth maps of M onto itself leaving $\boldsymbol{g}$ invariant [1]. The subscript m denotes the dimension of G_m as a Lie group.

Now use natural units so that $c = G = 1$, and let l be the unit of length. Each physical geometric-object field $\boldsymbol{\Phi}$ (e.g., $\boldsymbol{g}$ or ϕ_A) has a *dimension* q (usually an integer) such that under the *scale transformation*

$$l' = e^{\alpha} l, \quad \text{where} \quad \alpha = \text{const}, \tag{2.1a}$$

$\boldsymbol{\Phi}$ transforms like

$$\boldsymbol{\Phi}' = e^{q\alpha} \boldsymbol{\Phi}. \tag{2.1b}$$

It is important here to view as *different* geometrical objects two tensors related by raising or lowering of indices with $\boldsymbol{g}$, because $\boldsymbol{g}$ carries dimension. Below, when we introduce a geometrical object, we shall specify a particular arrangement of indices; e.g., $\boldsymbol{g}(g_{\mu\nu})$ has $q = 2$, but $\boldsymbol{g}^{-1}(g^{\mu\nu})$ has $q = -2$.

The spacetime interval ds has $q = 1$; we take $q = 2$ for $\boldsymbol{g}$. Coordinates x^μ must have $q = 0$, since generally they possess no physical significance. (Sometimes one adopts a particular coordinate that might deserve $q = 1$, e.g., proper time t of a congruence of observers; but allowing $q = 1$ generally for coordinates leads to chaos. It is better always to let $\boldsymbol{g}$ carry the dimension.) Then, e.g., it follows that $q = 0$ for *Riem* $(R^\mu{}_{\nu\varrho\sigma})$, $q = -1$ for a velocity vector $\boldsymbol{u}(u^\mu = dx^\mu/ds)$, and $q = 0$ for the matter stress-energy tensor $\boldsymbol{T}(T_{\mu\nu})$. The energy density measured by an observer with velocity $\boldsymbol{u}$, $\varepsilon \equiv \boldsymbol{u} \cdot \boldsymbol{T} \cdot \boldsymbol{u}$, has $q = -1 + 0 - 1 = -2$.

A smooth map $M \to M$ under which $\boldsymbol{g}$ suffers the effect of a constant scale transformation, Eq. (2.1),

$$\boldsymbol{g} \to \boldsymbol{g}' = e^{2\alpha} \boldsymbol{g}, \qquad \alpha \text{ constant on } M, \tag{2.2a}$$

is a *similarity* of $(M, \boldsymbol{g})$; we shall say that $(M, \boldsymbol{g})$ is *self-similar* if it admits such a map. (The same concept is called a *homothetic motion* in differential geometry [7].) It will be understood throughout (unless specified otherwise) that the matter fields ϕ_A suffer the same scale transformation. So for any physical field $\boldsymbol{\Phi}$ of dimension q,

$$\boldsymbol{\Phi} \to \boldsymbol{\Phi}' = e^{q\alpha} \boldsymbol{\Phi}, \qquad \alpha \text{ constant on } M. \tag{2.2b}$$

Repeating, a similarity is a map of spacetime onto itself, the only physical effect of which is a *uniform* change in the scale of length.

For $\alpha = 0$, a similarity reduces to an isometry; a *non-trivial* similarity has $\alpha \neq 0$. The similarities of $(M, \boldsymbol{g})$ form a Lie group H_n (n = dimension

[1] We shall ignore discrete isometries and similarities throughout. All work in this section is local; in particular, a family of group orbits is taken to be a family of submanifolds of constant dimension.

of H_n) [7]. Throughout, H_n is the similarity group of $(M, \boldsymbol{g})$ and G_m the isometry group. Then $G_m \subseteqq H_n$. If $G_m \neq H_n$, then H_n is *non-trivial.* The groups G_m or H_n may act transitively on surfaces of any dimensionality in M.

The infinitesimal generators $\boldsymbol{u}$ of H_n are vector fields on M. They form the Lie algebra $\boldsymbol{H}_n$; each $\boldsymbol{u} \in \boldsymbol{H}_n$ obeys the infinitesimal versions of Eq. (2.2):

$$\mathscr{L}_{\boldsymbol{u}} \boldsymbol{g} = 2\langle b, \boldsymbol{u}\rangle \boldsymbol{g} , \tag{2.3 a}$$

and more generally

$$\mathscr{L}_{\boldsymbol{u}} \boldsymbol{\Phi} = q\langle b, \boldsymbol{u}\rangle \boldsymbol{\Phi}, \qquad \boldsymbol{u} \in \boldsymbol{H}_n . \tag{2.3 b}$$

Here $\langle b, \boldsymbol{u}\rangle$ is a constant (independent of point $p \in M$), depending on the choice of $\boldsymbol{u}$; i.e., $\langle b, \ \rangle$ is a linear functional on $\boldsymbol{H}_n$ (linear, because $\mathscr{L}\boldsymbol{\Phi}$ is linear on $\boldsymbol{H}_n$). In particular, each $\boldsymbol{v} \in \boldsymbol{G}_m$ is a Killing vector field,

$$\mathscr{L}_{\boldsymbol{v}} \boldsymbol{g} = 0, \qquad \boldsymbol{v} \in \boldsymbol{G}_m .$$

So

$$\boldsymbol{G}_m = \text{Kernal}\langle b, \ \rangle ; \tag{2.4}$$

therefore $m = m - 1$ if H_n is non-trivial. (I.e., $\boldsymbol{G}_n$ is exactly the linear subspace of the vector space $\boldsymbol{H}_n$ which is "orthogonal" to the covector b.) Roughly, $(M, \boldsymbol{g})$ admits at most one independent, non-trivial similarity. It is then no surprise that the commutator of two similarities is an isometry: For $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}_n$,

$$\begin{aligned} \mathscr{L}_{[\boldsymbol{u},\boldsymbol{v}]} \boldsymbol{\Phi} &= [\mathscr{L}_{\boldsymbol{u}}, \mathscr{L}_{\boldsymbol{v}}] \boldsymbol{\Phi} \\ &= q[\langle b, \boldsymbol{u}\rangle \langle b, \boldsymbol{v}\rangle - \langle b, \boldsymbol{v}\rangle \langle b, \boldsymbol{u}\rangle] \boldsymbol{\Phi} \\ &= 0 \end{aligned}$$

$$\Rightarrow [\boldsymbol{u}, \boldsymbol{v}] \in \boldsymbol{G}_m ;$$

i.e.,

$$[\boldsymbol{H}_n, \boldsymbol{H}_n] \subseteqq \boldsymbol{G}_m . \tag{2.5}$$

Let us turn to the existence and construction of a self-similar spacetime with given H_n. Choose H_n with subgroup $G_{n-1} \subset H_n$ satisfying the necessary condition, Eq. (2.5). Now employ a powerful Ansatz: Construct, if possible, some spacetime $(M, {}_0\boldsymbol{g})$ (not necessarily physical) with isometry group ${}_0G_k$, such that $H_n \subseteqq {}_0G_k$. Try to construct the physical metric $\boldsymbol{g}$ by a conformal change of metric

$$\boldsymbol{g} = e^{2\psi} {}_0\boldsymbol{g} ; \tag{2.6}$$

Eq. (2.3 a) requires for the scalar field ψ, precisely

$$\mathscr{L}_{\boldsymbol{u}} \psi = \langle b, \boldsymbol{u}\rangle, \qquad \boldsymbol{u} \in \boldsymbol{H}_n ; \tag{2.7 a}$$

in particular,

$$\mathscr{L}_v \psi = 0, \quad v \in G_{n-1}. \tag{2.7b}$$

Equations (2.7) admit a solution ψ if, and only if, u is pointwise linearly independent on M of G_{n-1}, that is, if and only if

$$\dim G_{n-1}(p) = \dim H_n(p) - 1, \tag{2.8}$$

where $H_n(p) \subsetneqq M$ denotes the orbit of $p \in M$ under the action of H_n, etc. If Eq. (2.8) holds, there are local coordinates with one particular coordinate function x such that

$$u = \partial_x, \quad x = \text{const on each } G_{n-1}(p).$$

A solution of Eq. (2.7) is then

$$\psi = \beta x, \quad \text{where} \quad \beta = \langle b, \partial_x \rangle = \text{const}. \tag{2.9}$$

Then (M, g) is given by Eqs. (2.6) and (2.9), and (M, g) admits H_n as similarity group with G_{n-1} as isometry subgroup. (The construction can always be arranged so that the similarity group of (M, g) is no *larger* than H_n.) Further, each self-similar field Φ of dimension q can be constructed from a dimensionless field ${}_0\Phi$, invariant under H_n, by

$$\Phi = e^{q\psi}\, {}_0\Phi. \tag{2.10}$$

A fundamental theorem due to Defrise-Carter, concerning conformal symmetries of spacetime, implies that this Ansatz gives *all* self-similar spacetimes, apart from a few exceptional cases. To summarize:

Proposition 1. *Each spacetime (M, g) with non-trivial similarity group H_n and isometry subgroup $G_m \subsetneqq H_n$ has these properties:*

1) $[H_n, H_n] \subseteqq G_m$; *and* $m = n - 1$.

2) *Either:*

a) *(The usual case) (M, g) is conformally related to a spacetime $(M, {}_0g)$, with isometry group ${}_0G_k$, such that $H_n \subsetneqq {}_0G_k$; and* $\dim G_m(p) = \dim H_n(p) - 1$, *$p \in M$; or*

b) *(The exceptional case) (M, g) is a (vacuum or non-vacuum) "plane-wave spacetime" (see Appendix A).*

Proof. Sufficiency of 1) and 2a), and necessity of 1), were established above. Necessity of 2): The fundamental result is:

Theorem 1. (Defrise-Carter [8]). *Each spacetime (M, g) with conformal group C_j is conformally related to another spacetime $(M, {}_0g)$, such that either:*

A) *C_j is the isometry group of $(M, {}_0g)$; or*

B) *$(M, {}_0g)$ is a non-flat, vacuum "plane-wave spacetime"; or*

C) *$(M, {}_0g)$ is Minkowski spacetime.*

Apply to the present case: $H_n \subsetneqq C_j$, so A) implies 2a). It is shown in Appendix A that B) or C) is equivalent to 2b). Q.E.D.

The significance of Proposition 1 is that the highly developed art form of constructing spacetimes with a given isometry group can be immediately applied to similarity groups; the exceptional cases for which this procedure fails are few enough to be finished off once and for all. (Proposition 1 has no direct connection with any field equations; it is *not* a device for obtaining "new solutions from old".)

3. Spatially Self-similar Cosmological Models: Geometry

A spacetime (M, g) with isometry group G_3 transitive on spacelike hypersurfaces S is a spatially homogeneous cosmological model[2] [9–12]. Let us generalize: A spacetime (M, g) with similarity group H_3 transitive on certain spacelike hypersurfaces S will be called a *spatially self-similar cosmological model* (briefly, a *self-similar cosmology*)[3]. Such a spacetime generally admits only the isometry group $G_2 \subset H_3$ and is therefore spatially inhomogeneous (unless H_3 is trivial, i.e., $H_3 \equiv G_3$). Nevertheless, the spatial geometry is quite simple: Each space slice S is a stack of 2-surfaces, each homogeneous, all geometrically similar to one another; the distance between successive 2-surfaces is strictly proportional to the size of the 2-surface. Therefore each slice S admits a self-similarity which maps the homogeneous 2-surfaces onto one another. Once initial conditions are given on one S, the time-evolution of the slices S and their successive stacking in time is determined by the Einstein equations, which reduce to ordinary differential equations in time, just as in the homogeneous case.

The (local) construction of the self-similar cosmologies involves two steps: First, algebraically determine the structure of possible H_3 and $G_2 \subset H_3$. Secondly, geometrically construct g in a suitable basis of 1-forms. This construction closely parallels that of the homogeneous cosmologies [9–12].

Group Structure. Each possible similarity group H_3 must be one of the Bianchi [15] groups, I–IX. Let $\{v_a\}$ be a basis of H_3, with $a, b, c, \ldots$ running over $1, 2, 3$; use the summation convention. The commutation

[2] This form of the definition omits one case: The spatially homogeneous Kantowski-Sachs-Thorne [13, 14] models with $G_4 = SO_3 \otimes A_1$ acting on spacelike hypersurfaces; here SO_3 acts on 2-spheres, and G_4 admits no transitive $G_3 \subset G_4$. Likewise, this section and the next omit the special case of self-similar, spherically symmetric models, with $H_4 = SO_3 \otimes A_1$ admitting no transitive $H_3 \subset H_4$. This special case has been treated extensively by Cahill and Taub [1].

[3] Again, all our work is local.

relations are

$$[\boldsymbol{v}_a, \boldsymbol{v}_b] = C^c{}_{ab}\boldsymbol{v}_c\,, \tag{3.1}$$

where the structure constants $C^c{}_{ab} = C^c{}_{[ab]}$ obey Jacobi's identities

$$C^c{}_{[ab}C^e{}_{d]c} = 0\,. \tag{3.2}$$

In the non-trivial case, the isometry subgroup $G_2 \subset H_3$ is characterized, through Eq. (2.4), by the linear functional $\langle b, \;\rangle$; let $\langle b, \;\rangle$ have components b_a in the basis $\{\boldsymbol{v}_a\}$. The necessary restriction, Eq. (2.5), reads

$$b_a C^a{}_{bc} = 0\,. \tag{3.3}$$

Decompose $C^c{}_{ab}$ into two pieces a_a and $n^{ab} = n^{(ab)}$:

$$C^c{}_{ab} = n^{cd}\varepsilon_{dab} + 2a_{[a}\delta^c_{b]}\,, \tag{3.4}$$

where

$$a_a = \tfrac{1}{2}C^b{}_{ab}, \qquad n^{ab} = \tfrac{1}{2}C^{(a}{}_{cd}\varepsilon^{b)cd}\,. \tag{3.5}$$

Here ε^{abc} and ε_{abc} are the unit alternating symbols. All objects bearing indicies represent (relative) tensors on $\boldsymbol{H}_3$; $C^c{}_{ab}$ and a_a are absolute tensors; ε^{abc} and n^{ab} are tensor densities. The necessary and sufficient restrictions, Eqs. (3.2) and (3.3), take the respective forms

$$n^{ab}a_b = 0\,, \tag{3.6a}$$

$$(n^{ab} + a_c\varepsilon^{acb})b_b = 0\,. \tag{3.6b}$$

Therefore, to find all possible distinct structures for H_3 and G_2, just find all solution sets $\{n^{ab}, a_a, b_a\}$ of Eq. (3.6) which are inequivalent under change of basis $\{\boldsymbol{v}_a\}$. These solution sets for all H_3, trivial ($b_a = 0$) as well as non-trivial ($b_a \neq 0$), are specified as follows.

Divide solutions into

Class A: $a_a = 0$, $\quad b_a = 0$;
Class B: $a_a \neq 0$, $\quad b_a = 0$;
Class C: $a_a = 0$, $\quad b_a \neq 0$;
Class D: $a_a \neq 0$, $\quad b_a \neq 0$.

The following particular subclass of Class D will acquire importance in Section 4:

Class D_0: $a_a = b_a \neq 0$.

Classes A and B are the homogeneous cosmologies; H_3 acts as isometry group. In Classes C and D, H_3 is the non-trivial similarity group, with isometry subgroup G_2. Subdivide the classes into *(Bianchi) types* according to the signature of n^{ab} (the overall sign of the tensor density n^{ab} is no significance). Finally, for certain of the types, one or two invariant, real parameters h or f are needed to specify a_a or b_a, respectively. Define h and f by

$$a_a a_b = \frac{h}{2} \varepsilon_{ace} \varepsilon_{bdf} n^{cd} n^{ef} \quad \text{(Classes } B, D\text{)}, \tag{3.7a}$$

$$b_a b_b = \frac{f}{2} \varepsilon_{ace} \varepsilon_{bdf} n^{cd} n^{ef} \quad \text{(Class } C\text{)}, \tag{3.7b}$$

$$b_a = f a_a \quad \text{(Class } D\text{)}. \tag{3.7c}$$

With these conventions, Table 1 lists all structures for H_3 and G_2, distinct under change of basis $\{\boldsymbol{v}_a\}$. A particular basis is adopted for each

Table 1. Distinct structures of similarity group H_3 and isometry subgroup G_2 for spatially self-similar cosmological models; see Section 3 for notation. Class D contains the important subclass D_0; subclass D_0 is also given separately

Class	Type	Signature of n	a_3	b_3	Dimension
A	I	000	0	0	0
	II	+00	0	0	3
	VI$_0$	$+-0$	0	0	5
	VII$_0$	$++0$	0	0	5
	VIII	$++-$	0	0	6
	IX	$+++$	0	0	6
B	V	000	1	0	3
	IV	+00	1	0	5
	III	$+-0$	1	0	5
	VI$_h$, $-1 \neq h < 0$	$+-0$	$(-h)^{1/2}$	0	6
	VII$_h$, $h > 0$	$++0$	$h^{1/2}$	0	6
C	${}_1$I	000	0	1	3
	${}_1$II	+00	0	1	5
	${}_f$VI$_0$, $f < 0$	$+-0$	0	$(-f)^{1/2}$	6
	${}_f$VII$_0$, $f > 0$	$++0$	0	$f^{1/2}$	6
D	${}_f$V, $f \neq 0$	000	1	f	4
	${}_f$IV, $f \neq 0$	+00	1	f	6
	${}_f$III, $f \neq 0$	$+-0$	1	f	6
	${}^*_1$III	$+-0$	1	[a]	7
	${}_f$VI$_h$, $f \neq 0$, $-1 \neq h < 0$	$+-0$	$(-h)^{1/2}$	$f(-h)^{1/2}$	7
	${}_f$VII$_h$, $f \neq 0$, $h > 0$	$++0$	$h^{1/2}$	$f h^{1/2}$	7
D_0	${}_1$V	000	1	1	3
	${}_1$IV	+00	1	1	5
	${}_1$III	$+-0$	1	1	5
	${}_1$VI$_h$, $-1 \neq h < 0$	$+-0$	$(-h)^{1/2}$	$(-h)^{1/2}$	6
	${}_1$VII$_h$, $h > 0$	$++0$	$h^{1/2}$	$h^{1/2}$	6

[a] For ${}^*_1$III, $b_a = (110)$.

type so that n^{ab} takes Sylvester canonical form and so that (except for Type $^{*}_{\mathrm{I}}$III) $a_1 = a_2 = 0 = b_1 = b_2$. For Type $^{*}_{\mathrm{I}}$III, $a_1 = a_2 = 0 = b_3$, $b_1 = b_2 = 1 = a_3$. Listed are class, type, signature of n^{ab}, a_3, b_3, and dimension (to be explained below). Some authors prefer to view Type III of Class B as the special case $h = -1$ of Type VI_h; but we will distinguish it. For each type except $^{*}_{\mathrm{I}}$III, G_2 is Abelian.

We have emphasized the distinctions among types by insisting on inequivalence under change of basis. Now, let us emphasize the generic relationships among the types, by fixing the basis $\{\boldsymbol{v}_a\}$, and allowing the solution set $\{n^{ab}, a_a, b_a\}$ to wander freely. All solutions sets form a 7-dimensional manifold; each type is a submanifold of dimension (given in Table 1) $\leqq 7$. One type can be viewed as a special case of another if the former forms part or all of the boundary of the latter (equivalent to Lie-group contraction [16]). These relations by specialization are depicted in Fig. 1 (compare MacCallum [17], Collins and Hawking [12]). It is disappointing that the homogeneous Type VIII and IX cosmologies do not generalize.

Construction. Construct, as follows, a standard basis of 1-forms $\{dz, \boldsymbol{\sigma}^a\}$ on a self-similar cosmology $(M, \boldsymbol{g})$. Label the space slices $S(z)$ that are invariant under H_3 by a scalar parameter z; then z is a scalar field on M, invariant under H_3, and freely redefinable by $z \to z' = z'(z)$ [constrained by the requirement $\nabla z \neq 0$, so that z is a nondegenerate

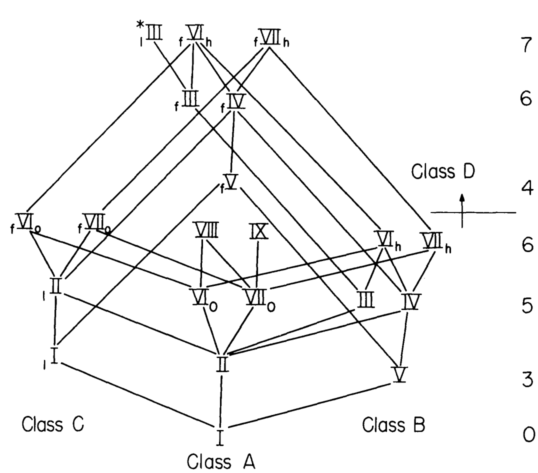

Fig. 1. Specialization diagram for spatially self-similar cosmological models. A line descending from one type to another indicates that the second type lies in the boundary of (i.e., "is a special case of") the first. The number on the right is the dimension of type, reading across horizontally. Classes *A* and *B* are homogeneous; Classes *C* and *D* are non-trivially self-similar. See Section 3

label for the $S(z)$]. Choose a point $p \in S(0)$ on some particular slice $S(0)$. Define a spatial basis of 1-forms $\boldsymbol{\sigma}^a(p)$ at p by

$$\langle \boldsymbol{\sigma}^a(p), \boldsymbol{v}_b(p) \rangle = \delta^a_b, \qquad \langle \boldsymbol{\sigma}^a(p), \nabla z(p) \rangle = 0,$$

for a basis $\{\boldsymbol{v}_a\}$ of H_3; here $\langle\ ,\ \rangle$ is inner product of 1-form and vector on M. Act with H_3 to drag the $\boldsymbol{\sigma}^a$ all over $S(0)$, $\mathscr{L}_{\boldsymbol{v}_a}\boldsymbol{\sigma}^b = 0$ on $S(0)$. The 1-form $b_a\boldsymbol{\sigma}^a$ on $S(0)$ is closed [Eq. (3.3)], $d(b_a\boldsymbol{\sigma}^a) = 0$; hence (locally) exact, $b_a\boldsymbol{\sigma}^a = d\psi$, for some scalar field ψ on $S(0)$, fixed uniquely by the condition $\psi(p) = 0$. Drag ψ all over M along integral curves of ∇z, $\langle d\psi, \nabla z \rangle = 0$ on M. Drag the $\boldsymbol{\sigma}^a$ all over M with the vector field $e^{2\psi}\,\nabla z$. (All these draggings are in fact integrable, by Jacobi's identities on the Lie algebra $\boldsymbol{H}_3 \otimes e^{2\psi}\,\nabla z$.) Then the standard basis of 1-forms is $\{dz, \boldsymbol{\sigma}^a\}$ with

$$d\psi = b_a\boldsymbol{\sigma}^a, \tag{3.8a}$$

$$d\boldsymbol{\sigma}^a = -\tfrac{1}{2}\, C^a{}_{bc}\boldsymbol{\sigma}^b \wedge \boldsymbol{\sigma}^c, \tag{3.8b}$$

$$\boldsymbol{g} = e^{2\psi}(-dz^2 + g_{ab}(z)\boldsymbol{\sigma}^a \otimes \boldsymbol{\sigma}^b). \tag{3.8c}$$

(The freedom to redefine z has been used up to set $g_{zz} = -e^{2\psi}$.)

Alternatively, derive Eqs. (3.8) by applying Proposition 1 to a spatially homogeneous cosmology $(M, {}_0\boldsymbol{g})$. Note that the timelike coordinate z is *not* a proper time, nor does the standard basis belong to a Gaussian-normal coordinate system. For many purposes, one would actually prefer a Gaussian-normal coordinate system, but for this paper, the standard basis will suffice.

Taub's plane-symmetric similarity solutions [2] are special cases of Types ${}_1$I and ${}_f$V.

4. Spatially Self-similar Cosmological Models: Dynamics

Einstein Equations. The Einstein tensor for the self-similar cosmologies is best calculated in an orthonormal basis of 1-forms $\{\omega^0, \omega^i\}$, obtained from the standard basis of Eqs. (3.8). (We follow the notation of [12] rather closely.) Split up the matrix $g_{ab}(z)$ of Eq. (3.8c) by

$$g_{ab} = e^{2\alpha}(e^{2\beta})_{ab}, \tag{4.1}$$

where $\alpha(z)$ is a scalar, and $\beta_{ab}(z)$ is a 3×3, traceless, symmetric matrix. Let

$$\boldsymbol{\omega}^0 = e^{\psi}\, dz, \tag{4.2a}$$

$$\boldsymbol{\omega}^i = e^{\psi}\, e^{\alpha}(e^{\beta})^i_a\,\sigma^a; \tag{4.2b}$$

then

$$\boldsymbol{g} = -\boldsymbol{\omega}^0 \otimes \boldsymbol{\omega}^0 + \boldsymbol{\omega}^i \otimes \boldsymbol{\omega}^i. \tag{4.3}$$

Here indices $i, j, k, \ldots = 1, 2, 3$, run over the orthonormal, spatial basis ω^i; upper and lower need not be distinguished for these spatial indices; and repeated indices are summed. A calculation yields for the Einstein tensor (see Appendix B):

$$e^{2\psi} G^0{}_0 = \tfrac{1}{2}\sigma_{ij}\sigma_{ij} - 3\alpha^{\cdot 2} - \tfrac{1}{2}\cdot {}^3_0R - 2b_k F_{kjj} + b_j b_j\,, \tag{4.4a}$$

$$e^{2\psi} G^0{}_k = \sigma_{ij} C_{ijk} - \sigma_{jk} C_{iij} - 2b_i \sigma_{ik} - 2b_k \alpha^{\cdot}\,; \tag{4.4b}$$

$$e^{2\psi} G^i{}_i = -6\alpha^{\cdot\cdot} - 9\alpha^{\cdot 2} - \tfrac{3}{2}\sigma_{ij}\sigma_{ij} - \tfrac{1}{2}\cdot {}^3_0R - 4b_k F_{kjj} + 5b_j b_j\,, \tag{4.4c}$$

$$\begin{aligned} e^{2\psi}(G^i{}_l - \tfrac{1}{3}\delta_{il} G^j{}_j) &= \sigma^{\cdot}_{il} + 3\alpha^{\cdot}\sigma_{il} + \sigma_{ij}\nu_{jl} - \nu_{ij}\sigma_{jl} \\ &\quad + ({}^3_0R_{il} - \tfrac{1}{3}\delta_{il}\,{}^3_0R) + 2(b_k F_{kil} + b_i b_l) \\ &\quad - \tfrac{2}{3}\delta_{il}(b_k F_{kjj} + b_j b_j)\,. \end{aligned} \tag{4.4d}$$

[*Riem* and $Ric(R_{\mu\nu})$ have dimension $q=0$; the mixed components $G^\mu{}_\nu$ have $q=-2$, since an index has been raised.] Here $^{\cdot} \equiv d/dz$, and

$$\sigma_{ij} = (e^\beta)^{\cdot}{}_{a(i}(e^{-\beta})_{j)}{}^a\,, \tag{4.5a}$$

$$\nu_{ij} = (e^\beta)^{\cdot}{}_{a[i}(e^{-\beta})_{j]}{}^a\,, \tag{4.5b}$$

$$C_{ijk} = (e^\beta)_{ia}(e^{-\beta})_j{}^b (e^{-\beta})_k{}^c C^a{}_{bc}\,, \tag{4.5c}$$

$$F_{ijk} = \tfrac{1}{2}(C_{ikj} + C_{jik} - C_{kji})\,, \tag{4.5d}$$

$$b_i = (e^{-\beta})_i{}^a b_a\,, \tag{4.5e}$$

$${}^3_0R_{il} = -C_{(jk)i}C_{jkl} - C_{(il)k}C_{jjk} + \tfrac{1}{4}C_{ijk}C_{ljk}\,. \tag{4.5f}$$

In Eqs. (4.5), $(e^{-\beta})_i{}^a$ denotes the matrix inverse to $(e^\beta)_{ia}$. The right-hand sides of Eqs. (4.4) are functions of z alone.

Let us presume that the equations of motion of the matter fields ϕ_A follow from the conservation relations $\boldsymbol{\nabla}\cdot\boldsymbol{T}=0$, where $\boldsymbol{T}$ is the stress-energy tensor of matter. (Analogous results can usually be obtained if this presumption is untrue.) The Einstein equations are

$$G^0{}_0 = 8\pi T^0{}_0\,, \tag{4.6a}$$

$$G^0{}_k = 8\pi T^0{}_k\,, \tag{4.6b}$$

$$G^i{}_i = 8\pi T^i{}_i\,, \tag{4.6c}$$

$$G^i{}_l - \tfrac{1}{3}\delta^i{}_l G^j{}_j = 8\pi(T^i{}_l - \tfrac{1}{3}\delta^i{}_l T^j{}_j)\,; \tag{4.6d}$$

these are ordinary differential equations in z. The initial-value problem has the usual structure: Given an initial space slice $S(0)$, the initial data $\{\alpha, \beta_{ab}, \alpha^{\cdot}, \beta^{\cdot}_{ab}, \phi_A(0)\}$ on $S(0)$ must satisfy the constraint equations, Eqs. (4.6a), (4.6b) (σ_{ab} may be given instead of $\beta^{\cdot}_{ab}$; here, $\phi_A(0)$ denotes initial values for the matter fields). Then the evolution equations, Eqs.

(4.6c), (4.6d), have a unique solution $\{\alpha, \beta_{ab}\}$ for some finite interval $z \in [0, z_1]$; and the constraint equations, Eqs. (4.6a), (4.6b), are necessarily satisfied over this entire interval. Alternatively, given the same conditions on initial data, if Eqs. (4.6a) and (4.6d) hold over $z \in [0, z_1]$, then Eqs. (4.6b) and (4.6c) necessarily hold. (All these results follow from the twice-contracted Bianchi identities and the conservation relations.)

Consider for a moment an arbitrary spacetime $(M, \boldsymbol{g})$ with a space slice $S(0)$. Give initial data $\{g_{ab}(x^c), K^a{}_b(x^c), \phi_A(0)\}$ on $S(0)$, satisfying the constraint equations, $G_{00} = 8\pi T_{00}$, $G_{0a} = 8\pi T_{0a}$ (here $\{x^a\}$ are coordinates on $S(0)$, g_{ab} the metric of $S(0)$, and $K^a{}_b$ the second fundamental form). If the initial data are self-similar under a transitive H_3, the discussion above shows that a self-similar solution to the evolution equations exists; by uniqueness of the Cauchy problem [18],

Proposition 2. *Self-similarity of initial data is preserved by the evolution equations of the Einstein equations, for the case H_3 transitive on initial space slice $S(0)$.*

It is reasonable to expect that this result generalizes to the case of an arbitrary self-similarity.

This result seems surprising at first; it is a familiar fact in classical hydrodynamics that an equation of state may exhibit a self-similarity which holds in a certain regime of conditions, but which fails eventually when new physical processes become important (e.g., radiative cooling sufficiently far behind a shock). So one wonders, why can't the matter in such a spacetime eventually enter such a new regime and spoil the self-similarity? The answer is that all regimes are *already* present in the initial slice, because of initial self-similarity; therefore any such "new" regimes would simply spoil self-similarity of the initial data. The implication is that, strictly speaking, only the simplest kinds of matter are allowed in a self-similar spacetime; e.g., dust, electromagnetic field, photon gas. Even mixtures of these are generally disallowed, since the boundary of a region where one component of the mixture dominates another would define an intrinsic scale, spoiling self-similarity. In previous work [1, 2], the possibilities are enriched by allowing matter to be self-similarly shocked, changing equation of state, as it passes a particular *timelike* $S(z)$; such a shock is necessarily due to a singularity in matter fields at some point. We will not attempt to discuss such self-similar shock surfaces.

Lest Proposition 2 seem trivial, let us point out that it seems that this crucial result fails miserably for any other generalized geometric symmetry. E.g., Schwarzschild spacetime admits a conformally flat, time-symmetric Cauchy slice; this slice therefore admits a group C_{10} of conformal symmetries of the initial data; but the conformal group of the spacetime is no larger than the isometry group, G_4.

Action Principle. The value of an action principle as a vehicle for intuition in the study of homogeneous cosmologies was demonstrated by Misner [19]; see also [20]. Let us construct the ADM [21] Hamiltonian action principle for the self-similar cosmologies, working for simplicity only in vacuum.

For homogeneous cosmologies, *such an action principle sometimes leads to wrong field equations* [22, 23]. The danger is that imposing the symmetry does not generally commute with varying the action. The origin of this danger is in spatial surface terms, which need not vanish automatically after imposing the symmetry, since homogeneous variations of the dynamical fields cannot generally be made to vanish on the spatial surface, if these variations are not to vanish identically throughout the interior of the volume of integration. For a complete discussion, see [23]. As for most other properties of homogeneous cosmologies, this danger generalizes to the self-similar case.

The action for a vacuum, self-similar cosmology is

$$16\pi I = \int d^4x\, L(x^\mu)\,, \tag{4.7}$$

where the Lagrangian density $L = (-{}^4g)^{1/2}\, {}^4R$ is a self-similar scalar density of dimension $q=2$, and is therefore of the form

$$L(x^\mu) = e^{2\psi_0} L(z)\,.$$

All dangerous, spatial surface terms derive, upon variation of I, from spatial divergence terms of the form

$$\nabla_a F^a = \nabla_a (e^{2\psi_0} F^a(z))$$

where F^a is a self-similar, spatial, vector-density field of dimension $q=2$, and ∇_a is here covariant differentiation in the metric $e^{2\psi} g_{ab}(z)$ of the space slices $S(z)$. One computes

$$\nabla_a F^a = 2(b_a - a_a) F^a\,,$$

so if $b_a = a_a$, then all dangerous spatial surface terms vanish. Conversely, from detailed inspection of the action principle, if $b_a \neq a_a$, then non-vanishing surface terms are present for all types of cosmologies:

Proposition 3. *The* ADM *action principle, Eqs.* (4.10) *below, gives correct field equations for a self-similar cosmology if and only if $a_a = b_a$; i.e., if and only if the cosmology is of Class A or Class D_0.*

This result generalizes the result of MacCallum and Taub [23]. It may sometimes be possible to "fix up" the Lagrangian for classes other than A and D_0 to cancel the offending surface terms, but it is unlikely that such a procedure can generally succeed.

To proceed with the details: work in the standard basis, Eqs. (3.8); g^{ab} is the inverse of g_{ab}; raise and lower with g_{ab}. The dynamical variables are to be $g_{ab}(z)$ and $\pi_{ab}(z)$, both of dimension $q=0$; $e^{2\psi}g_{ab}(z)$ is the physical metric of the space slices $S(z)$, while the tensor density π_{ab} on $S(z)$ is defined by

$$\pi_{ab}=\tfrac{1}{2}g^{1/2}(\dot{g}_{ab}-g_{ab}g^{cd}\dot{g}_{cd})\,; \tag{4.8}$$

here $g=\det g_{ab}$. Also needed are the Ricci tensor ${}^3_0R_{ab}$ belonging to the (unphysical) metric g_{ab}, and the Ricci scalar ${}^3_0R$:

$${}^3_0R_{ab}=-C^{(cd)}{}_aC_{cdb}-C_{(ab)c}C_d{}^{dc}+\tfrac{1}{4}C_{acd}C_b{}^{cd}\,, \tag{4.9a}$$

$${}^3_0R={}^3_0R_{ab}g^{ab}\,. \tag{4.9b}$$

Follow the procedure of ADM [21] to rewrite Eq. (4.7) in first-order form, and then integrate over a unit volume of the space slices $S(z)$, ignoring all surface terms. There results the action

$$16\pi I_{\mathrm{ADM}}=\int dz\,\pi^{ab}\dot{g}_{ab}\,, \tag{4.10a}$$

to be varied subject to the constraints

$$C^0=0=C^a\,, \tag{4.10b}$$

where

$$C^0=-g^{1/2}[{}^3_0R+(8a_a-2b_a)b^a+g^{-1}(\tfrac{1}{2}\pi^a_a\pi^b_b-\pi^{ab}\pi_{ab})]\,, \tag{4.10c}$$

$$\begin{aligned}C^a&=-2\,{}_0\nabla_b\pi^{ab}-4b_b\pi^{ab}+2b^a\pi^b_b\\&=2g^{ac}[(\pi^d_b-\tfrac{1}{3}\delta^d_b\pi^e_e)C^b{}_{dc}+2\pi^b_c(a_b-b_b)\\&\quad+\pi^d_d(b_c-\tfrac{2}{3}a_c)]\,;\end{aligned} \tag{4.10d}$$

here, ${}_0\nabla_a$ denotes covariant differentiation in a $S(z)$ with respect to g_{ab}. Equations (4.10) (with $a_a=b_a$) constitute the ADM action principle for vacuum, self-similar cosmologies of Classes A and D_0.

Example. Let us only sketch the further development of the ADM method, for only the particular case of vacuum, Type ${}_1\mathrm{VII}_h$, Class D_0 (this case is "generic" in Class D_0: see Fig. 1). The limiting case as $h\to 0$ in Class A is Type VII_o, which has been studied [20].

Let the 3×3 matrix N denote n^{ab}, G denote g_{ab}, and Π denote $\pi^a{}_b$. Choose a standard basis, Eqs. (3.8), so that for all z

$$N=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix},\qquad a_a=b_a=(0,0,h^{1/2})\,. \tag{4.11}$$

Now solve the spatial constraints, $C^a = 0$, explicitly: Diagonalize $G(0)$ in the initial slice $S(0)$. The spatial constraints are equivalent to

$$GM^T G^{\cdot} = G^{\cdot} MG, \quad \text{where} \quad M = N + h^{1/2} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.12}$$

The general solution of Eq. (4.12) for all z is given by transforming to a rotating basis [which still preserves Eq. (4.11)]:

$$G(z) = \Theta\, G^* \,\Theta^T, \qquad \Pi(z) = \Theta\, \Pi^* \,\Theta^T, \tag{4.13}$$

where $G^*(z)$ is diagonal; we write G^* in terms of dynamical coordinates $\beta_\pm(z)$, $\Omega(z)$ as

$$G^* = \exp 2\, \mathrm{diag}(\beta_+ + \sqrt{3}\,\beta_- - \Omega,\, \beta_+ - \sqrt{3}\,\beta_- - \Omega,\, -2\beta_+ - \Omega)\,; \tag{4.14}$$

and where

$$\Theta(z) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}; \tag{4.15}$$

here the angle of rotation $\theta(z)$ is not a dynamical coordinate, but is fixed in terms of $\beta_\pm$ and Ω by

$$\theta(0) = 0, \qquad d\theta = -h^{1/2}\, \mathrm{csch}^2(2\sqrt{3}\,\beta_-)\,(d\beta_+ - d\Omega)\,. \tag{4.16}$$

The momentum $\Pi^*(z)$ takes the general form (dictated by the requirement that $p_\pm$, p_Ω be canonically conjugate to $\beta_\pm, \Omega$)

$$\begin{aligned} 12\Pi^* = {} & \mathrm{diag}(p_+ + \sqrt{3}\,p_- - 2p_\Omega,\, p_+ - \sqrt{3}\,p_- - 2p_\Omega,\, -2p_+ - 2p_\Omega) \\ & + h^{1/2}\, \mathrm{csch}(2\sqrt{3}\,\beta_-) \\ & \begin{pmatrix} -h^{1/2}\,\mathrm{csch}(2\sqrt{3}\,\beta_-) & \exp(-2\sqrt{3}\,\beta_-) & 0 \\ \exp(2\sqrt{3}\,\beta_-) & -h^{1/2}\,\mathrm{csch}(2\sqrt{3}\,\beta_-) & 0 \\ 0 & 0 & 0 \end{pmatrix} (p_+ + p_\Omega)\,. \end{aligned} \tag{4.17}$$

With the substitutions, Eqs. (4.13)–(4.17), the ADM action, Eqs. (4.10), takes the form

$$16\pi I_{\mathrm{ADM}} = \int (p_+\, d\beta_+ + p_-\, d\beta_- + p_\Omega\, d\Omega)\,, \tag{4.18a}$$

to be varied subject to the "Hamiltonian" constraint, $C^0 = 0$, which is now

$$0 = p_+^2 + p_-^2 - p_\Omega^2 - \frac{h}{3}\,\mathrm{csch}^2(2\sqrt{3}\,\beta_-)\,(p_+ + p_\Omega)^2 + e^{-4\Omega}\, V(\beta_\pm)\,, \tag{4.18b}$$

where

$$V(\beta_\pm) = 48\, e^{4\beta_+}\, \sinh^2(2\sqrt{3}\,\beta_-)\,. \tag{4.18c}$$

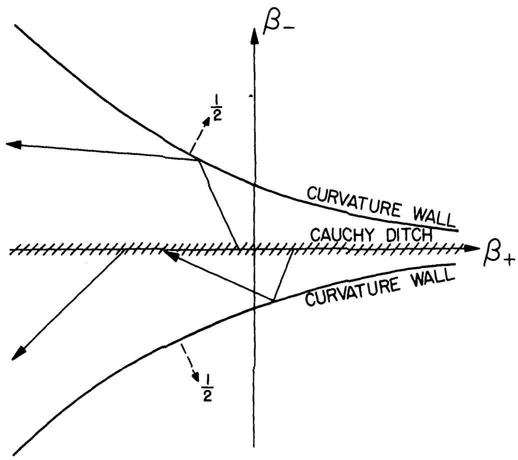

Fig. 2. Dynamical histories of vacuum, Class D_0; Type ${}_1\mathrm{VII}_h$, self-similar cosmologies, traced (lines with arrowheads) in the plane of the dynamical coordinates $\beta_\pm$ (see Section 4). Far from the walls or ditch, the system point moves at velocity 1. The "curvature walls" (heavy lines) move with velocity 1/2 (dotted arrows). The "Cauchy ditch" (hatched line) remains fixed; it represents the boundary of our coordinate patch

The usual methods of exegesis (see [20] and references cited therein) may now be employed to discover the qualitative behavior of the of the cosmology. Take the Hamiltonian $H \equiv -p_\Omega$ to be defined by Eq. (4.18b); then the equations of motion in Ω of $\beta_\pm$ and $p_\pm$ are given by $\delta I_{\mathrm{ADM}} = 0$, yielding Hamilton's equations.

Two loci in the $\beta_\pm$-plane (space of dynamical coordinates) are of great importance (Fig. 2):

1) The "curvature wall" is defined by $e^{-4\Omega} V(\beta_\pm) = 1$ (it moves at velocity 1/2). The system point moves at velocity $\leqq 1$; it must lie to the left of the wall. If the system point encounters the wall, it "bounces".

2) The Hamiltonian H is singular at $\beta_- = 0$; this fixed locus we call the "Cauchy ditch". If the system point encounters the Cauchy ditch, it fall in and disappears (see further discussion below).

Then the history of the system point takes three generic forms (sketched in Fig. 2). Emerging from the Cauchy ditch, the system point can: 1) coast infinitely far to the left, never hitting the curvature wall, or 2) bounce once off the wall and coast infinitely far to the left, or 3) bounce once off the wall and fall back into the ditch. As is well known, coasting far to the left represents a "Kasner-like" [24] or "velocity-dominated" [25] singularity in spacetime.

The "Cauchy ditch" has the following significance: When we constructed (locally) a coordinate patch for the spatially self-similar cosmologies in Section 3, we assumed that the hypersurfaces $S(z)$ of transitivity of H_3 were spacelike. But there is no necessary global geometric restriction on the causal structure of the $S(z)$; in an inextensible spacetime $(M, \boldsymbol{g})$, self-similar under H_3, some of the $S(z)$ may be spacelike and some timelike, with a null hypersurface $S(z_0)$ at the boundary between regions (see Section 5 for examples). From the dynamical point of view, this null $S(z_0)$ is a Cauchy horizon [18] of the initial slice $S(0)$. The vacuum spacetime is not singular on the Cauchy horizon, but there is generally no unique extension through the horizon. If one assumes analyticity (i.e., assumes "no news" on the other side), it should be possible to uniquely or almost uniquely extend a spatially self-similar cosmology through a Cauchy horizon (as it is possible for certain homogeneous cosmologies [26]), but this procedure has not been carried out. In the present formalism, "falling in the Cauchy ditch" in Fig. 2 is equivalent to the spacetime evolving up to such a Cauchy horizon, and thus reaching the boundary of the present coordinate patch.

5. Discussion

Examples. Let us discuss the self-similarity of some simple cosmological models; here, $\boldsymbol{g}$ is the physical metric, and $\boldsymbol{u}$ a generator of similarities. (Most of these models are treated in [18].)

Minkowski Spacetime:

$$\boldsymbol{g} = -dt^2 + dx^i\,dx^i \qquad (i = 1, 2, 3)\,; \tag{5.1a}$$

$$\boldsymbol{u} = t\,\partial_t + x^i\,\partial_i\,. \tag{5.1b}$$

The vector field $\boldsymbol{u}$ is unique only up to spatial translations $\boldsymbol{u} \to \boldsymbol{u} + a^i\,\partial_i$ (true also for examples below), and time translations $\boldsymbol{u} \to \boldsymbol{u} + b\,\partial_t$. The similarity group is $H_{11} \supset G_{10}$; here G_{10} = Poincaré group.

$k = 0$ Robertson-Walker:

$$\boldsymbol{g} = -\,dt^2 + R^2(t)\,dx^i\,dx^i, \qquad tR^{\cdot}/R = \text{const}\,; \tag{5.2a}$$

$$\boldsymbol{u} = t\,\partial_t + (1 - tR^{\cdot}/R)x^i\,\partial_i\,. \tag{5.2b}$$

Here $^{\cdot} \equiv d/dt$, and the restriction $tR^{\cdot}/R = \text{const}$ is equivalent to $R \propto t^p$, where p is a constant exponent. For $p \geqq 2/3$, these are solutions of the Einstein equations for hydrodynamic matter with equation of state $P = (\gamma - 1)\varrho$, where $\gamma = 2/(3p)$. Other equations of state generally break self-similarity. E.g., the standard, $k = 0$, "hot big-bang", model of our universe is asymptotically self-similar before and after onset of "matter dominance," but not during. The similarity group of the spacetime, Eqs. (5.2), is $H_7 \supset G_6$; H_7 is transitive on spacetime. Note that $\boldsymbol{u}$ is spacelike in some regions and timelike in others, as is the case in all these examples.

$k \neq 0$ Robertson-Walker:

None of these models admit exact, non-trivial self-similarities (except for unphysical equation of state, $P = -\varrho/3$). As is well known, these models approximate $k=0$ models sufficiently close to the big bang; we may say that at times sufficiently early that the intrinsic scale defined by spatial curvature is dynamically unimportant, the models are asymptotically self-similar. At large times, $k=-1$ models approach Minkowski spacetime, and are again asymptotically self-similar.

Kasner:

$$\boldsymbol{g} = -dt^2 + \Sigma_i t^{2p_i}(dx^i)^2, \qquad \Sigma_i p_i = 1 = \Sigma_i p_i^2\,; \tag{5.3a}$$

$$\boldsymbol{u} = t\,\partial_t + \Sigma_i(1-p_i)x^i\,\partial_i\,. \tag{5.3b}$$

Here, the p_i are constants. These vacuum spacetimes are exactly self-similar, with $H_4 \supset G_3$. A wide class of spacetime singularities [24, 25] are approximated by this metric sufficiently near the singularity. Therefore, such singularities are asymptotically self-similar: there are no dynamically important, intrinsic scales, sufficiently close to the singularity.

Heckmann-Schücking:

$$\boldsymbol{g} = -dt^2 + \Sigma_i t^{2p_i}(t+t_0)^{4/3-2p_i}(dx^i)^2, \qquad t_0 = \text{const},$$

$$p_i = \text{const} \quad \text{as in (5.3a)}.$$

These anisotropic, dust universes have only a G_3, with no non-trivial self-similarities. But for $t \ll t_0$ (matter dynamically unimportant) they are approximated by Kasner models; for $t \gg t_0$ (anisotropy dynamically unimportant) they are approximated by $k=0$ Friedmann (dust Robertson-Walker) models. In each of these two regimes of t, the Heckmann-Schücking models are asymptotically self-similar.

Mixmaster:

These are vacuum, Type IX homogeneous cosmologies; see [19, 20, 27]. Generally, these remarkable models never settle down to asymptotic self-similarity near the "oscilliatory" [27] singularity. Each "bounce" represents an evanescently important scale; there are an infinite number of dynamically important scales, tending to zero. Perhaps the average conservation of Misner's Hamiltonian signifies an approximate, stochastic scale invariance, analogously to the case of hydrodynamic turbulence.

Self-Similar Cosmologies that Admit $k=0$ Robertson-Walker:

The $k=0$ Robertson-Walker spacetimes, Eqs. (5.2), are special cases of self-similar cosmologies of Types ${}_f$III, ${}_f$V, and ${}_f$VII$_h$ [for $b_3 = (1 - tR^{\cdot}/R)^{-1}$]. Therefore, exploration of "nearby" self-similar cosmologies may possibly be of astrophysical relevance; compare [12]. Since $\boldsymbol{u}$, Eq. (5.2b), is timelike in some regions, Cauchy horizons appear in these self-similar cosmologies.

A Conservation Law. Let $\boldsymbol{u}$ generate a non-trivial self-similarity on a spacetime $(M, \boldsymbol{g})$, endowed with a matter stress-energy tensor $\boldsymbol{T}$. Here, $\boldsymbol{T}$ need *not* be self-similar; e.g., $\boldsymbol{T}$ may belong to any arbitrary distribution of test matter. Define the current $\boldsymbol{P}$,

$$P^\mu = T^{\mu\nu} u_\nu . \tag{5.4}$$

From Eqs. (2.3), and the conservation relations $\nabla \cdot \boldsymbol{T} = 0$,

$$\nabla \cdot \boldsymbol{P} = \langle b, \boldsymbol{u} \rangle T, \qquad \text{where} \quad T = \text{Trace}(\boldsymbol{T}) . \tag{5.5}$$

So if $T \equiv 0$ (matter of zero rest-mass), *then* $\boldsymbol{P}$ *is conserved.* In particular, there is a conserved quantity $\boldsymbol{k} \cdot \boldsymbol{u}$ along an (affinely parametrized) null geodesic with tangent $\boldsymbol{k}$ in any self-similar spacetime:

$$\nabla_{\boldsymbol{k}}(\boldsymbol{k} \cdot \boldsymbol{u}) = 0 . \tag{5.6}$$

Even if $T \neq 0$ in Eq. (5.5), this equation may be of use. In particular, for a timelike geodesic with tangent $\boldsymbol{k}$, $\boldsymbol{k} \cdot \boldsymbol{k} = -1$,

$$\nabla_{\boldsymbol{k}}(\boldsymbol{k} \cdot \boldsymbol{u}) = -\langle b, \boldsymbol{u} \rangle \Rightarrow \boldsymbol{k} \cdot \boldsymbol{u} = -\langle b, \boldsymbol{u} \rangle t + \text{const} , \tag{5.7}$$

where t = proper time along geodesic. The first integral, Eq. (5.7), helps one to integrate the geodesic equations.

Let us examine the physical consequences of this conservation law, as applied to null geodesics, Eq. (5.6). Consider a $k = 0$ Robertson-Walker model, Eqs. (5.2). The deceleration parameter,

$$q_0 = -RR^{\cdot\cdot}/R^{\cdot 2} ,$$

is a dimensionless physical observable, and therefore must be constant on M, because the similarity group H_7 acts transitively on M. But let us forget this argument, and rederive the result, $q_0 = \text{const.}$, using self-similarity *only* in the form of Eq. (5.6). Adopt coordinates so that our galaxy lies at $x = 0$, $t = t_0$. Observe other, "standard-candle", galaxies at various distances along a single, narrow pencil $\mathscr{P}$ of past-pointing null geodesics, with tangent $\boldsymbol{k}$. Assign thereby pairs (r_L, z) to these galaxies, where r_L = luminosity-distance, and z = redshift. Consider r_L as independent variable; therefore express the observational data as a function $z(r_L)$. One has

$$1 + z = R(t_0)/R(t) , \tag{5.8a}$$

$$r_L = R(t_0)\, x, \quad \text{where} \quad x \equiv \sqrt{(x^i x^i)} , \tag{5.8b}$$

or

$$d r_L = -(1 + z)\, dt \quad \text{along } \mathscr{P} ; \tag{5.8c}$$

and

$$k = (1+z)(-\partial_t + R^{-1} x^{-1} x^i \partial_i) . \tag{5.8d}$$

By Eqs. (5.8), $q_0(r_L)$ along our past light cone can be extracted directly from the observational data:

$$q_0 = (1+z) z''/z'^2 - 1 , \tag{5.9a}$$

where $z' \equiv dz/dr_L$, etc. Now let

$$\sigma \equiv k \cdot u = (1 + z - r_L z') t + r_L ; \tag{5.9b}$$

if the universe is self-similar, i.e., if

$$t \dot{R}/R = \text{const.} , \tag{5.10}$$

then σ is conserved along $\mathscr{P}$ [Eq. (5.6)]. From Eqs. (5.9),

$$t q_0' = -[(1+z)\sigma'/r_L z'^2]' . \tag{5.11}$$

Therefore, self-similarity of the universe implies $\sigma' = 0$, which in turn implies by Eq. (5.11) *constancy of* q_0 *along our past light cone.*

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Appendix A. Exceptional Self-similar Spacetimes

Here we complete the discussion and proof of Proposition 1, Section 2.

A (vacuum or non-vacuum) *plane-wave spacetime* (M, g) is a specetime with metric form [28]

$$\begin{gathered} g = 2K_{AB}(u) x^A x^B du^2 + 2 du\, dv + \delta_{AB} dx^A dx^B , \\ A, B, \ldots = 2, 3 , \end{gathered} \tag{A.1}$$

in some local coordinates. Let $K(u)$ denote the 2×2-matrix function of u with components $K_{AB}(u)$. (M, g) is vacuum if and only if $K(u)$ is identically traceless; (M, g) is always conformal to a vacuum plane-wave spacetime. (M, g) is conformally flat if an only if $K(u)$ is a pure trace, $K(u) = f(u) I$, where $f(u)$ is an arbitrary scalar function, and I is the 2×2 identity. (M, g) is flat if and only if $K(u) \equiv 0$.

Generally, (M, g) admits a G_5 transitive on the null hypersurfaces $u = \text{const.}$ [28]. Generally, (M, g) admits a non-trivial H_6, the additional generator being

$$w = 2v\,\partial_v + x^A \partial_A .$$

(The conformal group C_6 is generally equal to H_6. Higher symmetries arise in special cases; H_n is always non-trivial.)

The physical nature of this similarity is simple. For each observer, $\boldsymbol{w}$ generates a transformation which is a composition of a boost in the direction of propagation of the wave, with a scale transformation which exactly cancels the Doppler shift induced by the boost, leaving the wave invariant. Since $\boldsymbol{w}u = 0$, the vector field $\boldsymbol{w}$ is pointwise linearly dependent upon the generators of G_m; therefore Eqs. (2.7) admit no solution, and $(M, \boldsymbol{g})$ cannot fall under Case 2a) of Proposition 1.

By long and tedious calculations based on Cases B) and C) of Theorem 1, one shows conversely: Any spacetime $(M, \boldsymbol{g})$ obeying the hypotheses of Proposition 1 that does not fall under Case 2a), necessarily is of the plane-wave form, Eq. (A.1) and therefore falls under Case 2b). (Such a simple result begs for a simple demonstration, but the author is unaware of one.) These are all the "exceptional self-similar spacetimes" that fall under Case 2b) (c denotes a real constant):

$G_5 \subset H_6$ (general case): $\boldsymbol{K}(u)$ not as below.

$G_6 \subset H_7$: Let $h(u)$ be a scalar function, not c or cu^{-2}; then $\boldsymbol{K}(u) = h(u)\,\boldsymbol{I}$.

$G_6 \subset H_7$: Let $\boldsymbol{R}(\theta)$ be a 2×2 rotation matrix,

$$R_{AB}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix};$$

let $\boldsymbol{L}$ be a 2×2 constant matrix, not a pure trace; then

$$\boldsymbol{K}(u) = \boldsymbol{R}(cu)\,\boldsymbol{L}\boldsymbol{R}^T(cu)\,.$$

$G_6 \subset H_7$: Let $\boldsymbol{R}(\theta)$ and $\boldsymbol{L}$ be as above; then

$$\boldsymbol{K}(u) = u^{-2}\,\boldsymbol{R}(c\ln u)\,\boldsymbol{L}\boldsymbol{R}^T(c\ln u)\,.$$

$G_7 \subset H_8$: Here

$$\boldsymbol{K}(u) = c\boldsymbol{I}\,, \qquad c \neq 0\,.$$

$G_7 \subset H_8$: Here

$$\boldsymbol{K}(u) = cu^{-2}\boldsymbol{I}\,, \qquad c \neq 0\,.$$

$G_{10} \subset H_{11}$ (Minkowski spacetime):

$$\boldsymbol{K}(u) \equiv 0\,.$$

Appendix B. Useful Formulae for Spatially Self-similar Cosmologies

It is not necessary to have explicit formulae for ψ and the σ^a; Eqs. (3.8a), (3.8b) are sufficient.

In the orthonormal basis, Eqs. (4.2), the connection forms $\omega^\mu{}_\nu$, defined by $d\omega^\mu = -\omega^\mu{}_\nu \wedge \omega^\nu$, $\eta_{\varrho[\mu}\omega^\varrho{}_{\nu]} = 0$, are

$$\omega^0{}_i = e^{-\psi} b_i \omega^0 + e^{-\psi} \kappa_{ij} \omega^j, \tag{B.1a}$$

$$\omega^i{}_j = -e^{-\psi} v_{ij} \omega^0 + e^{-\psi}[F_{ijk} - b_i \delta_{jk} + b_j \delta_{ik}]\, \omega^k; \tag{B.1b}$$

the curvature forms, $\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\lambda \wedge \omega^\lambda{}_\nu$, are

$$\begin{aligned}\Omega^0{}_i = e^{-2\psi}[&\kappa^{\cdot}_{ik} + \kappa_{ij}\kappa_{jk} + \kappa_{ij} v_{jk} - v_{ij}\kappa_{jk} \\ &+ b_j F_{j(ik)} + b_i b_k - b_j b_j \delta_{ik}]\, \omega^0 \wedge \omega^k \\ &+ e^{-2\psi}[\kappa_{jk} F_{jil} - \tfrac{1}{2}\kappa_{ij} C_{jkl} - b_j \kappa_{jk}\delta_{il}]\, \omega^k \wedge \omega^l,\end{aligned} \tag{B.2a}$$

$$\begin{aligned}\Omega^i{}_j = e^{-2\psi}[&(2\kappa_{il}F_{klj})_{[ij]} + \kappa_{kl} C_{lij} - (2b_l \kappa_{lj}\delta_{ik})_{[ij]}]\, \omega^0 \wedge \omega^k \\ &+ e^{-2\psi}[\kappa_{il}\kappa_{jm} - \tfrac{1}{2} F_{ijk} C_{klm} - F_{ikl} F_{jkm} \\ &- 2(b_k F_{k(jl)}\delta_{im} + b_j b_l \delta_{im})_{[ij]} - \delta_{il}\delta_{jm} b_k b_k]\, \omega^l \wedge \omega^m;\end{aligned} \tag{B.2b}$$

Eqs. (4.4) follow from $\Omega^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\varrho\sigma}\omega^\varrho \wedge \omega^\sigma$, $R_{\nu\sigma} = R^\mu{}_{\nu\mu\sigma}$.

Here, notation follows Eqs. (4.5), plus

$$\kappa_{ij} = \sigma_{ij} + \delta_{ij}\alpha^{\cdot}. \tag{B.3}$$

For subclass $D_0(a^a = b^a \neq 0)$ and Class $A(a^a = 0 = b^a)$, the Einstein tensor is very simple; in the standard basis, Eqs. (3.8),

$$e^{2\psi} G^0{}_0 = \tfrac{1}{2}\sigma^a{}_b \sigma^b{}_a - 3\alpha^{\cdot 2} + \tfrac{1}{2} U, \tag{B.4a}$$

$$e^{2\psi} G^0{}_c = \sigma^a{}_b (n^{bd} + \varepsilon^{bdf} a_f)\, \varepsilon_{dac} - 2\alpha^{\cdot} a_c, \tag{B.4b}$$

$$e^{2\psi}(G^a{}_a + 3G^0{}_0) = -6\alpha^{\cdot\cdot} - 18\alpha^{\cdot 2} + 2U, \tag{B.4c}$$

$$e^{2\psi}(G^a{}_b - \tfrac{1}{3}\delta^a{}_b G^c{}_c) = (\sigma^{\cdot})^a{}_b + 3\alpha^{\cdot}\sigma^a{}_b + U^a{}_b; \tag{B.4d}$$

where

$$\sigma^a{}_b = (\beta^{\cdot})^a{}_b, \tag{B.5a}$$

$$U = g^{-1}(n^{ab} n^{cd} - \tfrac{1}{2} n^{ac} n^{bd})\, g_{ac} g_{bd}, \tag{B.5b}$$

$$U^{ab} = \delta U/\delta g_{ab} - \tfrac{1}{3} g^{ab} g_{cd}\, \delta U/\delta g_{cd}; \tag{B.5c}$$

so

$$\begin{aligned}U^a{}_b = g^{-1} n^{ac} n^{de} g_{ce} g_{bd} + \tfrac{1}{2} n^{cd} n^{ef} \varepsilon_{ceg}\varepsilon_{dfb} g^{ag} \\ - \tfrac{1}{6} g^{-1} \delta^a{}_b (n^{cd} n^{ef} + n^{ce} n^{df})\, g_{ce} g_{df};\end{aligned} \tag{B.5d}$$

g_{ab} is as in Eq. (4.1). Observe that a_a enters only through Eq. (B.4b). Therefore, in the ADM formulation, a_a enters only in the spatial constraints, Eq. (4.10d). The Hamiltonian constraint and the evolution equations are the same for a subclass D_0 cosmology as for the corresponding, limiting Class A cosmology.

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D. M. Eardley
California Institute of Technology
Pasadena, California 91109, USA