On the Nature of "Strong" Bogoliubov Transformations for Fermions*

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Abstract. Properties of the unitary operator implementing a general Bogoliubov transformation on a system of fermions are examined. The nature of this operator (and that of the transformed vacuum state) is exhibited for the case in which the operators appearing in the "diagonal" part of the transformation do not have an inverse. This case turns out to be very different from either the case where the inverses exist or the case with Bogoliubov transformations on systems of bosons. In particular, such transformations can be unitarily implementable in the initial Fock-Hilbert space even though the new vacuum is orthogonal to the initial vacuum. Particles are then created with certainty (i.e. probability 1) in the initial vacuum state and even the new charge operator could be different from the initial one.

Introduction

In many problems where a quantized field is acted upon by an external field, the differential equation of motion for the quantized field operator valued distribution is linear and homogeneous. This is the case, for example, when a Dirac electron-positron field, $\psi(t, x)$, is modified through minimal coupling with a classical electromagnetic field $A(t, x)$ according to

$$(-i\gamma \cdot \partial + m)\,\psi(t, x) = e\gamma \cdot A(t, x)\,\psi(t, x)\,.$$

In such problems, one is interested in knowing about the existence and nature of the S-matrix or of the unitary time evolution operator at finite times for the field operator variables or the one for the energy quanta creation and annihilation operator variables. Thus one is led to examine linear homogeneous relations, of the type described in Eq. (1.1) below[1],

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[1] We will study here explicitly theories with distinct particles and antiparticles. The results for theories with only one kind of particle can be obtained in a similar fashion. They will be mentioned at the end.

between different creation and annihilation operators i.e. to examine Bogoliubov transformations. (For more details on how exactly these transformations come about in external field problems, see for example: Part V of Friedrichs' book [1], Capri [2], Schroer, Seiler, and Swieca [3], and Seiler [4].) Stated more precisely, one needs the answers to the following questions: when is a Bogoliubov transformation unitarily implementable in the Fock-Hilbert space determined by the action of the untransformed creation operators on the untransformed vacuum? and what is the nature (i.e. its expression in terms of untransformed operators for example) of the unitary operator implementing such a transformation?

In the case of boson fields, the answers to these questions can be found in the Sections 23, 24, 25 of the Part V of Friedrichs' book [1] and in theorems proved in the articles by Shale [5] and Seiler [4]. The answer to the first question is: there exists a unitary operator T, such that $T^\dagger$ {all creation operators $b^\dagger, d^\dagger$} T = {all creation operators $B^\dagger, D^\dagger$ which are related to the $b^\dagger, d^\dagger$ according to Eq. (1.1)}, if and only if the operators M_i's appearing in (1.1) are such that

$$\| M_1^{-1} M_2 \|_{\text{H.S.}}^2 = \int d\alpha \, d\beta \, | \int d\gamma \, M_1^{-1}(\gamma, \alpha) \, M_2(\gamma, \beta)|^2 < \infty \ .$$

(In particular, the condition $\| M_2 \|_{\text{H.S.}} < \infty$ is sufficient for the existence of T.) As for the unitary operator T, different forms for it are given by Friedrichs. It is always such that the transformed vacuum state $|\phi_0\rangle$ is

$$|\phi_0\rangle = \eta \exp - \int d\alpha \, d\beta \, b^\dagger(\alpha) \, (M_1^{-1} M_2) \, (\alpha, \beta) \, d^\dagger(\beta) |0\rangle$$

where $|0\rangle$ is the initial vacuum and η a finite normalization constant. For theories with only one kind of particle, Eq. (1.1) is replaced by

$$B(\gamma) = \int d\alpha \, M_1(\gamma, \alpha) \, b(\alpha) + \int d\alpha \, M_2(\gamma, \alpha) \, b^\dagger(\alpha)$$

and the necessary and sufficient condition for the existence of T is also $\| M_1^{-1} M_2 \|_{\text{H.S.}} < \infty$. The transformed vacuum is then

$$|\phi_0\rangle = \eta \exp - \int d\alpha \, d\alpha' \, b^\dagger(\alpha) \, (M_1^{-1} M_2) \, (\alpha, \alpha') \, b^\dagger(\alpha') |0\rangle \ .$$

For fermion fields, one remarks that it is usually not clear in the literature on external field theory what conditions are necessary and sufficient for the existence of a unitary operator implementing the transformation. However, it was mentioned by Bongaarts [6] and Seiler [4] that such conditions can be found in theorems by Shale and Stinespring [7]. (We will explain these conditions below.) The lack of a

precise result in the other treatments arises from the fact that it is possible for the operators M_1 and M_4^* appearing in (1.1) not to have an inverse and this possibility has not been analyzed completely.

The fermion transformations, indeed, have the particularity that the operators M_1 and M_4^* are not guaranteed, by only the relations (1.7) and (1.9) below, to have an inverse. This was guaranted in the case of bosons because a minus sign appeared in front of $M_2 M_2^*$ and $M_3 M_3^*$ in the corresponding relations. Those transformations where M_1^{-1} and $(M_4^*)^{-1}$ exist can be treated in a similar manner as in the boson case: this was done by Friedrichs. He gave different forms for the corresponding unitary operators and these are in fact of the same type as those for bosons. The question of existence of the unitary operator in the general case was completely settled by Shale and Stinespring, as mentioned already. The result is: there exists a unitary operator T implementing a Bogoliubov transformation [as (1.1) below] in the Fock-Hilbert space determined by the untransformed operators and the untransformed vacuum if and only if $\|M_2\|_{\text{H.S.}} < \infty$ and $\|M_3\|_{\text{H.S.}} < \infty$. [$M_2$ and M_3 being the operators appearing in the off-diagonal part of (1.1).]

Our own contribution in the present note consists in giving explicit expressions for the transformed vacuum state and for the unitary operator implementing the transformation when M_1 and M_4^* do not have an inverse (these are the transformations which we refer to as "strong" transformations).

The method used to find such expressions yields, at the same time, an alternative proof (by construction) of the above-mentioned result concerning the existence conditions. We will proceed as follows.

We first explain the notation which we will use; it is essentially the one used in the external field problem where the Fermi particles and antiparticles are different. The transformation (1.1) will be considered to be a general transformation of the type encountered in such problems.

We then prove that the conditions $\|M_2\|_{\text{H.S.}} < \infty$ and $\|M_3\|_{\text{H.S.}} < \infty$ are necessary for the existence, in the initial Fock-Hilbert space, of a vacuum state annihilated by all the transformed annihilation operators. The existence of such a state is certainly necessary for the two sets of creation and annihilation operators (transformed and untransformed) to be unitarily related. The above conditions on M_2 and M_3 are therefore necessary conditions for the unitarity implementability of the Bogoliubov transformation (1.1).

Supposing that these conditions are satisfied, we then decompose the general transformation in a product of simpler transformations. The properties of the unitary operator, in the general case, are thus made evident. It will be constructed easily with the help of the results given by Friedrichs for the case of "weak" transformations.

1. Definition

We consider transformations of the type

$$B(\gamma) = \int d\alpha\, M_1(\gamma, \alpha)\, b(\alpha) + \int d\beta\, M_2(\gamma, \beta)\, d^\dagger(\beta)$$

$$D^\dagger(\lambda) = \int d\alpha\, M_3(\lambda, \alpha)\, b(\alpha) + \int d\beta\, M_4(\lambda, \beta)\, d^\dagger(\beta) \tag{1.1}$$

where the four infinite sets of indices $\{\alpha\}, \{\beta\}, \{\gamma\}$, and $\{\lambda\}$ are not necessarily the same and each one can be discrete or continuous or have both continuous and discrete elements. Integrals, like

$$\int d\alpha, \quad \text{mean} \quad \underset{\text{(over continuous } \alpha\text{'s)}}{\int d\alpha} \quad + \quad \underset{\substack{\alpha \\ \text{(over discrete } \alpha\text{'s)}}}{\sum}$$

In Eq. (1.1), $b(\alpha)$ and $d^\dagger(\beta)$ are operators such that with their adjoints they satisfy the anticommutation relations

$$\{b(\alpha), b^\dagger(\alpha')\} = \delta_{\alpha\alpha'} \qquad \{d(\beta), d^\dagger(\beta')\} = \delta_{\beta\beta'} \tag{1.2}$$

with all other anticommutators vanishing. It is understood that for continuous values of the indices, these are operator-valued distributions i.e. they are well defined operators only when smeared with a square integrable function of the index.

They are defined on the Hilbert space $\mathcal{H}$ consisting of states

$$|\psi\rangle = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{\sqrt{n!\,m!}} \int d\alpha_n\, d\beta_m\, \psi^{nm}(\alpha_n; \beta_m)\, |\alpha_n; \beta_m\rangle \tag{1.3}$$

such that

$$\langle\psi|\psi\rangle = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int d\alpha_n\, d\beta_m\, |\psi^{nm}(\alpha_n; \beta_m)|^2 < \infty \tag{1.4}$$

and where by definition

$$|\alpha_n; \beta_m\rangle = |\alpha_1, \alpha_2, ..., \alpha_n; \beta_1, \beta_2, ..., \beta_m\rangle$$

$$= b^\dagger(\alpha_1)\, b^\dagger(\alpha_2) ... b^\dagger(\alpha_n)\, d^\dagger(\beta_1)\, d^\dagger(\beta_2) ... \tag{1.5}$$

$$... d^\dagger(\beta_m)|0\rangle$$

and the vacuum state $|0\rangle$ is such that

$$b(\alpha)|0\rangle = 0 \rightarrow \alpha \qquad d(\beta)|0\rangle = 0 \rightarrow \beta. \tag{1.6}$$

We will use the notation

$$\left(\frac{\alpha_n}{\hat{i}}\right) = (\alpha_1, \alpha_2, ..., \alpha_{i-1}, \alpha_{i+1}, ..., \alpha_n).$$

The anticommutation relations of the operators B, D will be of the same type as for the operators b, d when the following relations hold between the M_i's:

$$\int d\alpha \, M_1(\gamma, \alpha) \, M_1^*(\gamma', \alpha) + \int d\beta \, M_2(\gamma, \beta) \, M_2^*(\gamma', \beta) = \delta_{\gamma\gamma'} \tag{1.7a}$$

$$\int d\alpha \, M_3(\lambda, \alpha) \, M_3^*(\lambda', \alpha) + \int d\beta \, M_4(\lambda, \beta) \, M_4^*(\lambda', \beta) = \delta_{\lambda\lambda'} \tag{1.7b}$$

$$\int d\alpha \, M_1(\gamma, \alpha) \, M_3^*(\lambda, \alpha) + \int d\beta \, M_2(\gamma, \beta) \, M_4^*(\lambda, \beta) = 0. \tag{1.7c}$$

By demanding that the transformation (1.1) had an inverse and using the relations (1.7) together with the similar relations for the inverse transformation, we can see that the inverse is the adjoint of the transformation (1.1), i.e.

$$b(\alpha) = \int d\gamma \, M_1^*(\gamma, \alpha) \, B(\gamma) + \int d\lambda \, M_3^*(\lambda, \alpha) \, D^\dagger(\lambda)$$
$$d^\dagger(\beta) = \int d\gamma \, M_2^*(\gamma, \beta) \, B(\gamma) + \int d\lambda \, M_4^*(\lambda, \beta) \, D^\dagger(\lambda) \tag{1.8}$$

with

$$\int d\gamma \, M_1(\gamma, \alpha) \, M_1^*(\gamma, \alpha') + \int d\lambda \, M_3(\lambda, \alpha) \, M_3^*(\lambda, \alpha') = \delta_{\alpha\alpha'} \tag{1.9a}$$

$$\int d\gamma \, M_2(\gamma, \beta) \, M_2^*(\gamma, \beta') + \int d\lambda \, M_4(\lambda, \beta) \, M_4^*(\lambda, \beta') = \delta_{\beta\beta'} \tag{1.9b}$$

$$\int d\gamma \, M_1(\gamma, \alpha) \, M_2^*(\gamma, \beta) + \int d\lambda \, M_3(\lambda, \alpha) \, M_4^*(\lambda, \beta) = 0. \tag{1.9c}$$

It is easily seen that the requirements (1.7) and (1.9) are meaningful if and only if the M_i's are all integral kernels of transformations of square integrable functions of the indices into other square integrable functions. This is also sufficient for the $B, B^\dagger, D, D^\dagger$ to be well defined operator valued distributions on all $\mathscr{H}$.

2. Necessary Conditions for Existence of a New Vacuum

We want to examine under what conditions a state $|\phi_0\rangle \in \mathscr{H}$ will exist such that

$$B(\gamma)|\phi_0\rangle = 0 \rightarrowtail \gamma \quad \text{and} \quad D(\lambda)|\phi_0\rangle = 0 \rightarrowtail \lambda. \tag{2.1}$$

Writing

$$|\phi_0\rangle = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{\sqrt{n!m!}} \int d\alpha_n \, d\beta_m \, \psi^{nm}(\alpha_n; \beta_m)|\alpha_n; \beta_m\rangle \tag{2.2}$$

and using $B(\gamma)$ as given in Eq. (1.1), we obtain that the following relations between the ψ^{nm}'s must exist in order for $B(\gamma)|\phi_0\rangle$ to be null.

$$\int d\alpha_j \, M_1(\gamma, \alpha_j) \, \psi^{nm}(\alpha_n; \beta_m) = \frac{1}{\sqrt{mn}} \sum_{i=1}^{m} (-1)^{n+i+j} M_2(\gamma, \beta_i)$$
$$\cdot \psi^{n-1\,m-1}\left(\frac{\alpha_n}{\hat{j}}; \frac{\beta_m}{\hat{i}}\right) \tag{2.3}$$

for $n = 1, 2, \ldots, m = 1, 2, \ldots, \alpha_j$ being any one of the α_n and

$$\int d\alpha_j M_1(\gamma, \alpha_j) \, \psi^{n0}(\alpha_n; -) = 0 \quad \text{for} \quad n = 1, 2, \ldots . \tag{2.4}$$

Similarly, $D(\lambda)|\phi_0\rangle = 0$ leads to

$$\int d\beta_j M_4^*(\lambda, \beta_j) \, \psi^{nm}(\alpha_n; \beta_m) = \frac{1}{\sqrt{mn}} \sum_{i=1}^{n} (-1)^{n+i+j+1} M_3^*(\lambda, \alpha_i)$$
$$\cdot \psi^{n-1\,m-1} \left(\frac{\alpha_n}{\hat{i}} \, ; \frac{\beta_m}{\hat{j}} \right) \tag{2.5}$$

for $n = 1, 2, \ldots, m = 1, 2, \ldots, \beta_j$ being any one of the β_m and

$$\int d\beta_j M_4^*(\lambda, \beta_j) \, \psi^{0m}(-; \beta_m) = 0 \quad \text{for} \quad m = 1, 2, \ldots . \tag{2.6}$$

We can find immediately a necessary condition for the Eqs. (2.3) and (2.5) to have square integrable solutions ψ^{nm}. [Square integrability of ψ^{nm} for each n and m being required by relation (1.4).]

From the Eq. (1.9), we have

$$\int d\gamma |\int d\alpha \, M_1(\gamma, \alpha) \, \psi(\alpha)|^2 \leqq \int d\alpha |\psi(\alpha)|^2 \tag{2.7}$$

$$\int d\gamma |\int d\beta \, M_2(\gamma, \beta) \, \psi(\beta)|^2 \leqq \int d\beta |\psi(\beta)|^2 . \tag{2.8}$$

Eq. (2.7) implies that ψ^{nm} can be square integrable only if the right hand side of (2.3) is square integrable, i.e. we need

$$\int d\gamma \, \frac{d\alpha_n}{\hat{j}} \, d\beta_m \left| \frac{1}{\sqrt{mn}} \sum_{i=1}^{m} (-1)^{n+i+j} M_2(\gamma, \beta_i) \cdot \psi^{n-1\,m-1} \left(\frac{\alpha_n}{\hat{j}} \, ; \frac{\beta_m}{\hat{i}} \right) \right|^2 < \infty .$$

This is

$$\frac{1}{mn} \int d\gamma \, \frac{d\alpha_n}{\hat{j}} \, d\beta_m \sum_{i=1}^{m} |M_2(\gamma, \beta_i)|^2 \left| \psi^{n-1\,m-1} \left(\frac{\alpha_n}{\hat{j}} \, ; \frac{\beta_m}{\hat{i}} \right) \right|^2$$
$$+ \frac{1}{mn} \sum_{\substack{i=1 \\ i \neq i'}}^{m} \sum_{i'=1}^{m} (-1)^{i+i'} \int d\gamma \, \frac{d\alpha_n}{\hat{j}} \, \frac{d\beta_m}{\hat{i}\hat{i}'}$$
$$\cdot \left(\int d\beta_i \, \psi^{n-1\,m-1} \left(\frac{\alpha_n}{\hat{j}} \, ; \frac{\beta_m}{\hat{i}'} \right)^* M_2(\gamma, \beta_i) \right)$$
$$\cdot \left(\int d\beta_{i'} \, M_2^*(\gamma, \beta_{i'}) \, \psi^{n-1\,m-1} \left(\frac{\alpha_n}{\hat{j}} \, ; \frac{\beta_m}{\hat{i}} \right) \right) < \infty$$

and supposing that $\psi^{n-1\,m-1}$ was non-zero and square integrable; because of Eq. (2.8), the second integral above is finite so that in fact we need the first term to be finite, i.e.

$$\int d\gamma \, d\beta |M_2(\gamma, \beta)|^2 < \infty . \tag{2.9}$$

Repeating the same argument with Eq. (2.5), we can see that we need also

$$\int d\lambda \, d\alpha \, |M_3(\lambda, \alpha)|^2 < \infty \, . \tag{2.10}$$

3. Properties of the Operators M_i

Let us now suppose that the necessary conditions found in the preceding section are satisfied. We will examine some of the properties of the operators M_i.

i) The kernel $(M_3^* M_3) = \int d\lambda \, M_3^*(\lambda, \alpha') \, M_3(\lambda, \alpha)$ is a non-null hermitian L^2 kernel. It therefore has a spectral decomposition

$$(M_3^* M_3)(\alpha', \alpha) = \sum_{n=1}^{\infty} \frac{1}{A_n^3} X_n^3(\alpha') X_n^3(\alpha)^* \tag{3.1}$$

where (A_n^3, X_n^3) is the characteristic system defined by

$$\int d\alpha (M_3^* M_3)(\alpha', \alpha) X_n^3(\alpha) = \frac{1}{A_n^3} X_n^3(\alpha') \, . \tag{3.2}$$

The Eq. (1.9 a) implies $1/A_n^3 = 1 - \| M_1 X_n^3 \|^2$ and Eq. (3.2): $1/A_n^3 = \| M_3 X_n^3 \|^2$ so that $0 \leq 1/A_n^3 \leq 1$. Since

$$\int d\alpha \, d\alpha' \, |(M_3^* M_3)(\alpha', \alpha)|^2 = \sum_{n=1}^{\infty} \left(\frac{1}{A_n^3} \right)^2 < \infty$$

and since each characteristic value in this sequence is included a number of times equal to its rank, there is necessarily only a finite number of different X_n^3's corresponding to a same given A_n^3. In particular, we shall call N the number of eigenfunctions with eigenvalue 1. We will choose the labels of the basis $\{X_n^3\}$ such that these correspond to $n = 1, 2, \ldots, N$. From (1.9 a) it is easy to see that also

$$\int d\alpha \, M_1(\gamma, \alpha) \, X_n^3(\alpha) = 0 \not{-} \gamma \quad \text{for} \quad n = 1, 2, \ldots, N \, . \tag{3.3}$$

ii) Similarly

$$(M_3^* M_3)(\lambda', \lambda) = \sum_{n=1}^{\infty} \frac{1}{A_n^1} X_n^1(\lambda') X_n^1(\lambda)^* \tag{3.4}$$

and there is a finite number of A_n^1 which are equal to 1. By (1.7 b), $X^1(\lambda)$ is such that

$$\int d\lambda (M_3^* M_3)(\lambda', \lambda) X^1(\lambda) = X^1(\lambda') \tag{3.5}$$

if and only if

$$\int d\lambda \, M_4(\lambda, \beta) \, X^1(\lambda) = 0 \not{-} \beta \, . \tag{3.6}$$

We shall now show that there are N such functions. We do this by showing that such functions can be obtained from the X_n^3, $n = 1, 2, ..., N$ and vice versa.

For each one of the $X_n^3(\alpha)$, $n = 1, 2, ..., N$, the function

$$[\int d\alpha M_3(\lambda, \alpha) X_n^3(\alpha)]^*$$

is easily seen to be non-zero. Such functions are orthogonal to each other; this follows from (3.2) where $\Lambda_n^3 = 1$ and the orthogonality of the X_n^3's. Each such function, because of (1.9 c) is such that

$$\int d\lambda M_4(\lambda, \beta) [\int d\alpha M_3^*(\lambda, \alpha) X_n^3(\alpha)^*] = -\int d\gamma M_2(\gamma, \beta) [\int d\alpha M_1^*(\gamma, \alpha) X_n^3(\alpha)^*]$$

and this is null by Eq. (3.3). For each $X_n^3(\alpha)$, $n = 1, 2, ..., N$, there is then a

$$X_n^1(\lambda) = [\int d\alpha M_3(\lambda, \alpha) X_n^3(\alpha)]^* \qquad (3.7)$$

such that this satisfies (3.6) or equivalently (3.5).

In turn the functions $X_n^3(\alpha)$, $n = 1, 2, ..., N$ are related to X_n^1, $n = 1, 2, ..., N$ as:

$$X_n^3(\alpha) = [\int d\lambda M_3(\lambda, \alpha) X_n^1(\lambda)]^* . \qquad (3.8)$$

iii) We shall need also the spectral decomposition of

$$(M_2 M_2^*)(\beta', \beta) = \int d\gamma M_2(\gamma, \beta') M_2^*(\gamma, \beta)$$

which is

$$(M_2 M_2^*)(\beta', \beta) = \sum_{n=1}^{\infty} \frac{1}{\Lambda_n^2} X_n^2(\beta') X_n^2(\beta)^* , \qquad (3.9)$$

with $0 \le 1/\Lambda_n^2 \le 1$ and we call $M(< \infty)$ the number of orthogonal X_n^2 which correspond to $\Lambda_n^2 = 1$. We have for these, from (1.9 b):

$$\int d\beta M_4^*(\lambda, \beta) X_n^2(\beta) = 0 \mathrel{\vcenter{\hbox{$\forall$}}} \lambda \quad \text{for} \quad n = 1, 2, ..., N . \qquad (3.10)$$

We also have

$$(M_2 M_2^*)(\gamma', \gamma) = \sum_{n=1}^{\infty} \frac{1}{\Lambda_n^4} X_n^4(\gamma') X_n^4(\gamma)^* \qquad (3.11)$$

where the first M X_n^4's correspond to $\Lambda_n^4 = 1$ and they satisfy

$$\int d\gamma M_1^*(\gamma, \alpha) X_n^4(\gamma) = 0 \mathrel{\vcenter{\hbox{$\forall$}}} \alpha \quad \text{for} \quad n = 1, 2, ..., M . \qquad (3.12)$$

They are related to the X_n^2's $n = 1, 2, ..., M$ according to:

$$X_n^4(\gamma) = [\int d\beta M_2^*(\gamma, \beta) X_n^2(\beta)]^* \qquad n = 1, 2, ..., M \qquad (3.13)$$

and

$$X_n^2(\beta) = [\int d\gamma M_2^*(\gamma, \beta) X_n^4(\gamma)]^* \qquad n = 1, 2, ..., M . \qquad (3.14)$$

4. The Unitary Operator

i) In order to see more clearly what happens in the transformation (1.1), we decompose it in a product of simpler transformations. We shall consider here the effect of the transformation on the following smeared creation and annihilation operators:

$$b_n = \int d\alpha \, X_n^3(\alpha)^* \, b(\alpha) \qquad d_n^\dagger = \int d\beta \, X_n^2(\beta) \, d^\dagger(\beta)$$

$$B_n = \int d\gamma \, X_n^4(\gamma)^* \, B(\gamma) \qquad D_n^\dagger = \int d\lambda \, X_n^1(\lambda) \, D^\dagger(\lambda) \,. \tag{4.1}$$

We recall that the functions $X_n^1(\lambda)$, $X_n^2(\beta)$, $X_n^3(\alpha)$, $X_n^4(\gamma)$ form bases in $L^2(\lambda)$, $L^2(\beta)$, $L^2(\alpha)$, $L^2(\gamma)$ respectively. Clearly, a complete description of the transformation can be obtained by examining its effect on the operators b_n, d_n, B_n, D_n for all values of the index n labelling the elements of any arbitrary bases in $L^2(\lambda)$, $L^2(\beta)$, $L^2(\alpha)$, $L^2(\gamma)$. In particular, it is obvious that $|\phi_0\rangle$ is such that

$$B(\gamma)|\phi_0\rangle = 0 \,\forall\, \gamma \quad \text{and} \quad D(\lambda)|\phi_0\rangle = 0 \,\forall\, \lambda$$

if and only if

$$B_n|\phi_0\rangle = 0 \,\forall\, n \quad \text{and} \quad D_n|\phi_0\rangle = 0 \,\forall\, n$$

when n labels bases in $L^2(\gamma)$ and $L^2(\lambda)$.

The inverse relations to (4.1) are

$$b(\alpha) = \sum_{n=1}^{\infty} X_n^3(\alpha) \, b_n \qquad d^\dagger(\beta) = \sum_{n=1}^{\infty} X_n^2(\beta)^* \, d_n^\dagger$$

$$B(\gamma) = \sum_{n=1}^{\infty} X_n^4(\gamma) \, B_n \qquad D^\dagger(\lambda) = \sum_{n=1}^{\infty} X_n^1(\lambda)^* \, D_n^\dagger. \tag{4.2}$$

Because of the properties of the functions X_n^1 which we have described in Section 3), one can see that the Bogoliubov transformation (1.1) is in fact:

$$B_n = d_n^\dagger \quad \text{for} \quad n = 1, 2, \dots, M \qquad D_n^\dagger = b_n \quad \text{for} \quad n = 1, 2, \dots, N \tag{4.3}$$

$$B_{M+n'} = \sum_{n=1}^{\infty} \{ (X_{M+n'}^4, M_1 X_{N+n}^3) \, b_{N+n} + (X_{M+n'}^4, M_2 X_{M+n}^2) \, d_{M+n}^\dagger \}$$

$$D_{N+n'}^\dagger = \sum_{n=1}^{\infty} \{ (X_{N+n'}^1, M_3 X_{N+n}^3) \, b_{N+n} + (X_{N+n'}^1, M_4 X_{M+n}^2) \, d_{M+n}^\dagger \} \tag{4.4}$$

$$\text{for} \quad n' = 1, 2, \dots, \infty \,.$$

The Eq. (4.4) itself is a Bogoliubov transformation between the two sets of operators $\tilde{b}_n = b_{N+n}$, $\tilde{d}_n = d_{M+n}$ $n = 1, 2, \dots, \infty$, and $\tilde{B}_n = B_{M+n}$,

$\tilde{D}_n = D_{N+n}$ $n = 1, 2, \ldots, \infty$. The kernels $\tilde{M}_1(n', n) \equiv (X^4_{M+n'}, M_1 X^3_{N+n})$ and $\tilde{M}_4(n', n)^* \equiv (X^2_{M+n}, M_4^* X^1_{N+n'})$ do not have any non-zero homogeneous solutions associated with them as is easily deduced from the properties of the X_n^i's. (4.4) is therefore a "weak" Bogoliubov transformation.

ii) Let us now look more particularly at "weak" Bogoliubov transformations. We recall that the unitary operators implementing such transformations are known (for the form used below, see Section 24 of the Part V of Friedrichs' book [1]). They can be written, for example, as

$$T = \eta\, e^{-dF_4 b}\, e^{-d^\dagger F_3 d}\, e^{-b^\dagger F_2 b}\, e^{-b^\dagger F_1 d^\dagger} \tag{4.5}$$

where expressions like $e^{-b^\dagger F_1 d^\dagger}$ stand for

$$\exp\left\{ -\sum_{n'=1}^\infty \sum_{n=1}^\infty b_{n'}^\dagger F_1(n', n)\, d_n^\dagger \right\}$$

and $\eta = \exp 1/2\, (\text{Trace}\, F_2 + \text{Trace}\, F_3)$ has finite norm.

The F_i's are given by

$$(e^{-F_2})(n, m) = M_1(n, m)$$
$$(e^{-F_3})(n, m) = [M_4(n, m)]^* ; \tag{4.6}$$

$F_1(n, m)$ is the solution of

$$\sum_{n=1}^\infty M_1(n', n)\, F_1(n, m) = -M_2(n', n)$$

and $F_4(n, m)$ of

$$\sum_{n'=1}^\infty M_1(n', n)\, F_4(m, n') = -M_3(m, n). \tag{4.7}$$

iii) In the case of the general transformation (4.3), (4.4), we can then define a unitary operator T_0 such that

$$\begin{aligned} T_0^\dagger b_{N+n} T_0 &= B_{M+n} \\ T_0^\dagger d_{M+n} T_0 &= D_{N+n} \end{aligned} \quad \forall n = 1, 2, \ldots, \infty \tag{4.8}$$

and

$$\begin{aligned} T_0^\dagger b_i T_0 &= b_i \quad \text{for} \quad i = 1, 2, \ldots, N, \\ T_0^\dagger d_i T_0 &= d_i \quad \text{for} \quad i = 1, 2, \ldots, M. \end{aligned} \tag{4.9}$$

That is: T_0 implements the "weak" Bogoliubov transformation (4.4) so that it exists as we saw in the previous section. It does not contain the operator variables not entering in (4.4) because of the requirement (4.9). It can be written in the form (4.5) for example, where the operators (b, d) are $\tilde{b}_n, \tilde{d}_n; n = 1, 2, \ldots$ and the M_i's are the $\tilde{M}_i$'s.

It is easy to check that the unitary operator

$$U = \left\{ \exp - i(M+N)\pi \sum_{n=1}^{\infty} [b_n^\dagger b_n + d_n^\dagger d_n] \right\} (d_M^\dagger - d_M) \ldots$$
$$\ldots (d_2^\dagger - d_2)(d_1^\dagger - d_1)(b_N^\dagger - b_N)(b_2^\dagger - b_2)(b_1^\dagger - b_1) \tag{4.10}$$

commutes with b_{N+n} and d_{M+n} for $n = 1, 2, \ldots, \infty$ and

$$U^\dagger b_i U = D_i \text{ for } i = 1, 2, \ldots, N, \quad U^\dagger d_i U = B_i \text{ for } i = 1, 2, \ldots, M. \tag{4.11}$$

The unitary operator

$$T = T_0 U = U T_0$$

will then transform the creation and annihilation operators as

$$T^\dagger b_n T = D_n \quad n = 1, 2, \ldots, N \qquad T^\dagger b_n T = B_n \quad n = N+1, \ldots, \infty$$
$$T^\dagger d_n T = B_n \quad n = 1, 2, \ldots, M \qquad T^\dagger d_n T = D_n \quad n = M+1, \ldots, \infty.$$

The vacuum $|\phi_0\rangle$ (i.e. the new zero particle state) is then simply

$$|\phi_0\rangle = T^\dagger |0\rangle = b_1^\dagger b_2^\dagger \ldots b_N^\dagger d_1^\dagger \ldots d_M^\dagger T_0^\dagger |0\rangle$$
$$= b_1^\dagger b_2^\dagger \ldots b_N^\dagger d_1^\dagger \ldots d_M^\dagger \frac{1}{\eta_0} \exp \sum_{n'=N+1}^{\infty} \sum_{n=M+1}^{\infty} b_{n'}^\dagger F_1^0(n',n) d_n^\dagger |0\rangle \tag{4.12}$$

where $F_1^0(n, m)$ is the unique solution of

$$\sum_{n=N+1}^{\infty} (X_{n'}^4, M_1 X_n^3) F_1^0(n, m) = -(X_{n'}^4, M_2 X_m^2)$$

$\leftarrow n' = M+1, M+2, \ldots, \infty$ and $m = M+1, M+2, \ldots, \infty$

and η_0 is the finite normalization constant corresponding to η in Eq. (4.5).

5. Conclusion

We examine what it would mean if such "strong" Bogoliubov transformations happened in the description of dynamical systems.

Let us consider a quantum mechanical system where operators $b, b^\dagger, d, d^\dagger$ are associated with particles at some time t_1 (which could be finite or $-\infty$) and $B, B^\dagger, D, D^\dagger$ are associated with the same particles at a later time t_2 (finite or $+\infty$); this, according to an equation of motion like (1.1) in the Heisenberg representation. We suppose that the Bogoliubov transformation is a "strong" transformation and is unitarily implementable.

i) The vacuum at time t_2 is orthogonal to the vacuum at time t_1.

This obviously would "complicate" a calculation of $|0\rangle_{t_2}(\equiv |\phi_0\rangle)$ through a perturbation series (in powers of the coupling constant for example) which would start from $|0\rangle_{t_1}(\equiv |0\rangle)$. The same remark evidently holds about a calculation of the unitary operator implementing the Bogoliubov transformation (which is a time evolution operator $U(t, t')$ or the S-matrix if t_1 is $-\infty$ and t_2 is $+\infty$) through a series starting with I.

ii) There are states of the system in which particles and antiparticles will be created with probability 1.

If the system is in the state $|0\rangle_{t_1}$, there are no particles present initially. To see what this state represents at time t_2, we observe that the state with zero particles at this time is $|0\rangle_{t_2}$. The state $T_0|0\rangle_{t_2}$ is one in which there is a non-zero probability of finding pairs of particles and anti-particles created by $B_{M+n}^\dagger$ and $D_{M+n}^\dagger$ $n = 1, 2, ..., \infty$. The state $|0\rangle_{t_1}$, which is

$$|0\rangle_{t_1} = T|0\rangle_{t_2} = B_M^\dagger ... B_2^\dagger B_1^\dagger D_N^\dagger ... D_1^\dagger T_0|0\rangle_{t_2},$$

is then a state in which M particles and N antiparticles have been created and in which there is a non-zero probability of finding also pairs of particles and antiparticles.

If, on the other hand, the system is in the state $|0\rangle_{t_2}$;

$$|0\rangle_{t_2} = b_1^\dagger b_2^\dagger ... b_N^\dagger d_1^\dagger ... d_M^\dagger T_0^\dagger |0\rangle_{t_1},$$

initially there are N particles and M antiparticles plus possibly some particle-antiparticle pairs. After the interaction, there are no particles present.

iii) If the particles and antiparticles are assigned a different charge, the total charge is not conserved unless $N = M$.

The charge operator at the different times is:

$$Q(t_1) = e \sum_{n=1}^\infty \{b_n^\dagger b_n - d_n^\dagger d_n\}$$

and

$$Q(t_2) = e \sum_{n=1}^\infty \{B_n^\dagger B_n - D_n^\dagger D_n\}.$$

By examining the action of these operators on any one of the two Fock bases (obtained by the action of $b^\dagger, d^\dagger$ on $|0\rangle_{t_1}$ or of $B^\dagger, D^\dagger$ on $|0\rangle_{t_2}$), one can see that they are related as

$$Q(t_2) = Q(t_1) + e(M - N).$$

It is easy to see that the results obtained apply also to the case of one operator variable transformations like

$$B(\gamma) = \int d\alpha M_1(\gamma, \alpha) b(\alpha) + \int d\alpha M_2(\gamma, \alpha) b^\dagger(\alpha).$$

In terms of smeared operators defined similarly as in the case discussed here, this transformation reduces to

$$B_n = b_n^\dagger \quad n = 1, 2, \ldots, M$$

and an ordinary Bogoliubov transformation between the other B_{M+n}'s and b_{M+n}'s. The unitary operator relating then has the form $T = T_0 U$ where T_0 is the usual one for the "weak" transformation and

$$U = \left\{ \exp - iM\pi \sum_{n=1}^{\infty} b_n^\dagger b_n \right\} (b_M^\dagger - b_M) \ldots (b_2^\dagger - b_2)(b_1^\dagger - b_1).$$

It is to be remarked that for a Bogoliubov transformation on a system of bosons or for a "weak" transformation with fermions, N and M above are always null. What we have described in i), ii), and iii) would then never happen in these cases.

We finally note that Akhiezer and Berestetsky have indicated (in the discussion at the end of Section § 17 of their book [8]) that external electromagnetic field theory could predict creation (or annihilation) of single electrons or positrons. This would happen when one does not restrict appropriately the strength of the potential which he switches on. For strong potentials, what gives rise to this is that some positive (negative) energy wave functions are transformed completely by the c-number evolution operator into negative (positive) energy wave functions. This can be prevented by demanding a condition like $|(\psi, e\gamma^0 \gamma A \psi)| < (\psi, |\gamma^0(i\gamma \cdot \partial + m)| \psi) \not\Rightarrow \psi$ for example. (This is the condition which leads to $Z \leqq 87$ for the Coulomb potential Ze/r; this is discussed in the Chapters V and VI of Kato's book [9].)

It would be interesting to examine other existing external field theories to see how a similar "weakness" condition comes in.

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