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Feynman Path Integrals

I. Linear and Affine Techniques
II. The Feynman-Green Function

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Abstract. Path integrals techniques are derived from a new definition [1] of Feynman path integrals. These techniques are used to establish that Feynman-Green functions for a given physical system are covariances of pseudomeasures suitable for its path integrals. The variance of a pseudomeasure is a more versatile tool than the Feynman-Green function it defines.

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I. Introduction. Feynman Space

In a previous paper [1], a new definition of Feyman path integrals has been proposed. In this paper, this definition is exploited to derive new techniques and obtain new properties of the Feynman-Green function.

In [1] Feynman path integrals were defined on the space $\mathscr{C}$ of continuous functions defined on the time interval $T=[t_a, t_b]$ and vanishing at one end point of the interval. The Feynman formulation of Quantum Mechanics and Quantum Field Theory sets up a much greater class of path integrals than the one studied in [1]: transition probabilities, expectation values, quantum fluctuation defined with respect to a given classical background field, for instance, are expressed as path integrals on a space of paths [histories] between given initial and final points [states of the system]. Moreover the computation of a given path integral may be easier on a space other than the space on which it is defined or with respect to a different pseudomeasure.

The mathematical analysis of a physical system in Feynman theory begins with the identification of the space of all its possible, non-equivalent, paths. Paths are said to be equivalent if they cannot be physically distinguished; for instance, fields related by a gauge transformation. The space of all the possible, non-equivalent, paths (histories) of a physical system will be called the *Feynman space of the system* and denoted $\mathbb{F}$.

In this paper, we shall consider spaces which can either be obtained from $\mathscr{C}$ or mapped into $\mathscr{C}$ by linear and affine mappings; thus we deal here only with Feynman integrals in non-relativistic Quantum Mechanics. However, the linear and affine transformation techniques are presented in a form suitable to path integrals for Quantum Field Theory. This paper is concerned with applications; mathematical subtleties which are not important in the present context are mentioned but not discussed.

II. Notations[1]. Basic Definitions and Properties

1. *Feynman Integral on* $\mathscr{C}_-$

In [1] paths integrals were defined on a space of paths vanishing at time $t=t_b$. We shall need henceforth to consider both the space of paths vanishing at t_b and the space of paths vanishing at t_a; we shall call

$\mathscr{C}_-$ the space of paths x defined on $(t_a, t_b]$ such that $x(t)\to 0$ when $t\to t_a$

$\mathscr{C}_+$ the space of paths x defined on $[t_a, t_b)$ such that $x(t)\to 0$ when $t\to t_b$

$\mathscr{C}$ the space of paths x defined on (t_a, t_b) such that $x(t)\to 0$ when $t\to t_a$ and when $t\to t_b$.

[1] To a large extent, the notation is the same as in [1]; a few superficial differences have been introduced.

The paths x are continuous functions; the topology on $\mathscr{C}_-, \mathscr{C}_+, \mathscr{C}$ is the norm topology induced by the uniform norm $\|x\| = \sup|x(t)|$ for all t in the range of x. When it is not necessary to distinguish the three ranges $(t_a, t_b]$, $[t_a, t_b)$, and (t_a, t_b) we shall simply speak of paths x on T; the letter T stands also for $t_b - t_a$.

The Feynman integral of a function F on $\mathscr{C}_-$ is a complex number written symbolically

$$K = \int_{\mathscr{C}_-} F(x)\, dw_-(x)\,.$$

w_- is the complex gaussian pseudomeasure on $\mathscr{C}_-$ of covariance infimum; i.e. w_- is the pseudomeasure whose Fourier transform $\mathscr{F}w_-$ is a function on the dual[2] $\mathscr{M}$ of $\mathscr{C}_-$ given by

$$\mathscr{F}w_- = \exp(-iW_-/2)$$

where W_- is the quadratic form on $\mathscr{M}$ defined by

$$W(\mu) = \int_T d\mu(r) \int_T d\mu(s) \inf(r - t_a, s - t_a)$$

$$\inf(r - t_a, s - t_a) = Y^+(r-s)\,(s - t_a) + Y^-(r-s)\,(r - t_a)\,.$$

Y^+ is Heaviside step-up function on T equal to 1 for positive arguments and equal to 0 otherwise; Y^- is Heaviside step-down function on T equal to 1 for negative arguments and 0 otherwise; $Y^+(t) = Y^-(-t)$.

W is called the variance of the gaussian[3] w.

We shall designate by the same letter in different types a bilinear form on $\mathscr{M} \times \mathscr{M}$ and its corresponding quadratic form on $\mathscr{M}$

$$\mathsf{W}(\mu, \nu) = \tfrac{1}{2}(W(\mu+\nu) - W(\mu) - W(\nu))\,,$$

$$\mathsf{W}(\delta_r, \delta_s) = \inf(r - t_a, s - t_a) \equiv G(r, s)\,.$$

$\mathsf{W}(\delta_r, \delta_s)$ is called the covariance of w.

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[2] Let X be a topological vector space, let X' be its dual, let $x \in X$ and $x' \in X'$; $x'(x)$ is denoted $\langle x', x\rangle$; the operational meaning of $\langle x', x\rangle$ depends on the space X considered; for instance:

a) Let $X = \mathscr{C}_-$, then $X' = \mathscr{M}$ is the space of bounded measures μ on the range of $x \in \mathscr{C}_-$ and $\langle \mu, x\rangle = \int_T x(t)\, d\mu(t)$.

Examples: Let λ be the Lebesgue measure $\langle \lambda, x\rangle = \int_T x(t)\, dt$;

let δ_r be the Dirac measure at r $\langle \delta_r, x\rangle = x(r)$.

b) Let X be the Hilbert space of real square integrable functions on T, then X' is isomorphic to X and $\langle f, g\rangle = \int_T f(t)\, g(t)\, dt$ for $f, g \in \mathscr{H}$. In a complex Hilbert space $\langle f, g\rangle = \int_T f^*(t)\, g(t)\, dt$. If μ is a regular measure $d\mu(t) = m(t)\, dt$ and $\langle \mu, x\rangle_{\mathscr{C}} = \langle m, x\rangle_{\mathscr{H}}$.

c) Let X be equal to $\mathbb{R}^n$, X' is isomorphic to X and $\langle x', x\rangle = \sum_{i=1}^{n} x'_i x^i$.

[3] In this paper, "gaussian" stands for "complex gaussian".

We shall assume that F is the limit of cylindrical functions:

$$F = \lim_{n=\infty} f \circ P_n$$

where P_n is a linear continuous mapping from $\mathscr{C}$ to $\mathbb{R}^n$ and f is a Γ-integrable function on $\mathbb{R}^n$; i.e. $\langle \Gamma, f \rangle_{\mathbb{R}^n} < \infty$ where Γ is a complex gaussian. Moreover we shall assume that

$$\lim \int_{\mathscr{C}_-} (f \circ P_n)(x)\, dw_-(x) = \int_{\mathscr{C}_-} \lim (f \circ P_n)(x)\, dw_-(x)\,.$$

When these conditions are satisfied, we say that F is Feynman integrable. This is not a definition of Feynman integrability because these conditions are sufficient but may not be necessary.

The Feynman integral of a function F on $\mathscr{C}_+$ is defined similarily, mutatis mutandis: the covariance W_+ of the pseudomeasure w_+ is

$$\begin{aligned} W_+(\delta_r, \delta_s) &\equiv G_+(r, s) \\ &= \inf(t_b - r, t_b - s) = Y^+(r-s)(t_b - r) + Y^-(r-s)(t_b - s)\,. \end{aligned}$$

The pseudomeasure induced on $\mathscr{C}$ either by W_- or W_+ is computed in Section III.

2. *Continuous Linear and Affine Mappings*

a) Transformation of a complex gaussian [pseudo] measure under a linear mapping. Let X and Y be two topological vector spaces, Hausdorff and locally convex; let X' and Y' be their topological duals; let P be a linear continuous mapping from X into Y; let $\tilde{P}$ be the transposed mapping from Y' into X' defined by $\langle \tilde{P} y', x \rangle = \langle y', Px \rangle$.

By definition, a complex gaussian [pseudo] measure on X of variance Q is the [pseudo] measure whose Fourier transform is the function $\exp(-iQ/2)$ on X' where Q is a positive quadratic form. Its image under P is the gaussian [pseudo] measure on Y whose Fourier transform is the function $\exp(-iQ \circ \tilde{P}/2)$ on Y'.

b) Transformation under an affine mapping. Let τ be the translation $x \mapsto x + y$ with $x, y \in X$. The Fourier transform at x' of the image of a gaussian with variance Q under the translation τ is

$$\exp(-i\langle x', y \rangle) \exp(-iQ(x')/2)\,.$$

It is no longer the exponential of a quadratic form, hence, according to the previous definition, it is not a gaussian. We shall call it a translated gaussian. A translated gaussian is often casually called a gaussian (of mean m); a proper gaussian for which $m = 0$ is then called a centered gaussian. This terminology originates from the equation

$$\int_{\mathbb{R}} \exp(-i\langle x', x\rangle) \frac{1}{\sqrt{2\pi i\alpha}} \exp(i(x-m)^2/2\alpha)\, dx$$
$$= \exp(-i\langle x', m\rangle \exp(-i\alpha x'^2/2)).$$

The image of the gaussian [pseudo] measure of variance Q by the affine mapping $\tau \circ P$ is the translated gaussian whose Fourier transform is defined by

$$\exp(-i\langle x', y\rangle) \exp(-iQ \circ \tilde{P}(x')/2).$$

c) Examples. Let $\mathbb{F}$ stand for either $\mathscr{C}_-$ or $\mathscr{C}_+$ or $\mathscr{C}$; the dual of $\mathbb{F}$ is $\mathscr{M}$.

In this paper we shall use the following linear continuous mappings: $P: \mathbb{F} \to \mathbb{F}$, $P_n: \mathbb{F} \to \mathbb{R}^n$ and $P: \mathscr{H} \to \mathbb{F}$ where $\mathscr{H}$ is a Hilbert space.

III. Paths with Given Boundary Values

In the first paragraph we compute the pseudomeasure on $\mathbb{R}_n$ induced by the pseudomeasure w_- on $\mathscr{C}_-$; in Paragraph 2 we consider path integrals over the space of paths with non vanishing Dirichlet boundary values; in Paragraph 3 we determine the pseudomeasure on $\mathscr{H}$ which induces w_- on $\mathscr{C}_-$.

1. Measures on $\mathbb{R}^n$

Let $P_n: \mathbb{F} \to \mathbb{R}^n$ by $x \mapsto u$ where u is the n-tuple $(u^i = \langle \mu_i, x\rangle)$ for $\mu_1 \ldots \mu_n \in \mathscr{M}$. Because the family $(\delta_t; t \in T)$ is a basis for $\mathscr{M}$, the mapping from $\mathbb{F}$ into $\mathbb{R}^n$ defined by $(\mu_i = \delta_{t_i})$ will be called the natural mapping into $\mathbb{R}_n$ for the basis (δ_t).

The transposed mapping $\tilde{P}_n: \mathbb{R}^n \to \mathscr{M}$ by $\zeta \mapsto \mu$ is defined by

$$\begin{aligned}\langle \tilde{P}_n \zeta, x\rangle &= \langle \zeta, P_n x\rangle_{\mathbb{R}^n} \\ &= \Sigma \zeta_i u^i \quad \text{where } (\zeta_i) \text{ are the coordinates of } \zeta \text{ in the dual basis} \\ &= \langle \Sigma \zeta_i \mu_i, x\rangle \\ \tilde{P}\zeta &= \Sigma \zeta_i \mu_i .\end{aligned}$$

Proposition 1. *The image on* $\mathbb{R}_n$ *by* P_n *of the pseudomeasure* w *on* $\mathscr{C}$ *is the gaussian measure* w_n *of covariance* $\mathscr{W} = (\mathscr{W}^{ij})$ *where*

$$\mathscr{W}^{ij} = W(\mu_i, \mu_j).$$

Proof.. The variance W_n of w_n is

$$\begin{aligned}W_n(\zeta) &= W \circ \tilde{P}_n(\zeta) = W(\Sigma \zeta_i \mu_i) \\ &= \Sigma \zeta_i \zeta_j W(\mu_i, \mu_j) = \Sigma \zeta_i \mathscr{W}^{ij} \zeta_j . \quad \blacksquare\end{aligned}$$

Remark. w_- and w_+ induces gaussian measures on $\mathbb{R}^n$ of covariance

$$\mathscr{W}_-^{ij} = W_-(\mu_i, \mu_j) \quad \text{and} \quad \mathscr{W}_+^{ij} = W_+(\mu_i, \mu_j) \quad \text{respectively}.$$

Application. Path integrals of cylindrical functions on $\mathbb{F}$ can be immediately reexpressed as integrals over $\mathbb{R}^n$:

$$\int_{\mathbb{F}} f \circ P_n(x)\, dw(x) = \int_{\mathbb{R}^n} f(u)\, dw_n(u)$$
$$dw_n(u) = \frac{du^1 du^2 \ldots du^n}{(2\pi i)^{n/2} (\det \mathscr{W})^{1/2}} \exp\left(\frac{i}{2} u^i u^j (\mathscr{W}^{-1})_{ij}\right). \tag{1}$$

Equation (1) generalizes a result[4] given in [1]. Many results previously derived [2–5] can be obtained readily from Eq. (1) by choosing $\mu_i = \delta_{t_i}$ or $\mu_i = \delta_{t_i} - \delta_{t_{i-1}}$ with $t_a = t_0 < t_1 < \cdots < t_n = t_b$. Equation (1) is used extensively in this paper.

2. *Path Integrals on* $\mathscr{C}^{ab}$

Transition probabilities and expectation values are computed from path integrals on a space $\mathscr{C}^{ab}$ of paths q on $T = [t_a, t_b]$ with given Dirichlet boundary values: $q(t_a) = a$ and $q(t_b) = b$. The space $\mathscr{C}^{ab}$ is not a linear space; $q^1, q^2 \in \mathscr{C}^{ab} \not\Rightarrow q^1 + q^2 \in \mathscr{C}^{ab}$. It is not an affine subspace of $\mathscr{C}_-$; $q \in \mathscr{C}^{ab} \Rightarrow q \notin \mathscr{C}_-$ unless $a = 0$. The pseudomeasure w_{ab} on $\mathscr{C}^{ab}$ induced by the pseudomeasure w on $\mathscr{C}$ can be obtained from the translation $\tau: \mathscr{C} \to \mathscr{C}^{ab}$ by $x \mapsto q = x + \bar{q}$ where $\bar{q}$ is the average path in $\mathscr{C}^{ab}$

$$\mathscr{F} w_{ab}(\mu) = \exp(-i\langle \mu, \bar{q} \rangle) \exp(-i W(\mu)/2). \tag{2}$$

Because w_{ab} on $\mathscr{C}^{ab}$ has the same variance as w on $\mathscr{C}$

$$\int_{\mathscr{C}^{ab}} F(q - \bar{q})\, dw_{ab}(q) = \int_{\mathscr{C}} F(x)\, dw(x).$$

In particular

$$iG(r,s) = \int_{\mathscr{C}^{ab}} \langle \delta_r, q - \bar{q} \rangle \langle \delta_s, q - \bar{q} \rangle\, dw_{ab}(q)$$
$$\int_{\mathscr{C}^{ab}} \langle \delta_r, q \rangle \langle \delta_s, q \rangle\, dw_{ab}(q) = iG(r,s) + \bar{q}(r)\bar{q}(s).$$

The average $\bar{q} \in \mathscr{C}^{ab}$ can be computed from the average $\bar{z}$ in the affine subspace $\mathscr{C}^{b-a} \subset \mathscr{C}^-$ defined by $\langle \delta_{t_b}, y \rangle = b - a$: indeed the translation $z \mapsto q = z + a$ yields $\bar{q}(t) = \bar{z}(t) + a$

$$\bar{z}(t) = N^{-1} \int_{\mathscr{C}^-} \chi(y) \langle \delta_t, y \rangle\, dw_-(y) \tag{3}$$

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[4] Last equation p. 62. Note a misprint in the first equation p. 63; it should read

$$\int_{\mathscr{C}} \langle x', x \rangle^{2n}\, dw(x) = \frac{1}{2^n} \frac{(2n)!}{n!} (iW(x'))^n.$$

where $\chi(y)$ is the characteristic function of $\mathscr{C}^{b-a} \subset \mathscr{C}_-$

$\chi(y) = 1$ *if* $\langle \delta_{t_b}, y \rangle = b - a, \chi(y) = 0$ otherwise;

and where the normalization N is given by

$$N = \int_{\mathscr{C}_-} \chi(y)\, dw_-(y)\,. \tag{4}$$

The integrands of N and $\bar{z}(t)N$ are cylindrical functions and can be reexpressed as integrals over $\mathbb{R}$ and $\mathbb{R}^2$ respectively by using Eq. (1):

Let $P_1 : y \mapsto u = \langle \delta_{t_b}, y \rangle$; *then* $\mathscr{W}_- = W(t_b) = T$ and

$$N = \int_{\mathbb{R}} \delta(u - b + a) \frac{du}{\sqrt{2\pi i T}} \exp(iu^2/2T) = \frac{1}{\sqrt{2\pi i T}} \exp(i(b-a)^2/2T)\,.$$

Similarly, let $P_2 : y \mapsto (u^1 = \langle \delta_t, y \rangle, u^2 = \langle \delta_{t_b}, y \rangle)$ then $\mathscr{W}_- = \begin{pmatrix} t - t_a & t - t_a \\ t - t_a & t_b - t_a \end{pmatrix}$ and

$$\int_{\mathscr{C}_-} \chi(y) \langle \delta_t, y \rangle\, dw_-(y) = N \frac{t - t_a}{T} (b - a)\,.$$

Finally $\bar{q}(t) = \dfrac{t - t_a}{T} b + \dfrac{t_b - t}{T} a.$

The average $\bar{q}$ can also be computed from the average path in the affine subspace of $\mathscr{C}_+$ defined by $\langle \delta_{t_a}, y \rangle = a - b$.

Proposition 2. *The pseudomeasure* w *on* $\mathscr{C}$ *induced by either the pseudomeasure* w_- *on* $\mathscr{C}_-$, *or the pseudomeasure* w_+ *on* $\mathscr{C}_+$, *is the gaussian pseudomeasure of covariance*

$$G(r,s) = W(\delta_r, \delta_s) = \inf((r - t_a)(t_b - s)T^{-1}, (s - t_a)(t_b - r)T^{-1})\,.$$

Proof. Let χ be the characteristic function of $\mathscr{C}^b \subset \mathscr{C}_-$ defined by $\chi(x) = 1$ for paths x such that $\langle \delta_{r_b}, x \rangle = b$ and 0 otherwise, then

$$iG(r,s) = N^{-1} \int_{\mathscr{C}_-} \chi(x) \langle \delta_r, x \rangle \langle \delta_s, x \rangle\, dw_-(x) - \bar{q}(r)\bar{q}(s)$$

where N is the normalization given by Eq. (3) and $\bar{q}$ is the average of the paths in $\mathscr{C}^b$ given by Eq. (3). The integrand is a cylindrical function and the integral over $\mathscr{C}_-$ is readily expressed as an integral over $\mathbb{R}^3$ [Eq. (1)] by using $P_n^- : \mathscr{C}_- \to \mathbb{R}^3$ by $u^i = \langle \delta_i, x \rangle$ with $\delta_1 = \delta_r$, $\delta_2 = \delta_s$, $\delta_3 = \delta_{t_b}$. We obtain Proposition 2 by a calculation similar to the calculation of N and $\bar{q}$ previously outlined.

Remark: In some cases it is preferable to solve a problem on $\mathscr{C}$ and then make a translation to $\mathscr{C}^{ab}$, in others it is preferable to set up the problem directly on $\mathscr{C}^{ab}$. A similar situation occurs in field theory: in some cases it is preferable to compute vacuum expectation values, in others it is preferable to consider quantum fluctuations with respect to a non zero classical background field.

3. Canonical Measure on $\mathscr{H}$

Definition. The canonical gaussian pseudomeasure on a Hilbert space $\mathscr{H}$ of real square integrable functions on T is the gaussian of variance

$$I(f) = \langle f, f \rangle_{\mathscr{H}} = \int_T f(t)\, f(t)\, dt\,.$$

Let P, P_-, P_+ be the primitive mappings defined by

$$P_- : \mathscr{H} \to \mathscr{C}_- \quad \text{by} \quad P_- f(t) = x(t) = \int_{t_a}^{t} f(r)\, dr$$

$$P_+ : \mathscr{H} \to \mathscr{C}_+ \quad \text{by} \quad P_+ f(t) = x(t) = \int_{t}^{t_b} f(r)\, dr$$

$$P : \mathscr{H} \to \mathscr{C} \quad \text{by} \quad P f(t) = x(t) = \frac{t_b - t}{T} \int_{t_a}^{t} f(r)\, dr - \frac{t - t_a}{T} \int_{T}^{t_b} f(r)\, dr$$

The transposed of the primitive mappings are

$$\tilde{P}_- \mu(r) = \int_T Y^-(r-t)\, d\mu(t)$$

$$\tilde{P}_+ \mu(r) = \int_T Y^+(r-t)\, d\mu(t)$$

$$\tilde{P} \mu(r) = \int_T d\mu(t) \left(\frac{t_b - t}{T}\, Y^-(r-t) - \frac{t - t_a}{T}\, Y^+(r-t) \right)$$

in particular

$$\tilde{P} \delta_s(r) = (t_b - s)\, T^{-1} Y^-(r-s) - (s - t_a)\, T^{-1} Y^+(r-s)\,.$$

Proposition 3. *The pseudomeasure w on $\mathscr{C}$ is the image by P of the canonical pseudomeasure on $\mathscr{H}$.*

Proof.

$$\begin{aligned} I \circ \tilde{P}(\mu, v) &= \int_T d\mu(r) \int_T dv(s) \int_T dt \left(\frac{t_b - r}{T}\, Y^-(t-r) - \frac{r - t_a}{T}\, Y^+(t-r) \right) \\ &\quad \cdot \left(\frac{t_b - s}{T}\, Y^-(t-s) - \frac{s - t_a}{T}\, Y^+(t-s) \right) \\ &= \int_T d\mu(r) \int_T dv(s)\, \inf((r - t_a)(t_b - s)\, T^{-1}, (s - t_a)(t_b - r)\, T^{-1}) \\ &= W(\mu, v)\,. \end{aligned}$$

Similarly one show that $I \circ \tilde{P}_- = W_-$ and $I \circ \tilde{P}_+ = W_+$; these last two equations generalize to w_- and w_+ a well-known property of the Wiener measure.

IV. The Feynman-Green Functions

1. Introduction

In this section we establish the relationship between the action S of a system and the pseudomeasures of its path integrals. The action is a function on the path space $\mathbb{F}$ of the system; the pseudomeasures are defined in terms of their Fourier transform, i.e. in terms of functions on the dual of $\mathbb{F}$. We cannot construct directly a "natural" mapping from $\mathbb{F}$ into its dual, nor vice versa, but we can do so via the natural mapping $P_n : \mathscr{C} \to \mathbb{R}^n$ and via the primitive mapping $P : \mathscr{H} \to \mathscr{C}$ defined in Section III because there exist canonical isomorphisms between $\mathbb{R}^n$ and its dual, and, between $\mathscr{H}$ and its dual.

The gaussian of covariance infimum on $\mathscr{C}$ is induced from the canonical gaussian on $\mathscr{H}$ by the primitive mapping P. The gaussian of covariance $\mathscr{W} = (\mathscr{W}^{ij} = W(\delta_{t_i}, \delta_t))$ on $\mathbb{R}^n$ is induced from the gaussian of covariance infimum on $\mathscr{C}$ by the mapping P^n, natural for the basis (δ_t) on $\mathscr{M}$.

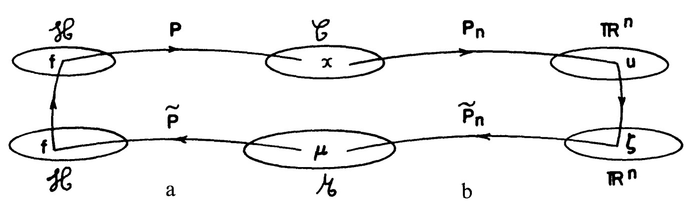

Fig. 1a, b. The gaussian of covariance infimum on $\mathscr{C}$ is induced from the canonical gaussian on $\mathscr{H}$ by the primitive mapping P. The gaussian of covariance $\mathscr{W} = (\mathscr{W}^{ij} = W(\delta_{t_i}, \delta_{t_j}))$ on $\mathbb{R}^n$ is induced from the gaussian of covariance infimum on $\mathscr{C}$ by the mapping P^n, natural for the basis (δ_t) on $\mathscr{M}$

Fig. 1a shows the construction of $P \circ J \circ \tilde{P} : \mathscr{M} \to \mathscr{C}$; the canonical isomorphism $J : \mathscr{H} \to \mathscr{H}$ is the identity when $\mathscr{H}$ is real and is the complex conjugation when $\mathscr{H}$ is complex. Figure. 1b shows the construction $\tilde{P}_n \circ J \circ P_n : \mathscr{C} \to \mathscr{M}$; in this section P_n is the natural mapping for the basis (δ_t) on $\mathscr{M}$, i.e. $P_n : x \mapsto (u^i = \langle \delta_{t_i}, x \rangle)$; J is the identity and will be omitted for brevity. To simplify the presentation we consider the case $\mathbb{F} = \mathscr{C}$; the results can immediately be restated for the cases $\mathbb{F} = \mathscr{C}_-$ and $\mathbb{F} = \mathscr{C}_+$; it will also be clear how they can be extended to other spaces.

2. *Definitions*

In this paper a Green function is defined in its strict sense:

Definition: The Green function for a positive (elliptic) second order linear differential operator D^2 with constant coefficients defined on an open set U is a kernel on $U \times U$ such that

$-D_r^2 G(r,s) = \delta_s$	G is an elementary kernel of $-D^2$
$G(r,s) = 0$ for $r \in \partial U, s \in U$	vanishing on the boundary
G is C^∞ on $\bar{U} \times U - \{r = s\}$	which is C^∞ except on the diagonal.

Definition. The small disturbance operator of a system S defined on $\mathscr{C}^{ab}$ is the operator D^2 defined on $\mathscr{C}$ by the equation

$$S''(\bar{q})\, x y \equiv \mathbf{S}(x,y) = -\int_T x(t)\, D^2 y(t)\, dt \quad \text{with} \quad \bar{q} \in \mathscr{C}^{ab} \quad \text{and} \quad x, y, \in \mathscr{C}$$

where S'' is the second derivative of the action S of the system S and where $\bar{q}$ is the classical path of S, i.e. $S'(\bar{q}) = 0$.

Definition. Let S_0 be a quadratic action, i.e.

$$S_0(x + \bar{q}) = S_0(\bar{q}) + \tfrac{1}{2} S_0''(\bar{q})\, x x \qquad \bar{q} \in \mathscr{C}^{ab}, x \in \mathscr{C}$$
$$S_0(x) = \tfrac{1}{2} S_0''(\bar{q})\, x x$$

the small disturbance operator of S_0 is a positive second order linear differential operator with constant coefficients defined on (t_a, t_b); its Green function will be called "*the* Green function of S_0" for brevity.

Definition. The Feynman-Green function for the system S is the expectation value of the time ordered product $x(r)\, x(s)$ in the transition from the initial state $A \equiv (x(t_a) = 0, t_a)$ to the final state $B \equiv (x(t_b) = 0, t_b)$ of the system S. It vanishes for either r or s equal to t_a or t_b.

3. *The Feynman-Green Function as a Covariance*

Proposition 4. *The covariance $W(\delta_r, \delta_s)$ of the gaussian pseudomeasure w on $\mathscr{C}$ is the Feynman-Green function $G(r,s)$ for the free particle.*

In general, given a system $S = S_0 + S_1$ where S_0 is a quadratic action, the Green function of S_0 is the covariance of a pseudomeasure suitable for the path integrals of the system S; i.e. the Feynman-Green function of the small disturbance operator S_0 is the covariance of a pseudomeasure suitable for the path integrals of the system S.

We shall give four proofs of the fact that $W(\delta_r, \delta_s)$ is the Feynman-Green function for the free particle in order to illustrate different aspects of the path integral formalism; three proofs are given in Paragraphs a, b, c below, the fourth is given in Paragraph 5.

a) Let $S_0 = \frac{1}{2} \int_T x^2(t)\, dt$, we have shown in [1] that the formal expression $\int_{\mathscr{C}} \exp(iS_0(x))\, \mathscr{D}x$ defined by Feynman is equal to $\int_{\mathscr{C}} dw(x)$. Hence

$$\langle T(x(r)\, x(s))\rangle_{S_0} \equiv \int_{\mathscr{C}} x(r)\, x(s) \exp(iS_0(x))\, \mathscr{D}x = \int_{\mathscr{C}} \langle \delta_r, x\rangle \langle \delta_s, x\rangle\, dw(x)\,.$$

b) One can check directly that $W(\delta_r, \delta_s)$ is *the* Green function of d^2/dt^2. ■

c) The third proof shows, on a particular case, the relationship between an action S and the pseudomeasures suitable for the path integrals of the system S. Let P_n be the natural mapping from $\mathscr{C}$ into $\mathbb{R}^n$ by $x \mapsto u = (u^i = \langle \delta_{t_i}, x\rangle)$ for $t_a = t_0 < t_1 < \cdots < t_n < t_{n+1} = t_b$. P_n maps w into the gaussian of covariance $\mathscr{W} = (\mathscr{W}^{ij})$ where $\mathscr{W}^{ij} = G(t_i, t_j)$.

$\mathscr{W}$ has the typical pattern of an "infimum matrix"; its inverse is a second order difference operator.

$$T\mathscr{W} = \begin{pmatrix} \alpha_1\beta_1 & \alpha_1\beta_2 \ldots\ldots\ldots\ldots & \alpha_1\beta_n \\ \alpha_1\beta_2 & \alpha_2\beta_2 \quad \alpha_2\beta_3 \ldots & \alpha_2\beta_n \\ \vdots & \alpha_2\beta_3 \quad \alpha_3\beta_3 \cdots & \alpha_3\beta_n \\ \vdots & \vdots \qquad \vdots & \\ \alpha_1\beta_n & \alpha_2\beta_n \quad \alpha_3\beta_n & \alpha_n\beta_n \end{pmatrix} \quad \text{where } \alpha_i = t_i - t_a, \quad \beta_i = t_b - t_i$$

$$\mathscr{W}^{-1} = \begin{pmatrix} \frac{1}{t_1 - t_a} + \frac{1}{t_2 - t_1} & \frac{-1}{t_2 - t_1} & & & \bigcirc \\ \frac{-1}{t_2 - t_1} & \frac{1}{t_2 - t_1} + \frac{1}{t_3 - t_2} & \frac{-1}{t_3 - t_2} & & \\ & \ddots & \ddots & \ddots & \\ & & & & \frac{-1}{t_n - t_{n-1}} \\ \bigcirc & & & \frac{-1}{t_n - t_{n-1}} & \frac{1}{t_n - t_{n-1}} + \frac{1}{t_b - t_n} \end{pmatrix}$$

$\mathscr{W}_{ij}^{-1} u^j = -(u^{i+1} - u^i)/(t_{i+1} - t_i) + (u^i - u^{i-1})/(t_i - t_{i-1})$ *together with* $u^0 = u^{n+1} = 0$.

Let Δ_i be the difference operator corresponding to the differential operator d^2/dt^2 defined on the space $\mathscr{C}$ of paths vanishing on the boundary, i.e. set $\mathscr{W}_{ij}^{-1} u^j = \langle -\Delta_i \delta, x\rangle$

$$\sum_{i,j,k} \zeta_i \mathscr{W}^{ij} (\mathscr{W}^{-1})_{jk}\, u^k = \sum_i \zeta_i u^i = \langle \zeta, u\rangle = \langle \mu, x\rangle\,,$$

using

$$\sum_i \zeta_i \mathscr{W}^{ij} = \left\langle \sum_i \zeta_i \delta_{t_i}(v), G(v, t_j)\right\rangle$$
$$= \langle \mu(v), G(v, t_j)\rangle$$

we obtain

$$\sum_j \langle -\langle \mu(v), G(v,t_j)\rangle \,\Delta_j \delta_t, x\rangle = \langle \mu, x\rangle \quad \forall \mu \in \mathcal{M} \quad \text{and} \quad \forall x \in \mathscr{C}$$

which can be written more explicitly, albeit symbolically

$$\int_T \left\{ \sum_{j=1}^{n} \left(\int_T d\mu(v)\, G(v,t_j) \right) (-\Delta_j \delta_t) \right\} (s)\, x(s) = \int_T d\mu(s)\, x(s)$$

for $\mu = \delta_r$, it comes

$$\int_T \left\{ \sum_j G(r,t_j)\,(-\Delta_j \delta_t) \right\} (s)\, x(s) = x(r)$$

hence

$$\sum_j G(r,t_j)\,(-\Delta_j \delta_t) = \delta_r \quad \text{or} \quad \sum_j (-\Delta_j G(r,t)\, \delta_{t_j} = \delta_r)$$

where Δ_j operates on the second argument of G, hence

$$-\Delta_j G(r,t) = \delta_r^{t_j} \quad \text{where} \quad \delta_r^{t_j} \text{ is the Kronecker } \delta\,.$$

This is the difference equation corresponding to

$$-\frac{d^2}{dt^2} G(r,t) = \delta(r-t) \ \textit{together with} \text{ the boundary condition } G(r,t_a) = G(r,t_b) = 0$$

G is the unique inverse of $-\delta''$ satisfying these properties. ■

Remark. This proof explains the remark that "the Feynman-Green function behaves like a finite matrix" [6].

Remark. Similar results can be derived for the covariances G_- and G_+:

$$(\mathscr{W}_-^{-1})_{ij} = (\mathscr{W}^{-1})_{ij} \quad \text{except for } i=j=n\,.$$

$$(\mathscr{W}_+^{-1})_{ij} = (\mathscr{W}^{-1})_{ij} \quad \text{except for } i=j=1\,.$$

The "first edge" term of $\mathscr{W}_+^{-1}$ and the "last edge" term of $\mathscr{W}_-^{-1}$ differ from the edge terms of $\mathscr{W}^{-1}$ because of the different boundary values of x in $\mathscr{C}_-$, $\mathscr{C}_+$ and $\mathscr{C}$.

$\mathscr{W}^{-1} u = 0$ is the difference equation corresponding to $-d^2x/dt^2 = 0$ *for* $x \in \mathscr{C}$

$\mathscr{W}_-^{-1} u = 0$ is the difference equation corresponding to $-d^2x/dt^2 = 0$ for $x \in \mathscr{C}_-$

$\mathscr{W}_+^{-1} u = 0$ is the difference equation corresponding to $-d^2x/dt^2 = 0$ for $x \in \mathscr{C}_+$.

Because difference operators are finite matrices, they have unique inverses; an equivalent statement is: because a difference equation has built-in boundary conditions, its solution is uniquely defined. Thus G, G_-, G_+ are unique solutions of difference equations defined respectively by $\mathscr{W}^{-1}, \mathscr{W}_-^{-1}$ and $\mathscr{W}_+^{-1}$.

The covariances G_- and G_+ are not Green functions; they are not expectation values of time ordered products because the paths in $\mathscr{C}_-$ and $\mathscr{C}_+$ do not have fixed values at both $t=t_a$ and $t=t_b$.

d) The proof c can be restated in more general terms to complete the proof of Proposition 4:

$$S(x+\bar{q}) = S_0(\bar{q}) + \tfrac{1}{2} S_0''(\bar{q})\, xx + S_1(x+\bar{q}) \qquad \bar{q} \in \mathscr{C}^{ab}, x \in \mathscr{C}$$

where $\bar{q}$ is the classical path of the action S_0 defined on $\mathscr{C}^{ab}$.

Let $G(r,s)$ be the Green function for S_0, let w be the gaussian pseudomeasure on $\mathscr{C}$ defined by the covariance G, then the pseudomeasure w_{ab} on $\mathscr{C}^{ab}$ induced by the measure w on $\mathscr{C}$ [see Eq. (2)] is suitable to compute the path integrals of the system S: The path $\bar{q}$ minimizes S_0, hence the bilinear form $\boldsymbol{S}_0$ on $\mathscr{C} \times \mathscr{C}$ defined by $\boldsymbol{S}_0(x,y) = S_0''(\bar{q})\, xy$ is positive, and the Green function of S_0 defines a positive variance:

$$W(\mu, v) = \int_T d\mu(r) \int_T dv(s)\, G(r,s)\,.$$

4. The Uhlenbeck-Ornstein Approximation

Given an action S, there are often different ways of separating out the quadratic terms S_0, leading to different approximations. For example, let

$S(x) = \int_T \left(\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}\omega^2 x^2 - Y(x)\right) dt$ we can either choose $S_0(x) = \tfrac{1}{2}\int_T \dot{x}^2\, dt$

and use the pseudomeasure w_{ab} of covariance

$$G(r,s) = \inf\left((r-t_a)(t_b-s)\, T^{-1}, (s-t_a)(t_b-r)\, T^{-1}\right)$$

or choose

$$S_0(x) = \tfrac{1}{2}\int_T (\dot{x}^2 - \omega^2 x^2)\, dt$$

and use the gaussian pseudomeasure w_{ab}^{ω} of covariance

$$\begin{aligned} G_\omega(r,s) = {} & Y^+(r-s)\frac{1}{\omega \sin\omega T} \sin\omega(s-t_a) \sin\omega(t_b-r) \\ & + Y^-(r-s)\frac{1}{\omega\sin\omega T} \sin\omega(r-t_a)\sin\omega(t_b-s)\,. \end{aligned}$$

The latter choice leads to perturbation expansions which converge more rapidly than the former [5].

5. *The White Noise Approximation*

We have shown in Paragraph III.3 that the pseudomeasure w on $\mathscr{C}$ is the image by the primitive mapping P of the *canonical* gaussian pseudomeasure on $\mathscr{H}$. It is natural then to call w a *white noise* pseudomeasure and to call white noise approximations the approximations consisting in a power expansion of the path integral $\int_{\mathscr{C}} \exp(iS(x))\, dw(x)$.

Although we have already proved three times that the covariance of w is the Green function of the free particle action, we shall prove it again here from the definition of w as a white noise pseudomeasure:

$$\begin{aligned} G(r, s) &= W(\delta_r, \delta_s) \\ &= \langle \tilde{P}\delta_r, J \circ \tilde{P}\delta_s \rangle \\ &= \langle \delta_r, P \circ J \circ \tilde{P}\delta_s \rangle \\ G(t, s) &= (P \circ J \circ \tilde{P}\delta_s)(t) . \end{aligned}$$

The mapping P does not have an inverse; nevertheless $P \circ J \circ \tilde{P}$ has an inverse, namely $-d^2/dt^2$ operating in $\mathscr{C}$, and $(P \circ J \circ \tilde{P})_t^{-1}\, G(t, s) = \delta_s$. ■

V. Covariances of Quantum Mechanics

In this section we establish the relationships between the covariances of the gaussian pseudomeasure used in the path integral formalism and some basic functions of Quantum Mechanics.

Let G be the covariance of a gaussian pseudomeasure w on the space $\mathscr{C}$ of paths vanishing on the boundary of T; let G_- and G_+ be the covariances of the pseudomeasures w_- and w_+ on the spaces $\mathscr{C}_-$ and $\mathscr{C}_+$ of paths with a free end point that induces w on $\mathscr{C}$. Let G, G_-, G_+ be elementary kernels of a second order linear positive differential operator with constant coefficients D^2.

1. *Advanced and Retarded "Green" Functions*

Let G^{adv} and G^{ret} be the advanced and retarded elementary kernels of D^2: $D_r^2 G^{\substack{adv\\ret}}(r, s) = \delta_s$; $G^{adv}(r, s) = 0$ when $r > s$; $G^{ret}(r, s) = 0$ when $r < s$. Then

$$\begin{aligned} G_-(r, s) &= G^{ret}(r, s) - G^{ret}(r, t_a) \qquad && r, s, \in (t_a, t_b] \\ G_+(r, s) &= G^{adv}(r, s) - G^{adv}(t_b, s) \qquad && r, s \in [t_a, t_b) . \end{aligned}$$

Proof. Set $F = G_- - G^{ret}$; $F(r, s)$ is a solution of $D_r^2 F(r, s) = 0$ which goes to 0 when $r \to t_a$ such that $F(r, s) - F(s, r) = G^{adv}(r, s) - G^{ret}(r, s)$ it follows that $F(r, s) = -G^{ret}(r, t_a)$.

2. *Van Vleck Determinant*

Let M be the Van Vleck determinant; for paths $q: T \to \mathbb{R}^3$ by $q(t) = (q^\alpha(t); \alpha = 1, 2, 3)$ such that $q^\alpha(t_a) = a^\alpha$ and $q^\alpha(t_b) = b^\alpha$, M is the determinant of the matrix

$$M_{\alpha\beta} = \frac{\partial^2 S(a, b, \overline{q})}{\partial a^\alpha \, \partial b^\beta}.$$

When $S(q) = \int_T \frac{1}{2}\dot{q}^2(t)\, dt$ with $q(t) \in \mathbb{R}$ then $M = T^{-1}$. In general let M be the Van Vleck determinant for the action whose small disturbance operator is the differential operator D^2, then

$$G(r, s) = G_-(r, s)\, M\, G_+(r, s).$$

Proof. Following the notation and the proof of Proposition 1, we obtain:

$$G(r, s) = W_-(\delta_r, \delta_s)\, [W_-(\delta_{t_b})]^{-1}\, (W_-(\delta_{t_b}) - W_-(\delta_s, \delta_{t_b}))$$
$$G(r, s) = G_-(r, s)\, M\, G_+(r, s). \quad \blacksquare$$

This result is readily generalized to pseudomeasures on space of paths with values on $\mathbb{R}^n$: The Feynman-Green function is then defined by

$$i G^{\alpha\beta}(r, s) = \int_{\mathscr{C}} \langle \delta_r, x^\alpha \rangle \langle \delta_s, x^\beta \rangle \, dw(x).$$

The covariances $G_-^{\alpha\beta}$ and $G_+^{\alpha\beta}$ are defined similarly and

$$G^{\alpha\beta}(r, s) = G_-^{\alpha\gamma}(r, s)\, M_{\gamma\delta} G_+^{\delta\beta}(r, s).$$

By expressing the covariances $G_-^{\alpha\beta}$ and $G_+^{\alpha\beta}$ in terms of the retarded and advanced "Green" functions, this equation yields a relationship derived by Bryce DeWitt [7] by a variational method for a large class of actions.

Remark. The determinant M was introduced in 1928 by Van Vleck [8] in his classic paper on the correspondence principle. It was later found [9] to be the normalization factor in Feynman integrals necessary to maintain the unitarity of the probability amplitudes. The product of n such factors, and its formal limit when $n \to \infty$, comes in the original definition of Feynman integral [9, Eq. (21)]. It can be shown to be equal to the determinant of the covariance $\mathscr{W}$ whose ij element is

$$W(\delta_{t_i}, \delta_{t_j}) = G(t_i, t_j).$$

3. *Uncertainty Principle*

An analysis of the uncertainty principle in the framework of the path formalism can be made by studying the product $q(r)\,\dot{q}(s)$ of the position and velocity of a particle: The uncertainty principle is a physical statement of the undefined value of $\dot{q}(s)$ for $q \in \mathscr{C}^{ab}$.

Let W be the variance of the pseudomeasure w_{ab} on $\mathscr{C}^{ab}$ corresponding to the free particle dimensionless action $\frac{m}{\hbar} \int_T \frac{1}{2} \dot{q}^2(t)\, dt$.

Formally

$$\langle b, t_b | T q(r)\, \dot{q}(s) | a, t_a \rangle = - \int_{\mathscr{C}^{ab}} \langle \delta_r, q \rangle \langle \delta'_s, q \rangle \, dw_{ab}(q)$$

$$= - i W(\delta_r, \delta'_s) - \bar{q}(r)\, \dot{\bar{q}}(s)$$

δ'_s is not in $\mathscr{M}$, hence $W(\delta_r, \delta'_s)$ is not defined; the extent to which it is not defined provides some insight in the uncertainty principle; for $r \neq s$ we obtain

$$W(\delta_r, \delta'_s) = -\frac{d}{ds} G(r, s) = \frac{\hbar}{mT} \left(Y^-(r-s)(r-t_a) - Y^+(r-s)(t_b - r) \right).$$

The discontinuity of W at $r = s$ is equal to $\hbar/m$. The expectation value of $q(r)\,\dot{q}(r)$ of a particle of mass m with known positions at t_a and t_b cannot be determined with an accuracy better than $\hbar/m$.

VI. The Diagram Technique

The diagram technique is derived from the new definition of the Feynman integrals so that it can be readily combined with the linear-affine techniques.

The Feynman rules are a prescription for the integration of

$$I = \int_{\mathscr{C}^{ab}} \exp(i S_1(q))\, dw_{ab}(q)$$

with $S_1(q) = \int_T V(q(t))\, dt = \langle \lambda, V(q) \rangle$ where λ is the Lebesgue measure.

The power expansion of S gives

$$I = \Sigma \frac{i^n}{n!} I_n \quad \text{with} \quad I_n = \int_{\mathscr{C}^{ab}} \langle \lambda, V(q) \rangle^n \, dw_{ab}(q).$$

1. Linear Continuous Potentials. $S_1(q) = \int_T f(t)\, q(t)\, dt$

Proposition 6. *Let V be a linear continuous mapping from $\mathscr{C}$ into $\mathscr{C}$, let V be its transposed mapping, then*

$$\int_{\mathscr{C}^{ab}} \langle \mu_1, V(q) - V(\bar{q}) \rangle \langle \mu_2, V(q) - V(\bar{q}) \rangle \cdots \langle \mu_n, V(q) - V(\bar{q}) \rangle \, dw_{ab}(q)$$

$$= i^{n/2} \Sigma W \circ \tilde{V}(\mu_{i_1}, \mu_{i_2})\, W \circ \tilde{V}(\mu_{i_3}, \mu_{i_4}) \ldots W \circ \tilde{V}(\mu_{i_{n-1}}, \mu_{i_n})$$

for n even and 0 *for n odd. The quantity*

$$I_n = \int_{\mathscr{C}_{ab}} \langle \mu_1, V(q)\rangle \langle \mu_2, V(q)\rangle \dots \langle \mu_n, V(q)\rangle \, dw_{ab}(q)$$

is readily obtained by multiplying out the lefthand side of this equation.

The proof of Proposition 6 is given in three steps

a)
$$\int_{\mathscr{C}} \langle \mu_1, x\rangle \langle \mu_2, x\rangle \dots \langle \mu_{2n}, x\rangle \, dw(x) = (i)^n \Sigma W(\mu_{i_1}, \mu_{i_2}) W(\mu_{i_3}, \mu_{i_4}) \dots W(\mu_{i_{2n-1}}, \mu_{i_{2n}})$$

where the sum Σ is taken over all partitions of $\{1, \dots 2n\}$.

$$\int_{\mathscr{C}} \langle \mu_1, x\rangle \dots \langle \mu_{2n+1}, x\rangle \, dw(x) = 0 .$$

These equations can be proved by using Eq. (1). They can also be proved by expressing the multilinear form $\langle \mu_1, x\rangle \dots \langle \mu_{2n}, x\rangle$ on $\mathscr{M} \times \dots \times \mathscr{M}$ as a linear combination

$$\langle \mu_i, x\rangle^{2n} - \Sigma \langle \mu_{i_1} + \dots + \mu_{i_{2n-1}}, x\rangle^{2n} + \dots - \Sigma \langle \mu_i, x\rangle^{2n} .$$

The integral of each term can be reexpressed in an integral over $\mathbb{R}$, it gives:

$$\int_{\mathscr{C}} \langle \mu_{i_1} + \dots \mu_{i_k}, x\rangle^{2n} \, dw(x) = \frac{1}{2^n} \frac{(2n)!}{n!} (i W(\mu_{i_1} + \dots + \mu_{i_k}))^n .$$

Using $W(\Sigma(\mu_i)) = \Sigma W(\mu_i) + 2\Sigma W(\mu_i, \mu_j)$, the desired result is obtained.

b)
$$\int_{\mathscr{C}_{ab}} \langle \mu_1, q - \bar{q}\rangle \langle \mu_2, q - \bar{q}\rangle \dots \langle \mu_{2n}, q - \bar{q}\rangle \, dw_{ab}(q) = (i)^n \Sigma W(\mu_{i_1}, \mu_{i_2}) \dots W(\mu_{i_{2n-1}}, \mu_{i_{2n}}) .$$

c) V being a linear mapping $V(q) - V(\bar{q}) = V(q - \bar{q}) = V(x)$. We can treat $V: \mathscr{C} \to \mathscr{C}$ as a change of variable; the pseudomeasure induced by this change of variable is the gaussian of variance $W \circ \tilde{V}$. ■

For easy comparison with results derived previously in the case $\mu_i = \delta_{t_i}$ we write out I_n; set $W \circ \tilde{V} \equiv W_V$

$$I_1 = \langle \mu, V(\bar{q})\rangle$$

$$I_2 = i W_V(\mu_1, \mu_2) + \langle \mu_1, V(\bar{q})\rangle \langle \mu_2, V(\bar{q})\rangle$$

$$I_3 = i \Sigma W_V(\mu_{i_1}, \mu_{i_2}) \langle \mu_{i_3}, V(\bar{q})\rangle$$

$$I_4 = -\Sigma W_V(\mu_{i_1}, \mu_{i_2}) W_V(\mu_{i_3}, \mu_{i_4}) + i \Sigma W_V(\mu_{i_1}, \mu_{i_2}) \langle \mu_{i_3}, V(\bar{q})\rangle \langle \mu_{i_4}, V(\bar{q})\rangle + \langle \mu_1, V(\bar{q})\rangle \langle \mu_2, V(\bar{q})\rangle \langle \mu_3, V(\bar{q})\rangle \langle \mu_4, V(\bar{q})\rangle .$$

Example. The forced harmonic oscillator

$$S=\int_T \left(\tfrac{1}{2}\dot{x}^2(t)-\tfrac{1}{2}\omega^2 x^2(t)+f(t)\,x(t)\right)dt=S_0+S_1$$

where

$$S_1(x)=\int_T V(x(t)\,dt=\int_T f(t)\,x(t)\,dt\,.$$

Let w^{ω}_{ab} be the gaussian pseudomeasure defined for the Uhlenbeck-Ornstein approximation in Paragraph IV.4.; let W_ω be the variance of w^{ω}_{ab}, the variance is:

$$W_\omega\circ\tilde{V}(\lambda)=\int_T dr\int_T ds\, f(r)\, f(s)\, G_\omega(r,s)\,.$$

The integrals I_n are conveniently recorded by the Feynman diagrams in which $f(r)\,f(s)\,G_\omega(r,s)$ is represented by a particle line from r to s. To each line is associated a propagator: the Feyman-Green function G_ω, to each vertex is associated the interaction f. The sum Σ gives all possible diagrams of a given order. The number of possible diagrams is $n!/(\sqrt{2})^n\cdot(n/2)!=1\cdot 3\cdots\cdot(n-1)$ as can be seen readily by setting $\mu_1=\mu_2=\cdots\mu_{2n}$ in the 2 integrals of paragraph a.

2. *Non-linear Potentials Satisfying* $\int_{\mathbb{R}} V(u)\exp(iu^2)\,du<\infty$

Set $\int_T V(x(t))\,dt=\int_T dt\;V(\langle\delta_t,x\rangle)$ and use

$$\int_{\mathscr{C}_{ab}}\left(\int_T dt\;V(\langle\delta_t,q-\bar{q}\rangle)\right)^n dw_{ab}(q)$$

$$=\int_T dt_1\ldots\int_T dt_n\frac{1}{(2\pi i)^{n/2}\,(\det\mathscr{W})^{1/2}}\int_{\mathbb{R}^n} du^1\ldots du^n\,V(u^1)\ldots$$

$$V(u^n)\exp\frac{i}{2}\,\Sigma u^i u^j(\mathscr{W}^{-1})_{ij}$$

where $\mathscr{W}$ is the matrix whose ij element is $G(t_i,t_j)$.

When $V(u)$ is a polynomial, the integration over $\mathbb{R}^n$ is straightforward and the computation of I_n can again be stated in terms of diagrams.

When V is an arbitrary non-linear interaction, the integral over $\mathbb{R}^n$ is usually not expressible in closed form.

VII. Conclusion. Variances Versus Feynman-Green Functions

A brief statement of the progress made in this paper could be this: the variance $W(\mu,\nu)$ is a more versatile tool than the Feynman-Green function $G(r,s)=W(\delta_r,\delta_s)$ it defines; a more sophisticated statement could be this: In the "definition" of Feynman integrals in terms of the action, the emphasis is on $x(t)$; in the definition of Feynman integrals in terms of a pseudomeasure, the emphasis is on $x \in \mathbb{F}$ where $\mathbb{F}$ is the Feynman space of the system. By shifting emphasis, one can use simple and reliable techniques to simplify and generalize previous results as well as to obtain new ones.

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