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The $\lambda\varphi_3^4$ Field Theory in a Finite Volume

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Abstract. The unnormalized doubly cutoff Schwinger functions converge as the ultraviolet cutoff is removed. The limits, the finite volume unnormalized Schwinger functions, are tempered distributions and are C^∞ in the coupling constant. They have asymptotic expansions given by perturbation theory. For λ sufficiently small they can be normalized and then they are the moments of a measure on $\mathscr{S}'_R(\mathbb{R}^3)$.

The $P(\varphi)_2$ models are the best behaved models studied in constructive field theory. The Wightman axioms have been verified for these theories (for weak coupling), firmly establishing their existence, and work related to $P(\varphi)_2$ is now largely aimed at determining physical properties and simplifying earlier proofs. The $\lambda\varphi_3^4$ model, which we are considering in this paper, is the next best behaved boson model. It differs from $P(\varphi)_2$ by having ultraviolet divergences and by requiring ultraviolet divergent mass and wave function as well as vacuum energy renormalizations. Work on $\lambda\varphi_3^4$ is still aimed at establishing its existence. The principal progress in this direction has been the proof of the existence [2] and semiboundedness [3] of the spatially cutoff Hamiltonian. In this paper we use the methods of [3] to show that the (unnormalized) spatially cutoff Schwinger functions exist, are tempered distributions, and are C^∞ in the coupling constant. If λ is small we can normalize the Schwinger functions and then they are the moments of a probability measure on $\mathscr{S}'_R(\mathbb{R}^3)$. The next step in the program might involve the use of methods developed for $P(\varphi)_2$ (see [4, 5]) to take the infinite volume limit and verify the Wightman axioms. Another open problem is that of determining if, as conjectured, the free and (spatially cutoff) interacting measures are mutually singular.

Readers are referred to [3] and [5] for further background material, notation and references and for details related to the inductive expansion.

We will be concerned solely with the Euclidean approach to φ_3^4. The free theory is given on the path space $L^2(\mathscr{S}'_R(\mathbb{R}^3), dq_0)$ where dq_0 is the Gaussian measure with mean zero and covariance $\mu^{-2} = (-\Delta + 1)^{-1}$.

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The Euclidean fields are the linear coordinate functions on $\mathscr{S}'_R(\mathbb{R}^3)$:

$$\Phi(f)(q) = \langle q, f \rangle$$

for all $q \in \mathscr{S}'_R(\mathbb{R}^3)$ and $f \in \mathscr{S}(\mathbb{R}^3)$. The partition function Z and unnormalized Schwinger functions ZS_n of the doubly cutoff interacting theory are just the mass and moments of the unnormalized doubly cutoff interacting measure $dq(\kappa, \lambda g)$.

$$Z(\kappa, \lambda g) = \langle 1 \rangle_{dq(\kappa, \lambda g)}$$

$$Z(\kappa, \lambda g)\, S_n(\kappa, \lambda g; f_1, \ldots, f_n) = \langle \Phi(f_1) \ldots \Phi(f_n) \rangle_{dq(\kappa, \lambda g)} .$$

The measure is given by

$$dq(\kappa, \lambda g) = e^{-V(\kappa, \lambda g)} dq_0$$

$$V(\kappa, \lambda g) = V_I(\kappa, \lambda g) + V_C(\kappa, \lambda g)$$

$$V_I(\kappa, \lambda g) = \lambda : \Phi_\kappa^4 : (g)$$

$$V_C(\kappa, \lambda g) = \tfrac{1}{2} \langle V_I^2(\kappa, \lambda g) \rangle_{dq_0} - \tfrac{1}{6} \langle V_I^3(\kappa, \lambda g) \rangle_{dq_0} - \tfrac{1}{2} \lambda^2 \delta m^2(\kappa) : \Phi_\kappa^2 : (g^2)$$

$$\delta m^2(\kappa) = -4^2 \times 6 \times (2\pi)^{-9} \int \delta^{(3)}(k_2 + k_3 + k_4) \cdot \prod_{i=2}^{4} \mu(k_i)^{-2} \kappa^2(k_i)\, d^3 k_i .$$

Here : : means Wick ordering with respect to $dq_0, \mu(k) \equiv (1 + \|k\|^2)^{\frac{1}{2}}$ and $\|k\|^2 = k^{(0)^2} + k^{(1)^2} + k^{(2)^2}$. We assume that the space cutoff $0 \leqq g \leqq 1$ is the product of a function in $C_0^\infty(\mathbb{R}^3)$ and the characteristic function of a union of unit lattice cubes. We also assume that the momentum cutoff κ is of the form

$$\kappa(k) = \prod_{i=0}^{2} \left[\eta\left(\frac{k^{(i)}}{\beta^{(i)}}\right) - \eta\left(\frac{k^{(i)}}{\alpha^{(i)}}\right) \right]$$

$$\alpha^{(i)} \leqq \beta^{(i)}$$

$$\alpha^{(i)}, \beta^{(i)} \in \{ M_0 = 0, M_j = M_1^{(1+\nu)^{j-1}} \text{ if } j \geqq 1 \}$$

where $M_1 > 1$ and $v > 0$ are constants given in [3] and η is a fixed $C_0^\infty(\mathbb{R}^1)$ function satisfying

$$\begin{aligned} \eta(x) &= 1 & |x| &\leqq \tfrac{1}{2} \\ 0 < \eta(x) &< 1 & \tfrac{1}{2} < |x| &< 2 \\ \eta(x) &= 0 & |x| &\geqq 2. \\ \eta(x) &= \eta(-x) \end{aligned}$$

By convention $\eta(k/0) \equiv 0$.

Note that the scalar counterterms in V_C are those suggested by the perturbation theory of the Euclidean Green's functions (i.e. Schwinger functions) and hence have a built-in wave function renormalization. See [3].

Theorem 1. a) *There exists a constant* $K_2(\lambda)$ *and a Schwartz space norm* $|\cdot|$, *such that*

$$|Z(\kappa, \lambda g)\, S_n(\kappa, \lambda g; f_1, \ldots, f_n)| \leqq n!\, |f_1| \cdots |f_n|\, e^{K_2 A(g)}$$

where $A(g)$ *is the volume of the set of points within a distance one of the support of* g.

b) $Z(1, \lambda g) = \lim_{\kappa \to 1} Z(\kappa, \lambda g)$ *and*

$$Z(1, \lambda g)\, S_n(1, \lambda g; f_1, \ldots, f_n) = \lim_{\kappa \to 1} Z(\kappa, \lambda g)\, S_n(\kappa, \lambda g; f_1, \ldots, f_n)$$

exist and obey the above bounds. By $\kappa \to 1$ *we mean*

$$\mathrm{glb}\{\|k\| \mid \kappa(k) \neq 1\} \to \infty\,.$$

c) $Z(1, \lambda g)$ *and* $Z(1, \lambda g)\, S_n(1, \lambda g; f_1, \ldots, f_n)$ *are* C^∞ *in* λ. *They have asymptotic expansions given by perturbation theory.* $Z(1, \lambda g) \neq 0$ *if* $0 \leqq \lambda < \lambda_0(A(g))$.

d) *If* $0 \leqq \lambda < \lambda_0$ *there exists a unique measure* $dq(1, \lambda g)$ *on* $\mathscr{S}'_R(\mathbb{R}^3)$ *such that*

$$S_n(1, \lambda g; f_1, \ldots, f_n) = \langle \Phi(f_1) \ldots \Phi(f_n) \rangle_{dq(1, \lambda g)}\,.$$

Theorems 1 a)–d) are corollaries of Theorems 2–5 respectively. These results are very much in the spirit of Symanzik's program to formulate field theory in terms of moments of probability measures [6].

We will also be dealing with expectation values of somewhat more complicated objects than the product of fields $\Phi(f_1) \ldots \Phi(f_n)$. These will be products of Wick monomials that have some contractions between different monomials. Each monomial of order n is represented in graph notation by a vertex with n legs. A contraction joins two legs, one from each vertex involved, to form a line. In general a graph G and its kernels w_v are used to represent the function $G(q)$ on $\mathscr{S}'_R(\mathbb{R}^3)$ given by

$$G(q)=\int \prod_{v\in V}\left[:\prod_{\ell\in U(v)}\hat{\Phi}(k_\ell):w_v\right]\left[\prod_{(\ell_1,\ell_2)\in\mathscr{C}_G}\delta^{(3)}(k_{\ell_1}+k_{\ell_2})\right]\prod_{\ell\in\cup_v L(v)}d^3k_\ell$$

$V=$ set of all vertices in G;

$L(v)=$ set of all legs of the vertex v;

$U(v)=$ set of all uncontracted legs of v;

$\mathscr{C}_G\subset\bigcup_{v_1\neq v_2}L(v_1)\times L(v_2)$ is the set of all contractions;

$\hat{\Phi}(k)=\Phi((2\pi)^{-3/2}\mu(k)\,e^{ik\cdot x})$ formally

$=A^*(k)+A(-k)$ in Fock space language;

w_v, the kernel of the vertex v, is a function of k_ℓ for all $\ell\in L(v)$. The kernel of $:\Phi^n_\kappa(g):$ is given by

$$w(k_1,\dots,k_n)=(2\pi)^{-3n/2}\,\tilde{g}(k_1+\dots+k_n)\,\kappa(k_1,\dots,k_n)\prod_{\ell=1}^{n}\mu(k_\ell)^{-1}$$

where $\tilde{g}$ is the Fourier transform of a space-time cutoff and κ is a momentum cutoff.

The notation G may refer, depending on context, to the topological graph G, the function $G(q)$ or the kernel $G(k_\ell)$. The last is the function of the momenta of G's external legs given by

$$\int\prod_{v\in V}w_v\prod_{(\ell_1,\ell_2)\in\mathscr{C}}\delta^{(3)}(k_{\ell_1}+k_{\ell_2})\,d^3k_{\ell_1}d^3k_{\ell_2}\,.$$

By choosing i of the external legs to be initial legs and the remainder to be final legs we can view $G(k_\ell)$ as the kernel of an integral operator from $L^2(\mathbb{R}^{3i})$ to $L^2(\mathbb{R}^{3f})$. $\|G\|_{i,f}$ is the norm of this operator. $\|G\|_{\text{H.S.}}$ is the Hilbert-Schmidt norm of the kernel.

We will be interested in two different estimates on $\langle G\rangle_{dq(\kappa,\lambda g)}$. The first emphasizes the kernel of G while the second emphasizes the space-time density (as opposed to the total number) of external G legs. There is a norm on the kernel of G appropriate to each.

Given $\delta>2\alpha>0$ we define

$$\|G\|_{1,\delta,\alpha}=\sup_{\mathscr{P}^e_\alpha}\sup_{\mathscr{C}}\|\mathscr{P}^e_\alpha\mathscr{C}M^\delta|G|\,\|_{\text{H.S.}}$$

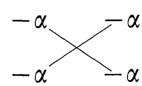
$$\|G\|_{2,\delta,\alpha}=\sup_{\mathscr{P}^e_\alpha}\sup_{\mathscr{C}}\sup_{D}\|\mathscr{P}^e_\alpha\mathscr{C}M^\delta|D\mathscr{T}G|\,\|_{\text{H.S.}}$$

$\mathscr{P}_\alpha^e, \mathscr{C}, D, \mathscr{T}, M^\delta$, and $|\cdot|$ are "operators" that modify the graph G and its kernel.

(1) M^δ multiplies each external leg ℓ by $K_1 \mu(k_\ell)^\delta$. K_1 is a constant to be chosen later.

(2) $\mathscr{C}$ is any contraction scheme on G's external legs. $\mathscr{C}$ need not contract up all the legs. We identify the contraction scheme $\mathscr{C}$ with the operator $\mathscr{C}$ that applies it.

(3) $|\cdot|$ takes the absolute value of the kernel to which it is applied.

(4) $\mathscr{P}_\alpha^e$ takes a collection of identical vertices 

(the $-\alpha$ means multiply that leg by $\mu^{-\alpha}$) and connects each to the graph G' on which it is operating. Each vertex may contract only to external G' legs but may have from one to four such contractions. They may not contract to any subgraph of G' that looks like —— . ⨯ has the kernel $F(k_1 + \cdots + k_4) \times \mu^{-1}(k_1) \ldots \mu(k_4)^{-1}$ where $F(k) = \prod_{\ell=0}^{2} \mu(k^{(\ell)})^{-1}$ and is effectively just a P^e vertex from the inductive expansion. In fact we include the $\mathscr{P}_\alpha^e$ operator in our norm so that we can handle the anomalous case in which a P^e vertex, instead of contracting to at least one P, C, or W vertex, contracts entirely to G legs. See Section 5 of [3].

(5) D is a monomial differential operator in the variables $\left\{k_\ell \,|\, \ell \in \bigcup_{v \in V} U(v)\right\}$ that is at most fourth order in $\{k_\ell^{(0)}, k_\ell^{(1)}, k_\ell^{(2)}\}$ for each fixed ℓ.

(6) $\mathscr{T}$ is a "translation" operator. Each vertex is thought of as having a space-time localization in some cube $\Delta_v \in \mathscr{D}$ centered at r_v. $\mathscr{D}$ is a cover of space-time by disjoint unit cubes. $\mathscr{T}$ multiplies the kernel by $\prod_v \prod_{\ell \in U(v)} e^{ik_\ell \cdot r_v}$ in effect translating the external legs to the origin[1].

The norms $\| \ \|_{i,\delta,\alpha}$ are very complicated and the role of each operator can really only be understood in the context of the proof of Theorem 2. Roughly speaking the $D\mathscr{T}$ operation will be used in the inductive expansion to provide distance convergence factors

$$\left(\text{cf. } \frac{d}{dk_\ell^{(j)}} e^{ik_\ell \cdot r_v} = i r_v^{(j)} e^{ik_\ell \cdot r_v}\right).$$

The M^δ operation will be used to provide energy convergence factors. $\mathscr{C}$ and $\mathscr{P}_\alpha^e$ appear because contractions and P^e vertices arise in the expansion.

[1] See Appendix 1 for another possible $\mathscr{T}$.

In practice we estimate the norms by using methods of [3] to decompose big graphs into little graphs. For example if a graph H is the union of subgraphs h_j

$$\|H\|_{\text{H.S.}} \leqq \prod_j \|h_j\|_{\text{H.S.}} .$$

This implies

$$\|G\|_{i,\delta,\alpha} \leqq \prod_{v \in V} \|v\|_{i,\delta,\alpha}$$

where v is the graph of the vertex v. If v has just a single leg

$$\begin{aligned} \|\Phi(f)\|_{1,\delta,\alpha} &= \|\text{—}K_1 \mu^\delta\|_{\text{H.S.}} \\ &= (2\pi)^{-3/2} K_1 \|(-\Delta+1)^{-1/2+\delta} f\|_{L_2} \\ &\equiv |f|_\delta . \end{aligned}$$

Theorem 2. *Suppose G_1 is a graph having N external legs and G_2 is a graph having $N(\Delta)$ external legs in Δ. Then there is a constant $K_2(\lambda, \delta_1, \delta_2, \alpha)$ such that*

$$|\langle G_1 G_2 \rangle_{dq(\kappa,\lambda g)}| \leqq N^N \prod_{\Delta \in \mathscr{D}} N(\Delta)^{N(\Delta)} \|G_1\|_{1,\delta_1,\alpha} \|G_2\|_{2,\delta_2,\alpha} e^{K_2 A(g)} .$$

The proof is a modification of the estimate of [3] and is delayed to later in the paper.

Corollary 2.1.

$$|Z(\kappa, \lambda g)\, S_n(\kappa, \lambda g; f_1, \ldots, f_n)| \leqq n!\, |f_1|_\delta \ldots |f_n|_\delta\, e^{K_2 A(g)} .$$

Proof. This is a direct application of Theorem 2. We have only used $n^n \leqq n!\, K_3^n$ and redefined K_1 to absorb K_3.

Remark 1. There is a class $L^2(\mathbb{R}^2) \subset H_\delta \subset H^{-1/2}(\mathbb{R}^2)$ of distributions on $\mathbb{R}^2$ for which $f \in H_\delta$ implies $|f(\boldsymbol{x})\, \delta(t-t_0)|_\delta < \infty$.

Corollary 2.2.

$$|\langle G_1 G_2 e^{\Phi(f)} \rangle_{dq(\kappa,\lambda g)}| \leqq N^N \prod_{\Delta \in \mathscr{D}} N(\Delta)^{N(\Delta)} \|G_1\|_{1,\delta_1,\alpha} \|G_2\|_{2,\delta_2,\alpha} e^{K_5 A(g)}$$

$$K_5 = K_5(|f|'_\delta, \lambda, \delta, \delta_i, \alpha)$$

$$|f|'_\delta = \max(1, |f|_\delta) .$$

This is proven by modifying Theorem 2. We also leave this to later in the paper.

Theorem 3. *Let* $\|G\|_{1,\delta,\alpha}$ *and* $|f|_{\delta_1}<\infty$ *for some* $0<\alpha<\alpha_0$ *(α_0 to be chosen later). Then* $\lim_{\kappa\to 1}\langle Ge^{\Phi(f)}\rangle_{dq(\kappa,\lambda g)}$ *exists and obeys the bounds of Theorem 2 and its corollaries.*

Proof. Let κ_a and κ_b be two momentum cutoffs of our standard form. Without loss of generality we can assume that $\kappa_b \geqq \kappa_a$ and that neither κ_a nor κ_b has a lower cutoff. We construct a sequence $\kappa_a=\kappa_0<\kappa_1 \ldots <\kappa_M=\kappa_b$ of such momentum cutoffs. To get κ_{i-1} from κ_i we lower the highest cutoff in κ_i (that has not yet reached its level in κ_a) one notch.

If $\kappa_i(s)=s\kappa_{i+1}+(1-s)\kappa_i$

$$|\langle Ge^{\Phi(f)}\rangle_{dq(\kappa_b,\lambda g)}-\langle Ge^{\Phi(f)}\rangle_{dq(\kappa_a,\lambda g)}|$$

$$\leqq \sum_{i=0}^{M-1} |\langle Ge^{\Phi(f)}\{e^{-V(\kappa_{i+1},\lambda g)}-e^{-V(\kappa_i,\lambda g)}\}\rangle_{dq_0}|$$

$$=\sum_{i=0}^{M-1}\left|\int_0^1 ds\left\langle Ge^{\Phi(f)}\left\{\frac{d}{ds}V(\kappa_i(s),\lambda g)\right\}e^{-V(\kappa_i(s),\lambda g)}\right\rangle_{dq_0}\right|.$$

$\frac{d}{ds}V(\kappa_i(s),\lambda g)$ is the sum of a finite number of P vertices. In addition we write $g=\sum_{\substack{\text{unit}\\ \text{cubes }\Delta}} g\chi_\Delta$ so that each vertex is localized on a cube of unit volume. Each P vertex has the property that its maximum lower cutoff λ_i is related to the minimum upper cutoff u_i of $\kappa_i(s)$ by $u_i \leqq O(1)\,\lambda_i^{1+\nu}$. We recall that for any given leg ℓ (vertex v)

$$\lambda_\ell(\lambda_v)\equiv \max_{(\ell),P}\{2,\alpha_\ell^{(P)}\}$$

$$u\equiv \min_{(\ell),P}\{\beta_\ell^{(P)}\}$$

where

$$\eta\left(\frac{k_\ell^{(P)}}{\beta_\ell^{(P)}}\right)-\eta\left(\frac{k_\ell^{(P)}}{\alpha_\ell^{(P)}}\right)$$

is the momentum cutoff function in the P^{th} space-time direction for the ℓ^{th} leg of v.

Now that each term contains a P vertex we perform a single C step of the $P-C$ expansion precisely as prescribed by rules (A) and (B) of [3] Section 2. (G vertices are considered old vertices, while $\Phi(f)$ vertices are considered C vertices.) This renormalizes the P vertex.

$$|\langle Ge^{\Phi(f)}\rangle_{dq(\kappa_b,\lambda g)} - \langle Ge^{\Phi(f)}\rangle_{dq(\kappa_a,\lambda g)}|$$

$$\leqq \sum_{i=0}^{M-1} \sum_{\sigma\in\bar{G}} \int_0^1 ds |\langle G_{\sigma,i} e^{\Phi(f)} e^{-V(\kappa_i(s),\lambda g)}\rangle_{dq_0}|\,.$$

Each graph $G_{\sigma,i}$ contains one G graph, one P vertex, at most 16 C vertices and at most 12 $\Phi(f)$ vertices. The only dependence of $G_{\sigma,i}$ on i is in the momentum cutoffs appearing in its kernel. Hence $\bar{G}$ is a finite index set and $|\bar{G}|$ depends on G and $A(g)$ but not on κ_a or κ_b.

The estimate leading to convergence is completed by first using the method of combinatoric factors to bound the number of terms in the above sum and then bounding the size of each term. λ_i, the maximum lower cutoff of the P vertex in $G_{\sigma,i}$, is either 2 or M_j where M_j is the upper cutoff of that one component of κ_i different from the corresponding component in κ_{i+1}. This implies that λ_i is independent of σ and is monotone non-decreasing in i. Furthermore λ_i can take on any given value at most three times. Then

$$\begin{aligned}\sum_i [\log\lambda_i]^{-1} &\leqq 3[\log 2]^{-1} + 3\sum_{j=1}^{\infty}[\log M_j]^{-1}\\ &\leqq 3[\log 2]^{-1} + 3[\log M_1]^{-1}\sum_{j=1}^{\infty}(1+\nu)^{-j+1}\\ &\leqq K_6(M_1,\nu)\end{aligned}$$

so that

$$\begin{aligned}&|\langle Ge^{\Phi(f)}\rangle_{dq(\kappa_b,\lambda g)} - \langle Ge^{\Phi(f)}\rangle_{dq(\kappa_a,\lambda g)}|\\ &\quad\leqq \sup_{i,\sigma} |\bar{G}|\, K_6 \log\lambda_i \int_0^1 ds |\langle G_{\sigma,i} e^{\Phi(f)}\rangle_{dq(\kappa_i(s),\lambda g)}|\\ &\quad\leqq \sup_{i,\sigma} K_7 \log\lambda_i \|G_{\sigma,i}\|_{1,\gamma,\alpha}\end{aligned}$$

where the constant K_7 depends on almost everything except the κ's. (The fact that $\kappa_i(s)$ is not quite in the standard form for a momentum cutoff is irrelevant as can be seen from the proof of Theorem 2.) We will choose γ later in the proof.

$\|\mathscr{P}_\alpha^e \mathscr{C} M^\gamma |G_{\sigma,i}|\,\|_{\text{H.S.}}$ and hence $\|G_{\sigma,i}\|_{1,\gamma,\alpha}$ may now be estimated by the methods of [3] Section 5. We divide $\mathscr{P}_\alpha^e \mathscr{C} M^\gamma |G_{\sigma,i}|$ into two subgraphs $\mathscr{P}_{\alpha,1}^e G$ and $\mathscr{P}_{\alpha,2}^e R$ where $\mathscr{P}_{\alpha,1}^e$ includes those P^e vertices that contract to G only and $\mathscr{P}_{\alpha,2}^e$ includes the rest. (We suppress the μ^γ's, σ's, etc. in our notation except when they are needed.) R is the graph containing the P, C, and $\Phi(f)$ vertices.

If P has four legs contracting to G legs (other than —) then choosing $\gamma = \delta/8$

$$\|\mathscr{P}_\alpha^e \mathscr{C} M^\gamma | G_{\sigma,i}|\,\|_{\text{H.S.}} = \left\| \mathscr{P}^e_{\alpha,1} G \begin{array}{c} (\gamma) \text{———} (\gamma) \\ \text{———} \\ \text{———} \\ \text{———} \end{array} P \right\|_{\text{H.S.}}$$

$$= \left\| \mathscr{P}^e_{\alpha,1} G \begin{array}{r} (2\gamma)+\alpha \text{———} -\alpha \\ \alpha \text{———} -\alpha \\ \alpha \text{———} -\alpha \\ \alpha \text{———} -\alpha \end{array} P \right\|_{\text{H.S.}}$$

$$\leqq O(1)\,\lambda_i^{-\alpha/2} \left\| \mathscr{P}^e_{\alpha,1} G \begin{array}{r} (2\gamma)+3\alpha/2 \text{———} -\alpha \\ 3\alpha/2 \text{———} -\alpha \\ 3\alpha/2 \text{———} -\alpha \\ 3\alpha/2 \text{———} -\alpha \end{array} P \right\|_{\text{H.S.}}$$

$$\leqq O(1)\,\lambda_i^{-\alpha/2} \left\| \mathscr{P}^e_{\alpha,1} G \begin{array}{r} \delta \text{———} -\alpha \\ \delta \text{———} -\alpha \\ \delta \text{———} -\alpha \\ \delta \text{———} -\alpha \end{array} P \right\|_{\text{H.S.}}$$

$$= O(1)\,\lambda_i^{-\alpha/2} \|\mathscr{P}^e_{\alpha,3} G\|_{\text{H.S.}}$$

$$\leqq O(1)\,\lambda_i^{-\alpha/2} \|G\|_{1,\delta,\alpha}\,.$$

The $\mu^{2\gamma}$ appears only if that leg is an external leg of $G_{\sigma,i}$ that is contracted by $\mathscr{C}$. $\mathscr{P}^e_{\alpha,3}$ contains all the P^e vertices in $\mathscr{P}^e_{\alpha,1}$ plus P viewed as a P^e vertex (which of course it really is).

If P has three legs contracting to G then[2] choosing $\gamma \leqq \max(\delta, 1/16)$

$$\|\mathscr{P}_\alpha^e \mathscr{C} M^\gamma | G_{\sigma,i}|\,\|_{\text{H.S.}} = \left\| \mathscr{P}^e_{\alpha,1} G \begin{array}{r} -\alpha \text{—}\rceil \\ -\alpha \text{—}\rceil \\ -\alpha \text{—}\rceil \\ \alpha \text{———} -\alpha \\ \alpha \text{———} -\alpha \; P \overset{\gamma}{\text{—}}(P^e) \\ \alpha \text{———} -\alpha \end{array} \right\|_{\text{H.S.}}$$

$$\leqq \|G\|_{1,\delta,\alpha} \left\| \begin{array}{l} -\alpha \\ -\alpha \\ -\alpha \end{array} \!\!> P \overset{\gamma}{\text{—}} \right\|_{3,1} \quad \text{or}$$

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[2] Most of our estimates on small graphs are either proven in [3] Section 6, or are simple extensions of those that are. The estimate on $\begin{array}{l}\gamma\text{—} \\ \gamma\text{—} \\ \gamma\text{—}\end{array}\!\!\Big\}\text{—}\gamma$ is proven in Appendix 2.

$$\|G\|_{1,\delta,\alpha}\left\|\begin{matrix}-\alpha\\-\alpha\\-\alpha\end{matrix}\!>\!P-\gamma\,P^e\!<\!\begin{matrix}-\alpha\\-\alpha\\-\alpha\end{matrix}\right\|_{\text{H.S.}}$$

$$\leqq O(1)\,\lambda_i^{-\alpha}\|G\|_{1,\delta,\alpha}$$

where we have used $\left\|\begin{matrix}-\alpha\\-\alpha\\-\alpha\end{matrix}\!>\!P-\gamma\right\|_{3,1}\leqq O(1)\,\lambda^{-\zeta}$ (provided $3\alpha-\zeta>0$, $\zeta+\gamma<\frac{1}{4}, \alpha<\frac{1}{2}$) and a similar estimate for $\left\|\begin{matrix}-\alpha\\-\alpha\\-\alpha\end{matrix}\!>\!P\overset{\gamma}{-}P^e\!<\!\begin{matrix}-\alpha\\-\alpha\\-\alpha\end{matrix}\right\|_{\text{H.S.}}$ This method is also used to bound P when it has four legs contracting to G but one of the G legs is $\bullet\!\!-$.

$$\left\|\begin{matrix}-\alpha\\-\alpha\\-\alpha\end{matrix}\!>\!P-\bullet\right\|_{\text{H.S.}}\leqq\|\bullet\!\!-\|_{\text{H.S.}}\left\|\begin{matrix}-\alpha\\-\alpha\\-\alpha\end{matrix}\!>\!P-\right\|_{3,1}.$$

If P has two or fewer legs contracting to G then $\mathscr{P}^e_{\alpha,2}R$ is precisely a P_c graph (in the notation of [3]) with a few factors of μ^γ thrown in. We write

$$\|\mathscr{P}^e_\alpha\mathscr{C}M^\gamma|G_{\sigma,i}|\,\|_{\text{H.S.}}\leqq\|G\|_{1,\delta,\alpha}\|\mathscr{P}^e_{\alpha,2}R\|_{\text{H.S.}}$$

and we estimate the second factor by the same algorithm as used in [3]. We use P_c to refer to R with the $\mathscr{P}^e_{\alpha,2}$ vertices, the μ^γ factors and the contractions in $\mathscr{C}$ added in.

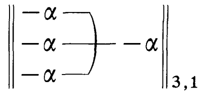
1) Define a "core" subgraph for P_c. If P is a cancelled mass diagram this is the core. If P contracts twice to a single C vertex then P and this C vertex form the core. If this is not the case P must contract to four different C vertices. Then P plus two of the C vertices form the core. We choose the C vertices to maximize first the number of Φ^4 vertices in the core, and then number of internal legs in the core.

2) Are there any non-core vertices with external legs left in P_c? If not, go to step 3. Otherwise remove one (giving P^e's first, outer C's second and inner C's third priority) using

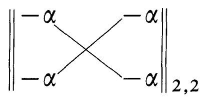
$$\|H_1H_2\|_{\text{H.S.}}\leqq\|H_1\|_{\text{H.S.}}\;\|H_2\|_{i,f}$$

where i is the number of H_2 legs that are internal to H_1H_2 and f is the number of H_2 legs external to H_1H_2. In our case $3\geqq i,f\geqq 1$ unless

$H_2 = \Phi(f)$. If we have removed a P^e vertex we use



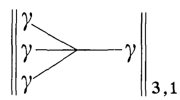
$< O(1)$



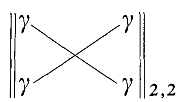
$< O(1)$.

If we have removed a C vertex we use

$$\|\gamma - M - \gamma\|_{\text{H.S.}} \leqq O(1) \log u_i \qquad \gamma < \tfrac{1}{4}$$



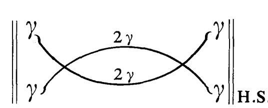
$\leqq O(1)\, u_i^{3\gamma} \qquad \gamma < \frac{1}{40}$



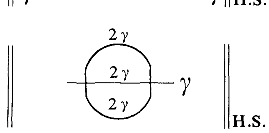
$\leqq O(1) \qquad \gamma < \frac{1}{16}$.

If we have removed a $\Phi(f)$ vertex we use $\|{-}\|_{\text{H.S.}} \leqq O(1)$. Now return to the beginning of Step 2.

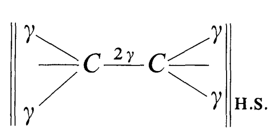
3) Are there two non-core Φ^4 vertices that are connected together? If not, go to Step 4. If there are, call these two vertices plus any vertices that contract only to them H_2. Remove from H_2 any $\Phi(f)$ and $-M-$ vertices using the method of Step 2. This leaves at most three Φ^4 vertices each of which is connected to the other(s). If there are three at least one must have external legs. It is removed as in Step 2. Remove a P^e vertex if there is a choice. The remaining cases are bounded by



$\leqq O(1) \qquad \gamma < \frac{1}{16}$



$\leqq O(1)\, u_i^{6\gamma} \qquad \gamma < \frac{1}{40}$



$\leqq O(1)\, u_i^{4\gamma} \qquad \gamma < \frac{1}{40}$

The diagram $\gtrless C - P^e \lessgtr$ does not arise since this P^e had to have legs external to H_2. Return to Step 2.

4) Are there two non-core vertices that are connected together? If not, go to Step 5. Otherwise define H_2 as in Step 3. Again remove any $\Phi(f)$ and $-M-$ vertices, other than the original two vertices. The latter are bounded by

$$\|{\cdot\!\!-\!\!<}\|_{\mathrm{H.S.}} \leqq \|{\cdot\!-}\|_{\mathrm{H.S.}} \, \|{-\!\!<}\|_{1,3}$$

$$\leqq O(1) \log u_i$$

$$\|{\cdot - M -}\|_{\mathrm{H.S.}} \leqq \|{\cdot -}\|_{\mathrm{H.S.}} \, \|{- M -}\|_{\mathrm{H.S.}}$$

$$\leqq O(1) \log u_i$$

$$\|{- M - M -}\|_{\mathrm{H.S.}} \leqq O(1) \log^2 u_i$$

$$\left\| \overset{\gamma}{-} M \overset{2\gamma}{-} C \begin{matrix} \nearrow \gamma \\ - \gamma \\ \searrow \gamma \end{matrix} \right\|_{\mathrm{H.S.}} \leqq \left\| \overset{\gamma}{-} M \overset{\gamma}{-} \right\|_{\mathrm{H.S.}} \left\| \gamma - C \begin{matrix} \nearrow \gamma \\ - \gamma \\ \searrow \gamma \end{matrix} \right\|_{1,3}$$

$$\leqq O(1) (\log u_i) \, u_i^{3\gamma}$$

$$\left\| M \begin{matrix} \overset{2\gamma}{=} \\ \underset{2\gamma}{=} \end{matrix} \!\!> \!\!< \begin{matrix} \gamma \\ \gamma \end{matrix} \right\|_{\mathrm{H.S.}} \leqq \left\| \overset{\gamma}{-} M \overset{\gamma}{-} \right\|_{\mathrm{H.S.}} \left\| \begin{matrix} \gamma \\ \gamma \end{matrix} \!\! > \!\!< \!\! \begin{matrix} \gamma \\ \gamma \end{matrix} \right\|_{2,2}$$

$$\leqq O(1) \log u_i \, .$$

Now return to Step 2.

5) This leaves only the core, plus some vertices that are fully contracted to the core. Note that since all the legs left were internal to $G_{\sigma,i}$ there are no factors of μ^γ involved. We remove all the extra $\Phi(f)$ and $-M-$ vertices. This leaves the following cases:

a) one or two vertex core:

$$\|{- M -}\|_{\mathrm{H.S.}} \leqq O(1) \, \lambda_i^{-\varepsilon_3} \quad \text{some} \quad \varepsilon_3$$

$$\|{= P = M}\|_{\mathrm{H.S.}} \leqq \|{= P =}\|_{2,2} \, \|{- M -}\|_{\mathrm{H.S.}}$$

$$\leqq O(1) \, \lambda_i^{-1/32} \log u_i$$

$$\|{= P = P =}\|_{\mathrm{H.S.}} \leqq O(1) \, \lambda_i^{-1/32} \, ,$$

b) three vertex core:

$$\|{=}\overset{\frown}{P{-}M \text{ or } \Phi(f){-}{-}{-}M} \text{ or } \Phi(f)\|_{\text{H.S.}}$$
$$\leqq O(1)\log^2 u_i \,\|{=}P{=}\|_{2,2}$$
$$\leqq O(1)\log^2 u_i\, \lambda_i^{-1/8}$$

$$\left\| \begin{matrix} \| \\ -P-C\equiv \end{matrix} \right\|_{\text{H.S.}}^2$$
$$= \begin{matrix} -P-C \\ \| \quad \| \! | \\ -P-C \end{matrix}$$
$$\leqq \|{-}\|_{\text{H.S.}}^2 \,\|{=}P{=}P{=}\|_{\text{H.S.}}\, \|{-}C\equiv C{-}\|_{\text{H.S.}}$$
$$\leqq O(1)\log^2 u_i\, \lambda_i^{-1/32}.$$

All the other three vertex cores are treated in [3] and yield the same results.

Combining all these results together gives us

$$\begin{aligned}
&\sup_{i,\sigma} K_7 \log\lambda_i \|G_{\sigma,i}\|_{1,\gamma,\alpha}\\
&\quad\leqq \max\{O(1)\log\lambda_i(\log u_i)^{m_1}\, u_i^{m_2\gamma}\lambda_i^{-\min(\varepsilon_3,1/64)},\\
&\qquad O(1)\,\lambda^{-\alpha/2}\} \quad \text{with } m_1, m_2 \text{ fixed integers}.\\
&\quad\leqq \max\{O(1)\log\lambda_i[\log\lambda_i]^{m_1}\, \lambda_i^{(1+\nu)m_2\gamma}\\
&\qquad\qquad \times \lambda_i^{-\min(\varepsilon_3,1/64)}, O(1)\,\lambda^{-\alpha/2}\}\\
&\quad\leqq O(1)\,\lambda_i^{-1/2\min(\varepsilon_3,1/64,\alpha)}
\end{aligned}$$

if we choose γ sufficiently small depending on ε_3, ν, and m_2. Note that since $\alpha<\gamma/2$ this places a restriction on α. $\alpha_0<\gamma/2$.

Convergence now follows immediately from the facts that $\lambda_i \geqq \lambda_0$ and $\lambda_0 \to \infty$ as $\operatorname{glb}\{\|k\|\,|\kappa_a(k) \neq 1\} \to \infty$. Q.E.D.

In view of Theorem 3 we define

$$\langle Ge^{\Phi(f)}\rangle_{\lambda g} = \lim_{\kappa\to 1}\langle Ge^{\Phi(f)}\rangle_{dq(\kappa,\lambda g)}.$$

Theorem 4. *Let* $\lambda \geqq 0$, $|f|_{\delta_1}<\infty$ *and* $\|G\|_{1,\delta,\alpha}<\infty$ *for all* $\alpha>0$. *Then (using a right derivative at* $\lambda=0$*)*

a) $\left|\dfrac{d^n}{d\lambda^n}\langle Ge^{\Phi(f)}\rangle_{dq(\kappa,\lambda g)}\right| \leqq F_n(\lambda)$. *The* F_n *are independent of* κ *and bounded on compact subsets of* $[0,\infty)$.

b) $\lim_{\kappa\to 1}\dfrac{d^n}{d\lambda^n}\langle Ge^{\Phi(f)}\rangle_{dq(\kappa,\lambda g)}$ *exists.*

c) $\langle Ge^{\Phi(f)}\rangle_{\lambda g}$ *is* C^∞ *in* λ *and*

$$\frac{d^n}{d\lambda^n}\langle Ge^{\Phi(f)}\rangle_{\lambda g}=\lim_{\kappa\to 1}\frac{d^n}{d\lambda^n}\langle Ge^{\Phi(f)}\rangle_{dq(\kappa,\lambda g)}.$$

d) $\langle Ge^{\Phi(f)}\rangle_{\lambda g}$ *has an asymptotic expansion at each* $\beta\in[0,\infty)$.*If*

$$\begin{aligned}G_n(\beta)&=\frac{d^n}{d\lambda^n}\langle Ge^{\Phi(f)}\rangle_{\lambda g}\Big|_{\lambda=\beta}\\&=\lim_{\kappa\to 1}\frac{d^n}{d\lambda^n}\langle Ge^{\Phi(f)}\rangle_{dq(\kappa,\lambda g)}\Big|_{\lambda=\beta}\end{aligned}$$

then for each $r>0$ *there exists a constant* $R(n,\beta,r)$ *such that*

$$\left|\langle Ge^{\Phi(f)}\rangle_{\lambda g}-\sum_{m=0}^{n}G_m(\beta)\frac{(\lambda-\beta)^m}{m!}\right|\leqq R(n,\beta,r)\,|\lambda-\beta|^{n+1}$$

for all $\max(0,\beta-r)\leqq\lambda\leqq\beta+r$.

Proof. Let $\kappa_a\leqq\kappa_b$ be two momentum cutoffs. (For the proof of a) we choose $\kappa_a\equiv 0$.) We construct a sequence of momentum cutoffs

$$\kappa_a=\kappa_0<\kappa_1\cdots<\kappa_m=\kappa_b$$

as in Theorem 3. We again have

$$\begin{aligned}&\langle Ge^{\Phi(f)}\rangle_{dq(\kappa_b,\lambda g)}-\langle Ge^{\Phi(f)}\rangle_{dq(\kappa_a,\lambda g)}\\&\quad=-\sum_{i=0}^{M-1}\int_0^1 ds\left\langle Ge^{\Phi(f)}\frac{d}{ds}V(\kappa_i(s),\lambda g)\right\rangle_{dq(\kappa_i(s),\lambda g)}\\&\quad=-\sum_{i=0}^{M-1}\sum_{\sigma\in G^0}\int_0^1 ds\langle G_{\sigma,i}e^{\Phi(f)}\rangle_{dq(\kappa_i(s),\lambda g)}\end{aligned}$$

where we have renormalized the P vertex. Note that each P vertex has a maximum lower cutoff $\lambda_i\geqq\lambda_a$ and that the graphs $G_{\sigma,i}$ depend on i only through the momentum cutoffs. Now

$$\begin{aligned}&\frac{d}{d\lambda}\langle Ge^{\Phi(f)}\rangle_{dq(\kappa_b,\lambda g)}-\frac{d}{d\lambda}\langle Ge^{\Phi(f)}\rangle_{dq(\kappa_a,\lambda g)}\\&\quad=-\sum_{i,\sigma}\int_0^1 ds\left\langle\frac{d}{d\lambda}G_{\sigma,i}e^{\Phi(f)}\right\rangle_{dq(\kappa_i(s),\lambda g)}\\&\qquad+\sum_{i,\sigma}\int_0^1 ds\left\langle G_{\sigma,i}e^{\Phi(f)}\frac{d}{d\lambda}V(\kappa_i(s),\lambda g)\right\rangle_{dq(\kappa_i(s),\lambda g)}.\end{aligned}$$

$\frac{d}{d\lambda}G_{\sigma,i}$ differs from $G_{\sigma,i}$ only in its dependence on λ. The second set of terms above come from differentiating $e^{-V(\kappa_i(s),\lambda g)}$. These terms have a second P vertex which, unlike the first, has no lower momentum cutoff. We use another C step to perform the renormalization cancellations for the second P vertex. We now have

$$\frac{d}{d\lambda}\langle Ge^{\Phi(f)}\rangle_{dq(\kappa_b,\lambda g)}-\frac{d}{d\lambda}\langle Ge^{\Phi(f)}\rangle_{dq(\kappa_a,\lambda g)}$$

$$=\sum_{i=0}^{M-1}\sum_{\tau\in G^1}\int_0^1 ds\langle G_{\tau,i}e^{\Phi(f)}\rangle_{dq(\kappa_i(s),\lambda g)}\,.$$

Continuing in this manner

$$\frac{d^n}{d\lambda^n}\langle Ge^{\Phi(f)}\rangle_{dq(\kappa_b,\lambda g)}-\frac{d^n}{d\lambda^n}\langle Ge^{\Phi(f)}\rangle_{dq(\kappa_a,\lambda g)}$$

$$=\sum_{i=0}^{M-1}\sum_{\varrho\in G^n}\int_0^1 ds\langle G_{\varrho,i}e^{\Phi(f)}\rangle_{dq(\kappa_i(s),\lambda g)}\,.$$

Due to our renormalization procedure each graph $G_{\varrho,i}$ contains at most logarithmic divergences and it contains one renormalized P vertex with maximum lower cutoff at λ_i. We can bound the sum using precisely the same argument as in Theorem 3. The only difference is that we now have (at most) n P subgraphs instead of one. The fact that none of the P vertices, except the first, have lower cutoffs and hence do not contribute further convergence factors is irrelevant. All we do is choose γ sufficiently small (depending on n) that the first P vertex provides enough convergence for all the P subgraphs. (Since $\alpha<\gamma/2$ we need to be able to make α small as well.)

This gives

$$\left|\frac{d^n}{d\lambda^n}\langle Ge^{\Phi(f)}\rangle_{dq(\kappa_b,\lambda g)}-\frac{d^n}{d\lambda^n}\langle Ge^{\Phi(f)}\rangle_{dq(\kappa_a,\lambda g)}\right|\leqq\lambda_a^{-\xi}F_n(\lambda)$$

for some $\xi>0$ and some ultraviolet cutoff independent function $F_n(\lambda)$. Parts a)–c) of our theorem follow. Part d) follows from Part a) and Taylor's theorem:

$$f(\lambda)-\sum_{m=0}^{n}\frac{f^{(m)}(\beta)}{m!}(\lambda-\beta)^m=\frac{1}{n!}\int_\beta^\lambda(\lambda-t)^n f^{(n+1)}(t)\,dt\,.\qquad \text{Q.E.D.}$$

Corollary 4.1. *The results of Theorem 4 apply to* $Z(1,\lambda g)$ *and* $ZS_n(1,\lambda g;f_1,\dots,f_n)$.

Corollary 4.2. *The results of Theorem 4 apply to* $S_n(1, \lambda g; f_1, \ldots, f_n)$ *provided we restrict* λ *to the interval* $[0, \lambda_0(A(g)))$ *(i.e. to the interval on which* $Z(1, \lambda g)$ *is known to be nonzero).*

$$C(\kappa, \lambda g; f) = Z^{-1}(\kappa, \lambda g) \langle e^{i\Phi(f)} \rangle_{dq(\kappa, \lambda g)} .$$

$(f \in \mathscr{S}_R(\mathbb{R}^3))$ is the characteristic function of the doubly cutoff measure. If λ is small the ultraviolet limit

$$C(1, \lambda g; f) = \lim_{\kappa \to 1} C(\kappa, \lambda g; f)$$

is well-defined and we have:

Theorem 5. *If* $0 \leqq \lambda \leqq \lambda_0(A(g))$, $C(1, \lambda g; f)$ *is the characteristic function of a unique measure* $dq(1, \lambda g)$ *on* $\mathscr{S}'_R(\mathbb{R}^3)$. *Furthermore*

$$\int F\, dq(1, \lambda g) = \lim_{\kappa \to 1} Z^{-1}(\kappa, \lambda g) \int F\, dq(\kappa, \lambda g)$$

for any F *in the sub* C^**-algebra of* $C(\mathscr{S}'_R(\mathbb{R}^3))$ *generated by* $\{e^{i\Phi(f)} \mid f \in \mathscr{S}_R(\mathbb{R}^3)\}$. *Also*

$$S_n(1, \lambda g; f_1, \ldots, f_n) = \langle \Phi(f_1) \ldots \Phi(f_n) \rangle_{dq(1, \lambda g)} .$$

Proof. $C(1, \lambda g; \cdot)$ is the limit of a sequence of characteristic functions so that it is normalized, $C(1, \lambda g; 0) = 1$, and positive,

$$\sum_{1 \leqq i, j \leqq M} \bar{\zeta}_i \zeta_j C(1, \lambda g; f_j - f_i) \geqq 0$$
$$\forall f_i \in \mathscr{S}_R(\mathbb{R}^3), \quad \zeta_i \in \mathbb{C} .$$

$C(1, \lambda g; f)$ is also continuous in f:

$$\begin{aligned}
&|C(1, \lambda g; f_2) - C(1, \lambda g; f_1)| \\
&\quad = \lim_{\kappa \to 1} Z^{-1}(\kappa, \lambda g) |\langle e^{i\Phi(f_2)} - e^{i\Phi(f_1)} \rangle_{dq(\kappa, \lambda g)}| \\
&\quad = \lim_{\kappa \to 1} Z^{-1}(\kappa, \lambda g) \left| \int_0^1 ds \frac{d}{ds} \langle e^{i\Phi(f_s)} \rangle_{dq(\kappa, \lambda g)} \right| \\
&\quad = \lim_{\kappa \to 1} Z^{-1}(\kappa, \lambda g) \left| \int_0^1 ds \langle \Phi(f_2 - f_1)\, e^{i\Phi(f_s)} \rangle_{dq(\kappa, \lambda g)} \right| \\
&\quad \leqq \left[\lim_{\kappa \to 1} Z^{-1}(k, \lambda g) \right] |f_2 - f_1|_\delta \sup_{0 \leqq s \leqq 1} K_9(\lambda, g, |f_s|'_\delta, \delta) \\
&\quad \to 0 \quad \text{as} \quad f_2 \to f_1 \quad \text{in} \quad \mathscr{S}_R(\mathbb{R}^3) .
\end{aligned}$$

By the Minlos theorem there is a unique measure $dq(1, \lambda g)$ on $\mathscr{S}'_R(\mathbb{R}^3)$ such that

$$C(1, \lambda g; f) = \langle e^{i\Phi(f)} \rangle_{dq(1, \lambda g)} .$$

Now by linearity and the definition of $C(1,\lambda g;f)$

$$\int H dq(1,\lambda g) = \lim_{\kappa\to 1} Z^{-1} \int H dq(\kappa,\lambda g)$$

for any H in the *-algebra generated by $\{e^{i\Phi(f)}\}$. However

$$\begin{aligned}&|\int F dq(1,\lambda g) - Z^{-1}\int F dq(\kappa,\lambda g)|\\ &\leqq 2\|F-H\|_\infty + |\int H dq(1,\lambda g) - Z^{-1}\int H dq(\kappa,\lambda g)|\end{aligned}$$

and for any $\varepsilon>0$ we can find an H with $\|F-H\|_\infty<\frac{\varepsilon}{6}$. Then we can choose κ close enough to 1 that the second term is bounded by $\varepsilon/3$. This completes the proof of the second statement of the theorem.

The final statement of the theorem follows from the fact that both $C(1,\lambda g;\mu f)$ and $\langle e^{i\mu\Phi(f)}\rangle_{dq(1,\lambda g)}$ are analytic in μ. See Fröhlich [1] for arguments along these lines.

Remark 2. From Remark 1 we see that for $f\in\mathscr{S}(\mathbb{R}^2)$ $\Phi(f\delta(\cdot-t_0))$ is defined almost everywhere on $\mathscr{S}'_R(\mathbb{R}^3)$ with respect to the measure $dq(1,\lambda g)$. These functions generate a sharp time t_0 subspace of $L^2(\mathscr{S}'_R(\mathbb{R}^3), dq(1,\lambda g))$. Alternatively, applying the argument of Theorem 5 to the functional

$$f\mapsto C(1,\lambda g; f\delta(\cdot-t_0))$$

on $\mathscr{S}_R(\mathbb{R}^2)$ shows that

$$S_n(1,\lambda g; f_1\delta(\cdot-t_0),\ldots,f_n\delta(\cdot-t_0))$$

is given by a measure $dq'(t_0,\lambda g)$ on $\mathscr{S}'_R(\mathbb{R}^2)$. Then $L^2(\mathscr{S}'_R(\mathbb{R}^2), dq'(t_0,\lambda g))$ is isomorphic to the sharp time t_0 subspace in the natural way.

Proof of Theorem 2. Theorem 2 is proven in the same manner as the estimate $|\langle e^{-V(\kappa,\lambda g)}\rangle_{dq_0}|\leqq e^{O(A(g))}$ was proven in [3]. We will just give the modifications that must be made.

First we write

$$\begin{aligned}\langle G_1' G_2'\rangle_{dq(\kappa,\lambda g)} &= N^N \prod_\Delta N(\Delta)^{N(\Delta)} \|G_1'\|_{1,\delta_1,\alpha}\|G_2'\|_{2,\delta_2,\alpha}\\ &\quad\times\langle G_1 G_2\rangle_{dq(\kappa,\lambda g)}\end{aligned}$$

where now $\|G_1\|_{1,\delta_1,\alpha}=N^{-N}$ and $\|G_2\|_{2,\delta_2,\alpha}=\prod_\Delta N(\Delta)^{-N(\Delta)}$. We will show that for this new G_1, G_2

$$|\langle G_1 G_2\rangle_{dq(\kappa,\lambda g)}|\leqq 1\cdot e^{K_2 A(g)}\,.$$

When the inductive expansion is applied to $\langle G_1 G_2 e^{-V(\kappa,\lambda g)}\rangle$ the G_1 and G_2 vertices are, to as large an extent as possible, ignored. They are not included in any vertex count (such as that used to terminate the $P_r - C_r$ expansion or to determine the cube size in the low momentum expansion). With one exception they remain completely passive (initiate no action). The exception is the low momentum contraction operation. We perform this on the G_1 and G_2 vertices after all $P_1 - C_1$ vertices have had all their low momentum legs contracted. In the r^{th} inductive step we use M_{r-1} as the boundary momentum for G_2 legs and $M_{r-2}(M_{-1} \equiv M_0 = 0)$ for the G_1 legs. This means that when a $G_1(G_2)$ leg initiates a contraction it can only contract to $G_1(G_1$ or $G_2)$ legs.

We must now make the two estimates that yield Theorem 2. The first, Lemma 4.1 (replacing Theorem 4.1 of [3]) estimates the number of terms in the expansion. The second, Lemma 5.1 (replacing [3] Theorem 5.1) bounds the size of each term.

Lemma 4.1. *The combinatoric bounds given in* [3] *Theorem 4.1 apply equally well to our case provided we include in addition a factor of*

i) $K_{10} N(\Delta)\, d_e(\Delta, \Delta')^4 \lambda^{6\varepsilon}$ *for each external* G_2 *leg localized in* Δ' *that contracts to a vertex localized in* Δ

ii) $K_{10} N \lambda^{8\varepsilon}$ *for each (external)* G_1 *leg.*

Here $d_e(\Delta, \Delta') = \max$ *(1, Euclidean distance between* Δ *and* Δ'*).* G_1 *vertices are not considered to have a localization so that a line joining a* G_1 *and a* G_2 *vertex does not have any distance factor* $d_e(\Delta, \Delta')$ *associated with it. Also since only external* G_i *legs enter we will use the expression* G_i *leg to apply only to external legs.*

Proof. There are two operations in the inductive expansion in which the presence of G_i legs leads to an increased number of terms. The first occurs when a P_r, C_r or W vertex initiates a contraction. The second is the application of the low momentum contraction scheme to the G_i vertices.

(a) Suppose that at some stage of the expansion we have a term T. Suppose a leg in Δ introduced in the r^{th} inductive step initiates a contraction. By the nature of the expansion it can contract only to the exponent or to a free leg in its own level or in a lower level. (We organize the vertices into levels by the ordering G_1 vertices, G_2 vertices, $P_1 - C_1$ vertices, W_1 vertices, $P_2 - C_2$ vertices ....) Hence

$$T = \sum_{\sigma \in A_e} T_e(\sigma) + \sum_{1 \leqq r' \leqq r} \sum_{\Delta'' \in \mathscr{D}_{r'}} \sum_{\sigma \in A(r', \Delta'')} T_{r' \Delta''}(\sigma)$$

$$+ \sum_{\Delta' \in \mathscr{D}} \sum_{n} \sum_{\sigma \in \bar{G}_2(\Delta', n)} T_{2, \Delta', n}(\sigma)$$

$$+ \sum_{n} \sum_{\sigma \in \bar{G}_1(n)} T_{1,n}(\sigma).$$

The $T_e(\sigma)$ are terms arising from contractions to the exponent. The $T_{r', \Delta''}(\sigma)$ are terms arising from contractions to vertices of the r'^{th} inductive step that are localized in the cube Δ'' of the r'^{th} step's space-time cover $\mathcal{D}_{r'}$. (Actually the situation is somewhat more complicated for $W_{r'}$ vertices but that does not concern us here.) The $T_{1,n}(\sigma)(T_{2,\Delta',n}(\sigma))$ arise from contractions to n^{th} generation G_1 legs (G_2 legs localized in Δ'). By n^{th} generation we mean the following. We call the original G_i legs that appear in $\langle G_1 G_2\rangle_{dq(\kappa,\lambda g)}$ 0^{th} generation legs. Just before the first low momentum expansion there is a squaring operation which replaces the 0^{th} generation graph by itself plus a duplicate image, which we call the 1^{st} generation graph. $\bar{G}_1(n)\,(\bar{G}_2(\Delta',n))$ is the set of free n^{th} generation G_1 legs (G_2 legs in Δ').

The $T_e(\sigma)$ and $T_{r'\Delta''}(\sigma)$ appear independently of G_1 and G_2 and in [3] Glimm and Jaffe found $c_e(\sigma)$ and $c_{r'\Delta''}(\sigma)$ satisfying

$$\sum_\sigma c_e^{-1}(\sigma)+\sum_{r'\Delta''\sigma} c_{r'\Delta''}^{-1}(\sigma)\leqq\tfrac{1}{4}\,.$$

If we set $c_{2,\Delta',n}(\sigma)=4Dd_e(\Delta,\Delta')^4\,2^{n+1}|\bar{G}_2(\Delta',n)|$

and

$$c_{1,n}(\sigma)=4\;\;2^{n+1}|\bar{G}_1(n)|$$

where

$$D=\sup_\Delta\sum_{\Delta'\in\mathcal{D}} d_e(\Delta,\Delta')^{-4}<\infty$$

we get

$$\begin{aligned}|T|&\leqq\Bigg\{\sum_\sigma c_e^{-1}(\sigma)+\sum_{r',\Delta'',\sigma} c_{r'\Delta''}^{-1}(\sigma)\\&\quad+\sum_{\Delta'\in\mathcal{D}}\sum_{n=0}^{\infty}\sum_{\sigma\in\bar{G}_2(\Delta',n)} c_{2,\Delta',n}^{-1}(\sigma)\\&\quad+\sum_{n=0}^{\infty}\sum_{\sigma\in\bar{G}_1(n)} c_{1,n}^{-1}(\sigma)\Bigg\}\sup|c(\sigma)\,T(\sigma)|\\&\leqq\Bigg\{\frac{1}{4}+(4D)^{-1}\left(\sum_{\Delta'} d_e(\Delta,\Delta')^{-4}\right)\left(\sum_{n=0}^{\infty}2^{-(n+1)}\right)\\&\quad+\frac{1}{4}\sum_{n=0}^{\infty}2^{-(n+1)}\Bigg\}\sup|c(\sigma)\,T(\sigma)|\\&\leqq\frac{3}{4}\sup|c(\sigma)\,T(\sigma)|\,.\end{aligned}$$

(The extra 1/4 will be used for the $\Phi(f)$ legs in the proof of Corollary 1.2.) Now

$$|c_{2,\Delta',n}(\sigma)| \leqq 4D\, d_e(\Delta, \Delta')^4\, 2^{n+1}\, 2^{\max(0,n-1)} N(\Delta')$$

$$\text{since } |\bar{G}_2(\Delta', n)| \leqq N(\Delta')\, 2^{\max(0,n-1)}$$

$$\leqq 4D\, d_e(\Delta, \Delta')^4\, 2^{2n+1} N(\Delta')$$

$$\leqq \begin{cases} 8D\, d_e(\Delta, \Delta')^4\, M_{n-1}^{\varepsilon} N(\Delta') & n > 2 \\ \quad \text{(cf. [3], Eq. 4.9)} & \\ 2^7 D\, d_e(\Delta, \Delta')^4\, N(\Delta') & n \leqq 2\,. \end{cases}$$

Any leg in $\bar{G}_2(\Delta', n)$ was free at the time of its introduction in the n^{th} inductive step. This implies that it must have been the image of a high momentum leg from the $(n-1)^{\text{st}}$ inductive step. This means that its low momentum cutoff λ_σ is at least M_{n-2} (or M_{n-3} for G_1 legs). Hence

$$M_{n-2} = M_{n-1}^{1/(1+\nu)} \geqq M_{n-1}^{1/2} \qquad (n > 2)$$

implies

$$|c_{2,\Delta',n}(\sigma)| \leqq 2^7 D d_e(\Delta, \Delta')^4\, N(\Delta')\, \lambda_\sigma^{2\varepsilon}\,.$$

Similarly

$$|c_{1,n}(\sigma)| \leqq 2^9 N \lambda_\sigma^{4\varepsilon}\,.$$

To keep track of these factors we assign them to the leg to which the contraction was made.

(b) Our first task in the low momentum contraction operation is to split each uncontracted leg into high and low momentum parts.

$$\begin{aligned} \kappa_{\text{leg}} &= \kappa^{(0)} \kappa^{(1)} \kappa^{(2)} \\ &= (\kappa_H^{(0)} + \kappa_L^{(0)})\,(\kappa_H^{(1)} + \kappa_L^{(1)})\,(\kappa_H^{(2)} + \kappa_L^{(2)}) \\ &= \kappa_L^{(0)} \kappa_L^{(1)} \kappa_L^{(2)} \text{ (low momentum part)} \\ &\quad + 7 \text{ other terms (high momentum part)}\,. \end{aligned}$$

Hence we require a combinatoric factor of 8 and we assign it to the leg we split. If a leg goes through j such splittings it acquires a total factor of 8^j. Since the leg was still uncontracted when it underwent its last splitting in the j^{th} inductive step it must have been a high momentum leg of the low momentum contraction operation of the $(j-1)^{\text{st}}$ inductive step. Then its low momentum cutoff must be at least M_{j-3} so that

$$8^j \leqq 2^{3j} \leqq \begin{cases} 2^9 & j \leqq 3 \\ M_{j-1}^{\varepsilon} \leqq M_{j-3}^{4\varepsilon} \leqq \lambda^{4\varepsilon} & j \geqq 4\,. \end{cases}$$

After splitting the legs into high and low momentum parts, we contract up all the low momentum legs. Since low momentum G_2 (G_1) legs can only contract to G_2 or G_1 (G_1) legs the analysis of (a) part shows that assigning $2^7 D d_e(\Delta, \Delta')^4 N(\Delta') \lambda^{2\varepsilon}$ ($2^9 N\lambda^{4\varepsilon}$) to contractee G_2 (G_1) legs is sufficient.

Combining the results of (a) and (b) completes the proof of the lemma.

Q.E.D.

Lemma 5.1. $|T(\sigma)|$ *is bounded above by a product of factors given by those of* [3] *and*

$$\|G_1\|_{1,\delta_1,\alpha} \quad \text{per } G_1 \text{ graph}$$

$$\|G_2\|_{2,\delta_2,\alpha} \quad \text{per } G_2 \text{ graph}$$

$$d_e(\Delta,\Delta')^{-4} \quad \text{per line joining a } \Delta' \; G_2 \text{ vertex and a } \Delta \; G_2, P, C \text{ or } W \text{ vertex}$$

$$K_{11} K_1^{-1} \lambda^{-\delta_i/4} \quad \text{per } G_i \text{ leg}.$$

Proof. We will focus our attention on the G_i vertices and legs. In fact we will ignore almost everything else. This is done solely to bring the notation within the realm of the imaginable.

$T(\sigma)$ is a vacuum graph so that it can be evaluated directly. After integrating out the delta functions arising from contractions we arrive at

$$T(\sigma) = \int \prod_{\substack{\text{lines} \\ \ell}} d^3 k_\ell \prod_{\substack{G_1 \\ \text{graphs} \\ p}} G_1((k)_p)\, \kappa_p((k)_p)$$

$$\prod_{\substack{G_2\,\text{graphs} \\ q}} G_2((k)_q)\, \kappa_q((k)_q) \prod_{\substack{P,C,W \\ \text{vertices}}} \dots .$$

Here the $\kappa(k)$ are momentum cutoffs that were introduced into the G_i graphs during the low momentum contraction operation. They were not in the original G_i graphs. $(k)_p$ and $(k)_q$ are the sets of momenta appropriate to the G_i graphs involved. In particular a contraction within a G_i graph (these may have been introduced in the inductive expansion) is manifest by two of the momenta in $(k)_p$ (or $(k)_q$) being negatives of each other.

In order to get a handle on the distance factors we translate each G_2 vertex to the origin:

$$T = \int \prod_\ell dk_\ell\, e^{\pm i r_\ell \cdot k_\ell} \prod G_1((k)_p)\, \kappa_p((k)_p) \prod \mathcal{T} G_2((k)_q)\, \kappa_q((k)_q)$$

$$\prod \dots .$$

$r_\ell = r_{v_1} - r_{v_2}$ is the contraction vector for ℓ i.e. the vector between the centres of the cubes on which v_1 and v_2 are located. We define $r_v = 0$ for G_1 vertices.

Writing $\prod^2$ for a product over all lines involving G_2 legs but not G_1 legs and $\prod^{2,1}$ for a product over all lines that in addition satisfy $d_e(\Delta, \Delta') > 1$ we have

$$\begin{aligned}
\left|\prod_\ell^2 d_{e\ell}^4 T\right| &\leqq \left|\prod_\ell^{2,1} d_{e\ell}^4 T\right| \\
&\leqq \left|\prod_\ell^{2,1} |r_\ell|^4\, T\right| \\
&\leqq \Big|\int \prod_\ell dk_\ell e^{\pm i r_\ell \cdot k_\ell} \prod_\ell^{2,1} (r_\ell^{(0)2} + r_\ell^{(1)2} + r_\ell^{(2)2})^2 \\
&\quad \cdot \prod_{G_1} \cdots \prod_{G_2} \cdots \prod \cdots \Big| \\
&= \left|\int \prod_\ell dk_\ell e^{\pm i r_\ell \cdot k_\ell} \Pi\, G_1 \kappa_p \prod_\ell^{2,1} (\nabla_\ell^2)^2\, \Pi\, \mathscr{T} G_2 \kappa_q \Pi \ldots\right|
\end{aligned}$$

$\nabla_\ell^2 \equiv \sum_{i=0}^{2} \partial^2/\partial k_\ell^{(i)2}$. If we now expand the differential operators and apply them via the product rule we get a sum of at most $\Pi^{2,1}(3^2 4^4)$ terms. [$(\nabla_\ell^2)^2$ is a sum of 3^2 fourth order monomial differential operators and each $\partial/\partial k_\ell^{(\alpha)}$ can find a $k_\ell^{(\alpha)}$ to act on at most in two $\mathscr{T}G_2$'s (or twice in the same $\mathscr{T}G_2$) and at most in two κ's (or twice in the same κ).] Furthermore because η was chosen to be a C_0^∞ function there is a constant $\bar{\eta}$ such that for any differential operator arising as above

$$|D\kappa_q((k)_q)| \leqq \bar{\eta}^{|q|} \bar{\kappa}_q((k)_q)$$

where $|q|$ is the number of legs in $(k)_q$ and $\bar{\kappa}_q$ is the characteristic function of the support of κ_q. (This follows from [3] Eq. 5.2.9 because $|n_2| \leqq 4$.) Thus far we have

$$\begin{aligned}
\left|\prod_\ell^2 d_{e\ell}^4 T\right| &\leqq \sup_{\{D_q\}} \int \prod_\ell dk_\ell \prod_p |G_1 \bar{\kappa}_p| \prod_q (3^2 4^4 \bar{\eta})^{|q|}\, \bar{\kappa}_q \\
&\quad \times |D_q \mathscr{T} G_2|\, \Pi \ldots
\end{aligned}$$

where D_q is a monomial differential operator that is at most fourth order in each k_ℓ of $\Pi^{2,1}$.

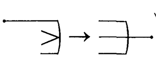
We now use the method of decomposing big graphs to estimate the above mess. To start we use subgraphs that consist of

1) a single G_i graph,
2) a single W vertex,
3) a P vertex and the C vertices it generated.

With this decomposition there are only two types of subgraphs that contain ultraviolet divergences:

$$P^e(\equiv P) \quad \text{and} \quad P_D(\equiv P— \quad \text{or} \quad \equiv P—\equiv P^e).$$

(The second type of P_D subgraph appears in later decompositions.) If the leading vertex in a P^e or P_D subgraph has an initial leg that contracts to a P or C vertex the divergent subgraph may be treated as in [3]. In particular we hook any such P^e subgraph onto the nearest P or W subgraph to which it contracts. However if all the initial legs contract to G_i subgraphs we must get the compensating convergence from the G_i subgraph.

a) P^e: If one of the G legs involved is of the form $\overline{\quad}\!\bullet$ we treat it as a C vertex thus converting our P^e subgraph into a P_D subgraph (i.e. ). Otherwise we first transfer an energy factor $\mu^{-\alpha}$ to each of the P^e legs from the G_i leg to which it contracts. In addition we take a factor of $\lambda^{-\min(\delta_1,\delta_2)/4}$ for each P^e leg. In diagrams

$$|G_i—P^e| \leqq |G_i \overset{\alpha+\delta_i/4 \quad -\alpha}{—\!—} O(1)\,\lambda^{-\min\delta_i/4}\,P^e|\,.$$

This $\lambda^{-\min\delta_i/4}$ provides the $\lambda^{-\varepsilon_1}$ (if P^e is P_1) or the $\alpha_1|\Delta|^{\varepsilon_1}$ (if P^e is P_r, $r>1$) for [3] Theorem 5.1. The latter case follows from

$$\begin{aligned}\lambda &\geqq M_{\ell(\Delta)} = M_{\ell(\Delta)+1}^{1/(1+\nu)} \\ &\geqq |\Delta|^{-1/(2+2\nu)} \quad ([3]\ \text{Eq. } 3.2.1)\,.\end{aligned}$$

We append the P^e vertex to one of the G_i subgraphs to which it contracts.

b) P_D: We transfer $\mu^{-\alpha}$ from the G_i vertices to the initial legs of the leading vertex of the P_D subgraph and use $\left\| \begin{smallmatrix} -\alpha \\ -\alpha \\ -\alpha \end{smallmatrix} \!> P— \right\|_{3,1} \leqq O(1)\,\lambda^{-\alpha}.$

This gives all the convergence we need.

We now have

$$\begin{aligned}\Big|\prod_{\ell}^{2} d_{e\ell}^4 T\Big| &\leqq \prod_p \|\mathscr{P}_\alpha^e(K_1^{-1}M^{\alpha+\delta_1/4})\,\mathscr{C}_p|G_1\bar{\kappa}_p|\,\|_{\text{H.S.}} \\ &\quad \times \sup_{\{D_q\}} \prod_q (3^2 4^4 \bar{\eta})^{|q|}\,\|\mathscr{P}_\alpha^e(K_1^{-1}M^{\alpha+\delta_1/4})\,\mathscr{C}_q\bar{\kappa}_q|D_q\mathscr{T}G_2|\,\|_{\text{H.S.}} \prod_{P,C,W} \cdots\end{aligned}$$

where $\mathscr{C}_p$ and $\mathscr{C}_q$ give the contractions introduced into the G_i by the expansion. Finally, since $\lambda_p^{\delta_i/4} \leqq K_{12}\mu_p^{\delta_i/4}$ on the support of $\bar{\kappa}_p$

$$\left|\left(\prod_{\ell}^{2} d_{e\ell}^{4}\right) \prod_{\substack{G_i \\ \text{legs}}} K_{11}^{-1} K_1 \lambda^{\delta_i/4} T\right|$$

$$\leqq \prod_p \|\mathscr{P}_\alpha^e \mathscr{C}_p M^{\delta_1/2+\alpha} |G_1 \bar{\kappa}_p| \|_{\text{H.S.}}$$

$$\cdot \prod_q \sup_{D_q} \|\mathscr{P}_\alpha^e \mathscr{C}_q M^{\delta_2/2+\alpha} |D_q \mathscr{T} G_2| \bar{\kappa}_q\|_{\text{H.S.}}$$

$$\leqq \prod_p \|G_1\|_{1,\delta_1,\alpha} \prod_q \|G_2\|_{2,\delta_2,\alpha}$$

where $K_{11} = K_{12} 3^2 4^4 \bar{\eta}$, $\delta_i/2 + \alpha < \delta_i$. Note that since we want the factors of $\lambda^{-\delta_i/4}$ even on legs contracted by $\mathscr{C}_{p,q}$ we must operate with M^{δ_i} before applying $\mathscr{C}_{p,q}$. Q.E.D.

Theorem 2 follows directly from Lemmas 4.1 and 5.1 simply by choosing $K_1 > K_{10} K_{11}$ and $\varepsilon < \delta_i/32$. Q.E.D.

Proof of Corollary 2.2. We expand $e^{\Phi(f)}$ in a power series and improve the estimates in Theorem 2 sufficiently to give the convergence of

$$\sum_n \frac{1}{n!} \langle G_1 G_2 \Phi(f)^n \rangle_{dq(\kappa,\lambda g)} .$$

We first write

$$\langle G_1 G_2 \Phi(f)^n \rangle_{dq} = n^n |f|_\delta'^n \left\langle G_1 G_2 \left(\frac{\Phi(f)}{n|f|_\delta'}\right)^n \right\rangle_{dq}$$

associating with each $\Phi(f)$ leg a factor of $(n|f|_\delta')^{-1}$. Now go through the inductive expansion with the following modifications.

1) Treat the $\Phi(f)$ vertices on a level between the G_2 vertices and the $P_1 - C_1$ vertices. In other words in the low momentum contraction operation use $M_{r-1}(M_{r-2}, M_{r-3})$ as the boundary momentum for $\Phi(f)(G_2, G_1)$ legs. This means K_{10} will be larger and $\lambda^{8\varepsilon}(\lambda^{6\varepsilon})$ will be replaced by $\lambda^{16\varepsilon}(\lambda^{12\varepsilon})$ in Lemma 4.1 but this is of no consequence.

2) Suppose two $\Phi(f)$ vertices contract together. As in Lemma 4.1 this requires a combinatoric factor of $2^7 n \lambda^{2\varepsilon}$. Instead of assigning this all to the contractee we assign $2^{7/2} n^{1/2} \lambda^\varepsilon$ to each $\Phi(f)$ vertex involved.

3) Suppose a $\Phi(f)$ and a G_i vertex contract. Since the $\Phi(f)$ vertices are in a higher level than the G_i vertices the $\Phi(f)$ vertex must have initiated the contraction. Hence the $\Phi(f)$ vertex does not have any combinatoric factor associated with it.

4) Finally suppose a $\Phi(f)$ and a P, C or W vertex contract. We then assign a factor of $(4K_3|f|_\delta')^{-1}$ (as well as the usual combinatoric factor)

to the $\Phi(f)$ vertex and a compensating factor of $(4K_3|f|'_\delta)$ to the P, C or W vertex. As far as the P, C, and W vertex is concerned $4K_3|f|'_\delta$ is just another factor of $O(1)$ and its only effect is to introduce a dependence on $|f|'_\delta$ into K_5.

5) Consider the squaring operation

$$|R(q)| \leqq \tfrac{1}{2}[\zeta^{-1} + \zeta \overline{R(q)}\, R(q)] .$$

This takes a term T with M $\Phi(f)$ vertices into a sum of two terms T_1 and T_2. T_1 has no $\Phi(f)$ vertices but is multiplied by ζ^{-1}. T_2 has $2M$ $\Phi(f)$ vertices and is multiplied by ζ. Since we are using the $\Phi(f)$ vertices to carry our convergence factors, T_1 appears to have inadequate convergence while T_2 has more than we need. We use ζ to even things up. If $M=0$ we use $\zeta=$ its value in [3] $\equiv \zeta_0$ (they call it δ). If $M \neq 0$ we use

$$\zeta = \{\max[n^{-\frac{1}{2}}, (4K_3|f|'_\delta)^{-1}]\}^{-M}\, \zeta_0 \equiv \zeta_1^{-M}\zeta_0 .$$

We keep track of the product of ζ_1's accumulated by each term separately and do not assign it to any vertex. We will show by induction on the number of squarings that the accumulated product is ζ_1^{-M+n}. This is certainly true if the term has gone through no squarings since then $M=n$. Suppose we have term T with $M=M_T$ and accumulated product $\zeta_1^{-M_T+n}$. Then the accumulated product for T_1 is $\zeta_1^{-M_T+n}\zeta_1^{M_T} = \zeta_1^n = \zeta_1^{-M_{T_1}+n}$ while that for T_2 is

$$\zeta_1^{-M_T+n}\zeta_1^{-M_T} = \zeta_1^{-2M_T+n} = \zeta_1^{-M_{T_2}+n} .$$

With all the above modifications

$|c(\sigma)\, T(\sigma)| \leqq$ contributions from G_i, P, C, and W vertices

$$\times\, \zeta_1^{-M_T+n} \prod_{\substack{\Phi(f)\\ \text{vertices}}} (n|f|'_\delta)^{-1} \begin{bmatrix} K_{10} n^{\frac{1}{2}} \lambda^{3\varepsilon} \\ 1\lambda^{2\varepsilon} \\ K_{10} n \lambda^{4\varepsilon} (4K_3|f|'_\delta)^{-1} \end{bmatrix} K_{11} K_1^{-1} \lambda^{-\delta/4} |f|_\delta$$

$$\leqq (\ )\, \zeta_1^{-M_T+n} \prod_{\substack{\Phi(f)\\ \text{vertices}}} \begin{bmatrix} n^{-\frac{1}{2}} \\ n^{-1} \\ (4K_3|f|'_\delta)^{-1} \end{bmatrix}$$

$$\leqq (\ )\, \zeta_1^{-M_T+n} \zeta_1^{M_T} .$$

Therefore

$$|\langle G_1 G_2 \Phi(f)^n \rangle_{dq(\kappa,\lambda g)}| \leqq (\ )\, n^n |f|'^n_\delta\, \zeta_1^n$$

$$\leqq (\ )\, [n^{n/2} |f|'^n_\delta + 4^{-n} K_3^{-n} n^n]$$

and since $n^n \leqq K_3^n n!$

$$|\langle G_1 G_2 e^{\Phi(f)}\rangle \, dq(\kappa, \lambda g)| \leqq (\ ) \sum_{n=0}^{\infty} [(n!)^{-\frac{1}{2}} (K_3^{\frac{1}{2}} |f|'_\delta)^n + 4^{-n}] .$$

The sum over n converges to some function of $|f|'_\delta$ which we again absorb in K_5. Q.E.D.

Appendix 1

An alternative, more natural translation operator $\mathscr{T}^1$ would multiply w_v by $\prod_{\ell \in L(v)} e^{ik_\ell \cdot r_v}$ thereby translating Δ_v to $\Delta_v - r_v$, i.e. the origin. However to use $\mathscr{T}_1$ in $\|\cdot\|_2$ we must also replace $|\cdot|$ by $|\cdot|^1$. $|\cdot|^1$ takes the absolute value of the w_v's. In other words, it takes the absolute value of the kernel before rather than after the internal contractions are made. If we use $\|\cdot\|^1_{i,\delta,\alpha}$ to represent the norms using $|\cdot|^1$ and $\mathscr{T}^1$ we have

$$\|G\|_{i,\delta,\alpha} \leqq \|G\|^1_{i,\delta,\alpha}$$

$$\|G\|^{(1)}_{1,\delta,\alpha} \leqq \|G\|^{(1)}_{2,\delta,\alpha} .$$

The $\|\cdot\|^1$ norms are the norms that are generally used in practice.

Appendix 2

We have used many estimates on one and two vertex graphs in the proofs of theorems two, three and four. They are mostly simple extensions of the estimates of [3] Section 6. One, however, is slightly more difficult than usual and we give its proof here.

Theorem 6. $\left\| \gamma \!-\! \overset{2\gamma}{\underset{2\gamma}{\bigcirc_{2\gamma}}} \!-\! \gamma \right\|_{\text{H.S.}} \leqq u^{6\gamma}$ *if* $\gamma < 1/40$ *where* u *is the smallest upper cutoff of any of the three internal lines.*

Proof.

$$\left| \gamma \overset{k_1}{-} \overset{2\gamma}{\underset{2\gamma}{\bigcirc_{2\gamma}}} \overset{k_5}{-} \gamma \right| \leqq O(1) \int dk_2 dk_3 dk_4 \frac{F(k_1 + k_2 + k_3 + k_4) F(-k_2 - k_3 - k_4 + k_5)}{\mu_1^{1-\gamma} \mu_5^{1-\gamma} (\mu_2 \mu_3 \mu_4)^{2-2\gamma}}$$

where $\mu_i = \mu(k_i)$.

If we use k_2, k_3 and $P = k_2 + k_3 + k_4$ as integration variables then

$$\int dk_2 dk_3 \mu_2^{-2+2\gamma} \mu_3^{-2+2\gamma} \mu^{-2+2\gamma}(P - k_2 - k_3)$$
$$\leqq O(1) \int dk_2 \mu_2^{-2+2\gamma} \mu^{-1+4\gamma}(P - k_2) \quad \text{if} \quad \gamma < 1/4 .$$

We split the k_2 integration into three regions:

$\mathrm{I}: |k_2| \leqq \tfrac{1}{2}|P| \qquad |P - k| \geqq \tfrac{1}{2}|P|$

$$\int_{\mathrm{I}} \leqq O(1)\, \mu^{-1+4\gamma}(P) \int_0^{|P|/2} \mu_2^{-2+2\gamma} dk_2$$
$$\leqq O(1)\, \mu^{-1+4\gamma}(P)\, \mu^{1+2\gamma}(P)$$
$$= O(1)\, \mu^{6\gamma}(P) .$$

$\mathrm{II}: \tfrac{1}{2}|P| \leqq |k_2| \leqq 2|P| \qquad |P - k_2| \leqq |P| + |k_2| \leqq 3|P|$

$$\int_{\mathrm{II}} \leqq O(1)\, \mu^{-2+2\gamma}(P) \int_{|P-k_2| \leqq 3|P|} \mu^{-1+4\gamma}(P - k_2) dk_2$$
$$\leqq O(1)\, \mu^{-2+2\gamma}(P)\, \mu^{2+4\gamma}(P)$$
$$= O(1)\, \mu^{6\gamma}(P) .$$

$\mathrm{III}: 2|P| \leqq |k_2| \qquad |P - k| \geqq |k_2|/2$

$$\int_{\mathrm{III}} \leqq O(1) \int \mu^{-3+6\gamma} dk_2$$
$$\leqq O(1)\, u^{6\gamma} \quad \text{if} \quad 6\gamma < 1 .$$

Performing the P integration gives (for some $\varepsilon_1 > 0$)

$$|\ominus| \leqq O(1)\, u^{6\gamma} F(k_1 + k_5)^{1-\varepsilon_1} (\mu_1 \mu_5)^{-1+\gamma}$$
$$+ O(1) (\mu_1 \mu_5)^{-1+\gamma} \int dP F(k_1 + P)\, F(k_5 - P)\, \mu(P)^{6\gamma}$$

where we have used [3] Proposition 6.1.5a) to bound the first integral.

$$\mu(P)^2 = 1 + P^{(0)^2} + P^{(1)^2} + P^{(2)^2}$$
$$\leqq (1 + P^{(0)^2}) \quad (1 + P^{(1)^2}) \quad (1 + P^{(2)^2})$$
$$= F^{-2}(P)$$
$$\therefore \mu(P)^{6\gamma} \leqq F^{-6\gamma}(P) .$$

Using a simple extension of [3] Corollary 6.1.7 with $\alpha_1 = \alpha_2 = -3\gamma$ and $\frac{1}{2} > \varepsilon_2 > \frac{1}{4} + 3\gamma$

$$\int dP\, F(k_1+P)\, F^{-6\gamma}(P)\, F(k_5-P) \leqq O(1)\, F^{1-\varepsilon_2}(k_1+k_5) \prod_i F^{-3\gamma}(k_i)$$

$$\leqq O(1)\, F^{1-\varepsilon_2}(k_1+k_5)\, \mu^{9\gamma}(k_1)\, \mu^{9\gamma}(k_5)$$

since

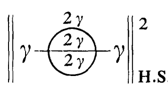
$$F^{-2}(k) = (1+k^{(0)^2})(1+k^{(1)^2})(1+k^{(2)^2})$$

$$\leqq \mu(k)^2\, \mu(k)^2\, \mu(k)^2 \leqq \mu^6(k).$$

$\therefore$  $\leqq O(1)\, u^{6\gamma} F(k_1+k_5)^{1-\varepsilon_1} (\mu_1\mu_5)^{-1+\gamma}$

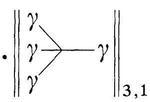
$$+ O(1)\, F(k_1+k_5)^{1-\varepsilon_2} (\mu_1\mu_5)^{-1+10\gamma}$$

$$\leqq O(1)\, u^{6\gamma} F(k_1+k_5)^{1-\varepsilon_2} (\mu_1\mu_5)^{-1+10\gamma}$$

$\therefore$ 

$\leqq O(1)\, u^{12\gamma} \int dk_1 dk_5 \dfrac{F(k_1+k_5)^{2-2\varepsilon_2}}{(\mu_1\mu_5)^{2-20\gamma}}$

$$\leqq O(1)\, u^{12\gamma} \quad \text{if} \quad \gamma < 1/40. \qquad \text{Q.E.D.}$$

Corollary 6.1. 

$\leqq O(1)\, u^{3\gamma}.$

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