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Martin-Dynkin Boundaries of Random Fields

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Abstract. Analogous of exit spaces of Dynkin [4] for Markov processes are constructed for random fields introduced by Dobrushin [2].

Let X be a finite set and let T be countable. Let $\Omega = X^T$ and let $\mathscr{B}_V$ be the σ-algebra generated by $\{\omega \in \Omega;\ \omega(t) = x\}_{t\in V, x\in X}$ for $V \subset T$. The σ-algebra $\mathscr{B}_T$ is denoted simply by $\mathscr{B}$. Let be given a system of *conditional distributions* $q_{V,\omega}(A)$, which satisfy the following conditions, where V is a finite subset of T, $\omega \in \Omega$ and $A \in \mathscr{B}_V$.

α) $q_{V,\omega}(\cdot)$ is a probability measure on $\mathscr{B}_V$.

β) $q_{V,\omega}(A)$ is a $\mathscr{B}_{V^c}$-measurable function of ω for $A \in \mathscr{B}_V$.

γ) If $V_1 \subset V_2$, then for $A \in \mathscr{B}_{V_1}$, $B \in \mathscr{B}_{V_2 \setminus V_1}$ and $\omega \in \Omega$

$$q_{V_2,\omega}(A \cap B) = \int_B q_{V_1, c(V_2;\omega',\omega)}(A)\, q_{V_2,\omega}(d\omega'),$$

where $c(V_2;\omega',\omega)(t) = \omega'(t)$ for $t \in V_2$, and $= \omega(t)$ for $t \notin V_2$.

A probability measure P on $(\Omega, \mathscr{B})$ is called a *random field* with conditional distribution q, if for $A \in \mathscr{B}_V$

$$P(A \mid \mathscr{B}_{V^c}) = q_{V,\omega}(A) \qquad \text{a.e. } (P).$$

Dobrushin [2] shows that the totality $\mathscr{P}$ of random fields with q is a non-empty, compact and convex set, if

δ) $\lim_{W\to T} \sup_{\omega'} |q_{V,\omega'}(A) - q_{V,c(W;\omega',\omega)}(A)| = 0$

for $A \in \mathscr{B}_V$ and $\omega \in \Omega$, which we assume throughout this note.

Let $V_1 \subset V_2 \subset \cdots$ be an increasing sequence of finite subsets V_n of T with $\cup V_n = T$. Let Ω_∞ be the set of ω for which there exists $\lim_{n\to\infty} q_{V_n,\omega}(A)$ for every cylindrical A.

Let $Q_\omega(A)$ be the limit. $Q_\omega(\cdot)$ is countably additive on $\mathscr{B}_V$ for every finite subset V. Therefore it is extended to a probability measure on $\mathscr{B}$, which we denote by the same Q_ω. It is easy to see $Q_\omega \in \mathscr{P}$.

Let $\mathscr{B}_\infty = \bigcap_V \mathscr{B}_{V^c}$, where V runs over the set of all finite subsets of T.

Lemma 1. *If $P \in \mathscr{P}$, then*

$$P(A \cap B) = \int_B Q_\omega(A)\, P(d\omega) \quad \text{for} \quad A \in \mathscr{B} \quad \text{and} \quad B \in \mathscr{B}_\infty\,,$$

i.e.,

$$P(A \mid \mathscr{B}_\infty) = Q_\omega(A) \quad \text{a.e. } (P)\,.$$

Proof. Taking in mind that $q_{V_n,\omega}(A) = P(A \mid \mathscr{B}_{V_n^c})$ is a martingale, we have $P(\Omega_\infty) = 1$ for every random field P by Doob's convergence theorem [3]. If $A \in \mathscr{B}_{V_n}$ and $B \in \mathscr{B}_\infty$, then $P(A \cap B) = \int_B q_{V_n,\omega}(A)\, P(d\omega)$ $= \int_{B \cap \Omega_\infty} q_{V_n,\omega}(A)\, P(d\omega)$. Letting $n \to \infty$, we have

$$P(A \cap B) = \int_{B \cap \Omega_\infty} Q_\omega(A)\, P(d\omega) = \int_B Q_\omega(A)\, P(d\omega)\,.$$

The equality holds also for non-cylindrical A.

Let $2_P = \{A; P(A) = 0 \quad \text{or} \quad 1\}$.

Lemma 2. (Theorem 3.4 in [6].) *P is extremal in $\mathscr{P}$ if and only if $\mathscr{B}_\infty = 2_p\,(\operatorname{mod} P)$.*

Proof. i) We assume $\mathscr{B}_\infty \neq 2_P (\operatorname{mod} P)$, then there exists $\tilde{\Omega} \in \mathscr{B}_\infty$ such that $0 < P(\tilde{\Omega}) < 1$. Let $P_{\tilde{\Omega}}(\cdot) = P(\tilde{\Omega})^{-1}\, P(\cdot \cap \tilde{\Omega})$. For every $A \in \mathscr{B}_V$ we have $P_{\tilde{\Omega}}(A \mid \mathscr{B}_{V^c}) = q_{V,\omega}(A)$, i.e., $P_{\tilde{\Omega}} \in \mathscr{P}$, because for every $B \in \mathscr{B}_{V^c}$,

$$\begin{aligned} P_{\tilde{\Omega}}(A \cap B) &= P(\tilde{\Omega})^{-1}\, P(A \cap B \cap \tilde{\Omega}) = P(\tilde{\Omega})^{-1} \int_{B \cap \tilde{\Omega}} P(A \mid \mathscr{B}_{V^c})\, P(d\omega) \\ &= \int_B q_{V,\omega}(A)\, P(\tilde{\Omega})^{-1}\, P(d\omega \cap \tilde{\Omega}) = \int_B q_{V,\omega}(A)\, P_{\tilde{\Omega}}(d\omega)\,. \end{aligned}$$

A measure $P_{\tilde{\Omega}^c} \in \mathscr{P}$ is defined analogously. Both $P_{\tilde{\Omega}}$ and $P_{\tilde{\Omega}^c}$ are distinct, since they are mutually singular. Therefore the sum $P = P(\tilde{\Omega}) P_{\tilde{\Omega}} + P(\tilde{\Omega}^c) P_{\Omega^c}$ is not extremal.

ii) Let $\mathscr{B}_\infty = 2_P (\operatorname{mod} P)$ and let $P = \lambda P_1 + (1-\lambda) P_2$, where $0 < \lambda < 1$ and $P_1, P_2 \in \mathscr{P}$. By Lemma 1, $Q_\omega(A) = P(A)$ a.e. (P) for each $A \in \mathscr{B}$, that is, $P\{\omega : P(A) = Q_\omega(A)\} = 1$, hence $P_i\{\omega : Q_\omega(A) = P(A)\} = 1$ for $i = 1, 2$, because the coefficients λ and $1-\lambda$ are both positive. Thus we have, by Lemma 1 again, $P_i(A) = \int Q_\omega(A)\, P_i(d\omega) = P(A)$, that is, $P_i = P$, therefore P is extremal.

Corollary. *P is extremal if and only if*

$$\lim_{V \to T} \sup_{B \in \mathscr{B}_{V^c}} |P(A \cap B) - P(A)\, P(B)| = 0 \quad \text{for all} \quad A \in \mathscr{B}\,.$$

Let $B_\omega = \{\omega'; Q_{\omega'} = Q_\omega\}$, which belongs to $\mathscr{B}_\infty$, as is easily seen. We have, by Lemma 1, $Q_\omega(B_\omega) = Q_\omega(B_\omega \cap B_\omega) = \int_{B_\omega} Q_{\omega'}(B_\omega)\, Q_\omega(d\omega')$ $= Q_\omega(B_\omega)^2$, so that $Q_\omega(B_\omega) = 0$ or 1. We call ω *regular*, if $Q_\omega(B_\omega) = 1$. Let Ω_r be the set of all regular ω.

Theorem 1. *P is extremal in $\mathscr{P}$ if and only if $P=Q_\omega$ for some $\omega\in\Omega_r$.* (Cf. Theorem 2.2 in [5].)

Proof. i) We assume that P is extremal, i.e., $\mathscr{B}_\infty=2_P$ (mod P) by Lemma 2. Since $Q_{\omega'}(A)$ is $\mathscr{B}_\infty$-measurable, $Q_{\omega'}(A)$ does not depend on ω' a.e. (P), so that there exists Ω_A with $P(\Omega_A)=1$ for whose elements ω_1 and ω_2 it holds $Q_{\omega_1}(A)=Q_{\omega_2}(A)$. Let $\bar{\Omega}=\bigcap_A\Omega_A$, where A runs over the set of all cylindrical subsets of Ω. The number of cylindrical subsets of Ω is countable, hence $P(\bar{\Omega})=1$. Take an arbitrary element ω of $\bar{\Omega}$, then $Q_{\omega'}=Q_\omega$ for almost all $(P)\,\omega'$. By Lemma 1 we have $P=\int Q_{\omega'}P(d\omega')=Q_\omega$.

If ω is not regular, then $P(B_\omega)=Q_\omega(B_\omega)=0$, i.e., $P(B_\omega^c)=P\{\omega';Q_{\omega'}\neq Q_\omega\}=P\{\omega';Q_{\omega'}\neq P\}=1$. Therefore we have $P\{\omega':Q_{\omega'}(A)\neq P(A)\}>0$ for some cylindrical A, which implies $P\{\omega':Q_{\omega'}(A)>P(A)\}>0$ or $P\{\omega':Q_{\omega'}(A)<P(A)\}>0$.

In case when $P\{Q_{\omega'}(A)>P(A)\}>0$, we have $P\{Q_{\omega'}(A)>P(A)\}=1$ by our assumption that $\mathscr{B}_\infty=2_P(\mathrm{mod}\,P)$. By Lemma 1, $P(A)=\int Q_{\omega'}(A)\cdot P(d\omega')=\int_{\{Q_{\omega'}(A)>P(A)\}}Q_{\omega'}(A)\,P(d\omega')>P(A)$, which is absurd. In case when $P\{Q_{\omega'}(A)<P(A)\}>0$, we are led to the same contradiction.

ii) Let $P=Q_\omega$ for some $\omega\in\Omega_r$. For $A\in\mathscr{B}_\infty$ we have

$$P(A)=Q_\omega(A\cap A\cap B_\omega)=\int_{A\cap B_\omega}Q_{\omega'}(A)\,Q_\omega(d\omega')=\int_A Q_\omega(A)\,Q_\omega(d\omega')$$
$$=Q_\omega(A)^2=P(A)^2\,.$$

Therefore $P(A)=0$ or 1 for $A\in\mathscr{B}_\infty$, i.e., $\mathscr{B}_\infty=2_P(\mathrm{mod}\,P)$.

Corollary 1. $P(\Omega_r)=1$ *for all* $P\in\mathscr{P}$.

Proof. The Choquet theorem [1] shows that any $P\in\mathscr{P}$ is represented in the form; $P=\int_{\Omega_r}Q_\omega\mu(d\omega)$. For any regular ω, $Q_\omega(\Omega_r)=1$, since $B_\omega\subset\Omega_r$. Therefore $P(\Omega_r)=\int_{\Omega_r}Q_\omega(\Omega_r)\,\mu(d\omega)=1$.

Corollary 2. *Extremal random fields of $\mathscr{P}$ are mutually singular.*

Let $\mathscr{B}_r$ be the σ-algebra on Ω_r generated by $\{Q_\omega(A);\ A\in\mathscr{B}\}$.

Lemma 3. *The σ-algebra $\mathscr{B}_r$ coincides with the family of sets in $\mathscr{B}_\infty$ which are representable as a (possibly uncountable) union of sets B_ω for regular ω.*

Proof. If $\cup B_\omega$ belongs to $\mathscr{B}_\infty$, then $Q_{\omega'}(\cup B_\omega)=\chi_{\cup B_\omega}(\omega')$. Therefore $\cup B_\omega\in\mathscr{B}_r$. On the other hand $\{\omega:Q_\omega(A)<a\}=\bigcup_{\omega;Q_\omega(A)<a}B_\omega\in\mathscr{B}_\infty$, from which follows our result.

Let $P|_r$ be the restriction of P on $\mathscr{B}_r$.

Theorem 2. *For any* $P \in \mathscr{P}$, $P = \int_{\Omega_r} Q_\omega P|_r(d\omega)$. *If* $P = \int_{\Omega_r} Q_\omega \mu(d\omega)$ *with a probability measure* μ *on* $(\Omega_r, \mathscr{B}_r)$, *then* $P \in \mathscr{P}$ *and* $\mu = P|_r$. (Cf. Proposition 3.5 in [6].)

Proof. $P = \int_{\Omega_r} Q_\omega P|_r(d\omega)$ is a direct consequence of Lemma 1 and Corollary 1 to Theorem 1. Let $P = \int_{\Omega_r} Q_{\omega'} \mu(d\omega')$ and let $\cup B_\omega \in \mathscr{B}_r$. We have $P(\cup B_\omega) = \int_{\Omega_r} Q_{\omega'}(\cup B_\omega)\, \mu(d\omega') = \int_{\Omega_r} \chi_{\cup B_\omega}(\omega')\, \mu(d\omega') = \mu(\cup B_\omega)$, hence $P = \mu$ on $\mathscr{B}_r$ by Lemma 3.

Let us consider a case where T is the ν-dimensional lattice Z^ν. For $\tau \in T$, let notations be as follows:

$$\tau V = \{\tau + v; v \in V\} \quad \text{for} \quad V \subset T,$$
$$\tau\omega(t) = \omega(t - \tau) \quad \text{for} \quad \omega \in \Omega \quad \text{and} \quad t \in T,$$
$$\tau A = \{\tau\omega; \omega \in A\} \quad \text{for} \quad A \in \mathscr{B}.$$

Let S be a subgroup of T and let conditional distributions $q_{V,\omega}$ be S-invariant, i.e., $q_{\tau V, \tau\omega}(\tau A) = q_{V,\omega}(A)$ for all $\tau \in S$. We slightly modify the definition of Ω_∞ and Q_ω; let Ω_∞ be the set of ω for which there exists $\lim_{n\to\infty} q_{\tau V_n, \omega}(A)$ for every cylindrical A and every $\tau \in S$ and these limits coincide with each other for all $\tau \in S$. The limit is denoted by Q_ω for $\omega \in \Omega_\infty$. The same convergence theorem as in the proof of Lemma 1 assures that $P(\Omega_\infty) = 1$ for each $P \in \mathscr{P}$. Corresponding modifications are made for definitions of B_ω, Ω_r, etc. The same argument as preceding one works for our modified Ω_∞, Ω_r, B_ω, etc. Obviously, $Q_\omega(\tau A) = \lim_{n\to\infty} q_{V_n,\omega}(\tau A) = \lim_{n\to\infty} q_{\tau^{-1}V_n, \tau^{-1}\omega}(A) = Q_{\tau^{-1}\omega}(A)$ for $\omega \in \Omega_\infty$ and $\tau \in S$. It is easy to see that Ω_r is S-invariant. Finally we remark that $P = \int_{\Omega_r} Q_\omega \mu(d\omega)$ is S-invariant if and only if μ is so.

References

1. Choquet, G.: Ann. Inst. Fourier **10**, 333—344 (1960)
2. Dobruschin, R. L.: Teor. Verojatnost. i Primenen **13**, 201—229 (1968) Russian
3. Doob, J. L.: Stochastic Processes 1953
4. Dynkin, E. B.: Actes congrès internat. math., tom 2, 507—512 (1970)
5. Georgii, H. O.: Commun. math. Phys. **32**, 107—118 (1973)
6. Lanford III, O. E., Ruelle, D.: Commun. math. Phys. **13**, 194—215 (1969)

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