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A Family of Equilibrium States Relevant to Low Temperature Behavior of Spin $\frac{1}{2}$ Classical Ferromagnets. Breaking of Translation Symmetry

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Abstract. An analysis of a family of equilibrium states is performed which, combined with our previous work, allows to describe all translation invariant equilibrium states of spin $\frac{1}{2}$ classical ferromagnetic systems with finite range interactions at low temperatures. A model is described with continuously many equilibrium states for low temperatures.

The ferromagnetic systems have several special features when compared with general lattice systems. In this note we exploit various inequalities of Griffiths, Kelly, and Sherman (GKS) to analyse the way in which the translation invariance of spin $\frac{1}{2}$ classical ferromagnetic systems can be broken. We also give an example of a system with continuously many ergodic equilibrium states.

Among the equilibrium states of the systems under consideration special role is played by the state ϱ^+ obtained as the limit of finite volume states with "+" boundary conditions. The rather straightforward analysis of Section 3 and the information obtained in [7] show that for a large family of ferromagnetic systems the translation symmetry can be broken only in the manner described by formula (8): the extremal non invariant states that enter into the decomposition are obtained from ϱ^+ by flipping spins at some lattice sites, and only flippings leaving invariant the energy are allowed. At the same time the description of all translation invariant equilibrium states is reduced to a description of all translation invariant measures on a symmetry group of the system.

This analysis suggests that for some systems the family of all ergodic equilibrium states (for a given temperature) can be large. This was

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presumably realized before for models with disconnected interactions. In Section 4 we describe a (connected) model with continuously many ergodic equilibrium states for low enough temperatures. We finish with some conjectures.

1. Notation. Equilibrium States

A configuration of spin $\frac{1}{2}$ classical system on the ν-dimensional lattice is a function from $\mathbb{Z}^\nu$ to $\{1, -1\}$. The set

$$\mathscr{X} = \{1, -1\}^{\mathbb{Z}^\nu}$$

of all configurations is made into a compact separable space by the product topology. The function on $\mathscr{X}$ which to each configuration assignes its value at the lattice site i is denoted by σ_i, and

$$\sigma_A = \prod_{i \in A} \sigma_i, \qquad A \in \mathscr{P}_f(\mathbb{Z}^\nu)$$

where $\mathscr{P}_f(\mathbb{Z}^\nu)$ is the family of all finite subsets of $\mathbb{Z}^\nu$. $\mathscr{X}_\Lambda$ will denote the set of configurations in $\Lambda \subset \mathbb{Z}^\nu$:

$$\mathscr{X}_\Lambda = \{1, -1\}^\Lambda .$$

An interaction is defined by a subfamily $\mathscr{B}$ of $\mathscr{P}_f(\mathbb{Z}^\nu)$ and by a function J on $\mathscr{B}$; the elements of $\mathscr{B}$ are called bonds. We consider mainly translation invariant interactions, i.e. both the family $\mathscr{B}$ and the function J are invariant under translations. We assume that $J(B) \geqq 0$ for all $B \in \mathscr{B}$ (ferromagnetic interactions) and that

$$\sum_{B \ni 0} J(B) < \infty .$$

The Gibbs state in Λ corresponding to a configuration Y outside of Λ ascribes to a configuration X in Λ the probability

$$\varrho_\Lambda^Y(X) = \frac{1}{Z_\Lambda^Y} \exp\left[\sum_{B \cap \Lambda \neq \emptyset} K(B)\, \sigma_B(X, Y)\right] \tag{1}$$

where $K(B) = \beta J(B)$ and

$$Z_\Lambda^Y = \sum_{X \in \mathscr{X}_\Lambda} \exp\left[\sum_{B \cap \Lambda \neq \emptyset} K(B)\, \sigma_B(X, Y)\right] .$$

Let $\mathfrak{A}$ denote the C^*-algebra of complex continuous functions on $\mathscr{X}$ and let $\mathfrak{A}_\Lambda$ be the subalgebra of $\mathfrak{A}$ of the functions depending on the restriction of a configuration to Λ only; $\mathfrak{A}_\Lambda$ is identified with $\mathscr{C}(\mathscr{X}_\Lambda)$.

A probability measure on $\mathscr{X}$, or, equivalently, a state of $\mathfrak{A}$ is called an equilibrium state (corresponding to K) if when restricted to $\mathfrak{A}_\Lambda$,

Λ finite, it is a combination of the states $\{\varrho_\Lambda^Y\}_Y$:

$$\varrho(f) = \int \varrho_\Lambda^Y(f)\, \tilde{\varrho}_\Lambda(dY), \qquad f \in \mathfrak{A}_\Lambda, \tag{2}$$

where $\tilde{\varrho}_\Lambda$ is a measure on $\mathscr{X}_{\mathbb{Z}^\nu \setminus \Lambda}$. The set of all equilibrium states is denoted by Δ; the dependence of Δ on the interaction and the temperature is not written explicitly. Δ is a compact space when equipped with the w^*-topology which in our case is defined by the family of functions

$$\varrho \mapsto \varrho(\sigma_A), \qquad A \in \mathscr{P}_f(\mathbb{Z}^\nu).$$

Δ is closed under forming convex combinations. It is a Choquet simplex, i.e. for each $\varrho \in \Delta$ there exists a (unique) measure carried by the set $\mathscr{E}(\Delta)$ of the extremal points of Δ with the resultant equal to ϱ [1].

2. Inequalities and the ϱ^+ State[2]

Let a finite system on a volume Λ be given, with bonds $\mathscr{B} \subset \mathscr{P}(\Lambda)$ and interaction J. Let ϱ be the corresponding Gibbs state:

$$\varrho(X) = Z^{-1} \exp \sum_{B \in \mathscr{B}} K(B)\, \sigma_B(X), \qquad K(B) = \beta J(B).$$

We list the inequalities that will be needed later.

If J is ferromagnetic then

$$\varrho(\sigma_A \sigma_B) \geqq \varrho(\sigma_A)\, \varrho(\sigma_B), \qquad A, B \subset \Lambda. \tag{G.1}$$

If ϱ' is the Gibbs state corresponding to an interaction J', not necessarily ferromagnetic but with the same set $\mathscr{B}$ of bonds as J, and if $|J'(B)| \leqq J(B)$ all B, then

$$\varrho'(\varrho_A) \leqq \varrho(\sigma_A) \tag{G.2}$$

with $J'(A) = 0$ if $A \neq B$ and $J'(B) = J(B)$ we obtain by direct computation: $\varrho'(\sigma_B) = \operatorname{th} K(B)$. Therefore, by inequality (G.2),

$$\varrho(\sigma_B) \geqq \operatorname{th} K(B), \quad \text{all} \quad B \in \mathscr{B}. \tag{3}$$

We now place ourself in the situation of Section 1 and we draw some consequences of the inequalities.

The state ϱ_Λ^Y fits into the framework above with $\{B \cap \Lambda\}_{B \in \mathscr{B}}$ as the set of bonds and

$$J^Y(B') = \sum_{B \cap \Lambda = B'} J(B)\, \sigma_{B \setminus B'}(Y)$$

[1] See [1], [4], [6], for the notion and properties of equilibrium state.

[2] For the material of this section see [8], [9], and references there; the inequality (G.2) is given in exercise 3 of [3]. However, some proofs, in particular the use of the maximality of ϱ^+, seem to be new.

as the interaction. It follows that the state ϱ_Λ^+ defined by: $Y_i = 1$ all $i \in \mathbb{Z}^\nu \backslash \Lambda$, corresponds to a ferromagnetic interaction, and that

$$|J^Y(B')| \leqq J^+(B'), \quad \text{all} \quad B', \quad \text{all} \quad Y.$$

Therefore by (G.2)

$$\varrho_\Lambda^+(\sigma_A) \geqq \varrho_\Lambda^Y(\sigma_A), \quad \text{all} \quad Y, \ A \subset \Lambda \tag{4}$$

and hence

$$\varrho_\Lambda^+(\sigma_A) \geqq \varrho(\sigma_A), \quad \text{all} \quad \varrho \in \Delta, \ A \subset \Lambda. \tag{5a}$$

By (G.1) and (3)

$$\varrho_\Lambda^+(\sigma_A \sigma_B) \geqq \varrho_\Lambda^+(\sigma_A)\, \varrho_\Lambda^+(\sigma_B) \qquad A, B \subset \Lambda, \tag{5b}$$

$$\varrho_\Lambda^+(\sigma_B) \geqq \operatorname{th} K(B) \qquad B \in \mathscr{B}, \ B \subset \Lambda. \tag{5c}$$

If $\Lambda' \supset \Lambda$ then $\varrho_{\Lambda'}^+$ restricted to Λ is of the form (2). Therefore by (4)

$$\varrho_{\Lambda'}^+(\sigma_A) \leqq \varrho_\Lambda^+(\sigma_A) \quad \text{for all} \quad A \subset \Lambda.$$

It follows that $\varrho_\Lambda^+(\sigma_A)$ converge when $\Lambda \to \infty$, and it is not hard to see that the limit defines an equilibrium state in the sense of (2). This state is denoted by ϱ^+. By (5) ϱ^+ has the following properties:

$$\varrho^+(\sigma_A) \geqq \varrho(\sigma_A) \qquad A \in \mathscr{P}_f(\mathbb{Z}^\nu), \ \varrho \in \Delta, \tag{6a}$$

$$\varrho^+(\sigma_A \sigma_B) \geqq \varrho^+(\sigma_A)\, \varrho^+(\sigma_B) \qquad A, B \in \mathscr{P}_f(\mathbb{Z}^\nu), \tag{6b}$$

$$\varrho^+(\sigma_B) \geqq \operatorname{th} K(B) \qquad B \in \mathscr{B}. \tag{6c}$$

ϱ^+ is an extremal equilibrium state and it is invariant under any affine transformation of $\mathbb{Z}^\nu$ leaving invariant the interaction.

For a proof, suppose that $\varrho^+ = \lambda \varrho_1 + (1-\lambda)\varrho_2$, $0 < \lambda < 1 \perp$, $\varrho_1, \varrho_2 \in \Delta$. If there exists $A \in \mathscr{P}_f(\mathbb{Z}^\nu)$ such that $\varrho_1(\sigma_A) \neq \varrho^+(\sigma_A)$ then by (6a) $\varrho_1(\sigma_A) < \varrho^+(\sigma_A)$. Since, again by (6a), $\varrho_2(\sigma_A) \leqq \varrho^+(\sigma_A)$ we arrived at a contradiction proving the extremality of ϱ^+.

A proof of the second part of the statement comes from the observation that for any affine transformation of $\mathbb{Z}^\nu$ the transform of ϱ^+ again satisfies (6a) and a state ϱ^+ satisfying (6a) is obviously unique.

3. States that Agree with ϱ^+ on the Group Generated by Bonds

It is convenient, and important for what follows, to introduce in $\mathscr{X}$ a group structure by regarding $\{1, -1\}$ as an abelian group and defining the group operation in $\mathscr{X}$ pointwise (elements of $\mathscr{X}$ are mappings from $\mathbb{Z}^\nu$ to $\{1, -1\}$). In this way $\mathscr{X}$ acquires the structure of a compact abelian group. The σ_A's, $A \in \mathscr{P}_f(\mathbb{Z}^\nu)$, are characters of $\mathscr{X}$ and the mapping $A \mapsto \sigma_A$ identifies $\mathscr{P}_f(\mathbb{Z}^\nu)$ (with the symmetric difference as the group operation and the discrete topology) with the group dual to $\mathscr{X}$. The subgroup of $\mathscr{P}_f(\mathbb{Z}^\nu)$ generated by $\mathscr{B}$ is denoted by $\bar{\mathscr{B}}$.

The (internal) symmetry group [3]

$$\mathscr{S} = \{X \in \mathscr{X} : \sigma_B(X) = 1, \quad \text{all} \quad B \in \mathscr{B}\}$$

is a closed subgroup of $\mathscr{X}$ and therefore compact.

For the Ising model $\bar{\mathscr{B}}$ consists of all the even subsets of $\mathbb{Z}^\nu$ and $\mathscr{S}$ has only two elements: E with all components equal to 1, and F with all components equal to -1.

For $Y \in \mathscr{X}$ the translation mapping $X \mapsto X + Y$ is a homeomorphism of $\mathscr{X}$; the state

$$f \mapsto \varrho(f_Y)$$

where $f_Y(X) = f(X + Y)$ will be denoted by ϱ_Y. Clearly

$$\varrho_Y(\sigma_A) = \sigma_A(Y)\, \varrho(\sigma_A)\,. \tag{7}$$

It follows directly from (2) that for an equilibrium state ϱ and $Y \in \mathscr{S}$ ϱ_Y is again an equilibrium state, and since the mapping $\varrho \mapsto \varrho_Y$ preserves convex combinations ϱ_Y is an extremal equilibrium state if ϱ is. For each $\varrho \in \Delta$ the mapping $Y \mapsto \varrho_Y$ from $\mathscr{S}$ to Δ is obviously continuous. For a measure μ on $\mathscr{S}$ we define

$$\varrho_\mu = \int \varrho_\mu^+\, \mu(dY)\,. \tag{8}$$

By (7)

$$\varrho_\mu(\sigma_A) = \mu(\sigma_A)\, \varrho^+(\sigma_A)\,. \tag{9}$$

Let $\mathscr{S}^+$ be the isotropy subgroup of ϱ^+:

$$\mathscr{S}^+ = \{Y \in \mathscr{S} : \varrho_Y^+ = \varrho^+\}\,.$$

Since $\mathscr{B}$ is $\mathbb{Z}^\nu$-invariant the same is true about $\mathscr{S}$. Similarly $\mathscr{S}^+$ is $\mathbb{Z}^\nu$-invariant, since ϱ^+ is $\mathbb{Z}^\nu$-invariant. This will alow us later to consider the action of $\mathbb{Z}^\nu$ on the factor group $\mathscr{S}/\mathscr{S}^+$.

The function: $Y \mapsto \varrho_Y^+$ is constant on $\mathscr{S}^+$-cosets. Therefore the integral in (8) can be transformed into an integral over $\mathscr{S}/\mathscr{S}^+$, or, equivalently, onto an integral over $\mathscr{S}$ with respect to a $\mathscr{S}^+$-invariant measure. We write $[Y] \mapsto \varrho_{[Y]}^+$ for the function on $\mathscr{S}/\mathscr{S}^+$ corresponding to $Y \mapsto \varrho_Y^+$, and we remark that, by the definition of $\mathscr{S}^+$, $[Y] \mapsto \varrho_{[Y]}^+$

[3] For finite systems the groups $\bar{\mathscr{B}}$ and $\mathscr{S}$ are introduced in [5]. The reader will find there a discussion of several examples.

is an injective mapping. We write

$$\varrho_\mu = \int \varrho^+_{[Y]} \mu(d[Y]) \tag{8'}$$

for the integral over $\mathscr{S}/\mathscr{S}^+$ that corresponds to (8).

Let

$$\mathscr{B}^+ = \{A \in \mathscr{P}_f(\mathbb{Z}^\nu) : \varrho^+(\sigma_A) \neq 0\}.$$

By (6b) $\mathscr{B}^+$ is a subgroup of $\mathscr{P}_f(\mathbb{Z}^\nu)$, and by (6c) $\mathscr{B}^+ \supset \mathscr{B}$. Therefore $\mathscr{B}^+$ contains $\bar{\mathscr{B}}$.

Let Δ^+ be the family of the equilibrium states that agree with ϱ^+ on $\bar{\mathscr{B}}$:

$$\Delta^+ = \{\varrho \in \Delta : \varrho(\sigma_A) = \varrho^+(\sigma_A) \quad \text{for all} \quad A \in \bar{\mathscr{B}}\}.$$

If $\varrho \in \Delta^+$ then $\varrho_Y \in \Delta^+$ for each $Y \in \mathscr{S}$, and (8) defines an element of Δ^+ for any μ.

Theorem. *All elements of Δ^+ are of the form* (8). *The mapping $\mu \mapsto \varrho_\mu$ from normalized measures on $\mathscr{S}/\mathscr{S}^+$ to equilibrium states is one-to-one, and* (8′) *gives the decomposition of ϱ_μ into extremal elements of Δ. μ is $\mathbb{Z}^\nu$-invariant if and only if ϱ_μ is $\mathbb{Z}^\nu$-invariant, and μ is ergodic if and only if ϱ_μ is.*

Proof. Let $\varrho \in \Delta^+$ and let

$$\varrho = \int_{\mathscr{E}(\Delta)} \varrho_\xi \, \mu(d\xi)$$

be the decomposition of ϱ into extremal elements of Δ. By (6a) $\varrho_\xi(\sigma_A) \leqq \varrho^+(\sigma_A)$ for all $A \in \mathscr{P}_f(\mathbb{Z}^\nu)$. If $A \in \bar{\mathscr{B}}$ then $\varrho(\sigma_A) = \varrho^+(\sigma_A)$ and therefore the set of $\xi \in \mathscr{E}(\Delta)$ for which $\varrho_\xi(\sigma_A) \neq \varrho^+(\sigma_A)$ is of μ-measure zero. Since the family $\mathscr{P}_f(\mathbb{Z}^\nu)$ is denumerable, $\varrho_\xi \in \Delta^+$ μ-almost everywhere. Therefore to prove the first part of the theorem it is enough to show that if $\varrho \in \Delta^+ \cap \mathscr{E}(\Delta)$ then there exists $Y \in \mathscr{S}$ such that $\varrho = \varrho^+_Y$.

Let $\varrho \in \Delta^+ \cap \mathscr{E}(\Delta)$ and let

$$\bar{\varrho} = \int \varrho_Y \, dY, \qquad \overline{\varrho^+} = \int \varrho^+_Y \, dY$$

where dY is the normalized Haar measure on $\mathscr{S}$. $\bar{\varrho} \in \Delta^+$ and $\bar{\varrho}(\sigma_A) = 0$ if $A \notin \bar{\mathscr{B}}$ since

$$\bar{\varrho}(\sigma_A) = \varrho(\sigma_A) \int \sigma_A(Y) \, dY$$

and σ_A is a character. Similarly $\overline{\varrho^+}(\sigma_A) = 0$ if $A \notin \bar{\mathscr{B}}$ and therefore $\bar{\varrho} = \overline{\varrho^+}$. This implies, by the uniqueness of the decomposition into extremal elements of Δ, that the intersection of $\{\varrho_Y\}_{Y \in \mathscr{S}}$ and $\{\varrho^+_Y\}_{Y \in \mathscr{S}}$ is not empty; i.e. there exist $Y', Y'' \in \mathscr{S}$ such that $\varrho_{Y'} = \varrho^+_{Y''}$. Hence $\varrho = \varrho^+_Y$, $Y = Y' - Y''$.

Since $\varrho^+_{[X]} \neq \varrho^+_{[Y]}$ if $[X] \neq [Y]$ the representation (8′) gives the unique decomposition of ϱ_μ into extremal equilibrium states and therefore the mapping $\mu \mapsto \varrho_\mu$ is one-to-one.

It follows from (9) that a translate of μ corresponds to a translate of ϱ_μ. Therefore invariant measures are in one-to-one correspondence with invariant elements of Δ^+. Since the mapping $\mu \mapsto \varrho_\mu$ is linear and bijective this implies that ergodic measures are in one-to-one correspondence with ergodic elements of Δ^+. The theorem is proved.

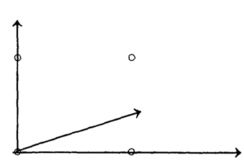
It was shown in [7] that for a large family of ferromagnetic systems all the translation invariant equilibrium states at low temperatures belong to Δ^+. Therefore the above theorem yields a description of all the translation invariant equilibrium states at low temperatures for those systems. For instance, in the case of a connected two-body interaction $\bar{\mathscr{B}}$ contains all the even subsets of $\mathbb{Z}^\nu$ ([7], Section 4.9) and therefore $\mathscr{S}$ has only two elements: E, F. Hence, by (8), only two ergodic equilibrium states are possible. For disconnected systems $\mathscr{S}$, $\mathscr{S}^+$, $\mathscr{S}/\mathscr{S}^+$ are products of the groups corresponding to the connected components.

4. A Non-Denumerable Family of Ergodic Equilibrium States

The system is three-dimensional. The bonds are the translates of the following ones:

$$B_1 = \{0, e_2, e_3, e_2 + e_3\}, \quad B_2 = \{0, e_1, e_3, e_1 + e_3\}, \quad B_3 = \{0, e_1, e_2, e_1 + e_2\}$$

where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$; B_2 is pictured below



We put $J(B) = 1$, all $B \in \mathscr{B}$. Let for $n \in \mathbb{Z}$, F_n denotes the configuration that is -1 at the lattice sites belonging to the plane: $x_1 = n$, and $+1$ at all other sites. Since intersections of the bonds with such planes are even $F_n \in \mathscr{S}$, all $n \in \mathbb{Z}$. The subgroup $\mathscr{F}$ of $\mathscr{X}$ generated by $\{F_n\}$ is isomorphic to the group of configurations of a one-dimensional lattice system with the action of $\mathbb{Z}^3$ on $\mathscr{F}$ reduced to the action of $\mathbb{Z}$ on the later.

For low enough temperatures the group $\mathscr{S}^+$ has at most two elements: E, F ($E_i = +1$, $F_i = -1$ for all $i \in \mathbb{Z}^3$).

For a proof it is enough to show that $\mathscr{B}^+$ contains all the Ising type bonds for low enough temperatures. Since $\mathscr{B}^+$ is invariant under rotations and translations, as ϱ^+ is, it is enough to show that $\{0, e_3\} \in \mathscr{B}^+$.

This, in turn, will be deduced from the existence of the spontaneous magnetization in the Ising model.

Let $\tilde{\mathscr{B}}$ be the set of bonds of $\mathscr{B}$ that are not contained in $\{x \in \mathbb{Z}^3 : x_3 \geqq 1\} \cup \{x \in \mathbb{Z}^3 : x \leqq 0\}$, and let $\tilde{J} = J \mid \tilde{\mathscr{B}}$. If $\tilde{\varrho}^+$ is the "+" state corresponding to the interaction $\tilde{J}$ then by the GKS inequalities

$$\varrho^+(\sigma_A) \geqq \tilde{\varrho}^+(\sigma_A) \quad \text{for all} \quad A \in \mathscr{P}_f(\mathbb{Z}^\nu).$$

Let $\Lambda_n = \{x \in \mathbb{Z}^3 : O \leqq x_i \leqq n,\ i = 1, 2, 3\}$. A configuration X in Λ_n can be identified with $(X_0, X_1, \ldots, X_n)$ where X_i is a configuration in $\Lambda_n^0 = \{x \in \mathbb{Z}^2 : O \leqq x_i \leqq n,\ i = 1, 2\}$. If A is a sum of a subset, say A_0, of the plane $x_0 = 0$ and of the translate of A_0 by e_3 then

$$\widetilde{\varrho_n^+}(\sigma_A) = \sum_{X_0, X_1} \sigma_A(X_0, X_1) \cdot \exp\Bigg[\beta \sum_{\substack{B \in \tilde{\mathscr{B}} \\ B \cap \Lambda_n \neq \emptyset}} \sigma_B(X_0, X_1)\Bigg] \Bigg/ \sum_{X_0, X_1} \exp\Bigg[\beta \sum_{\substack{B \in \tilde{\mathscr{B}} \\ B \cap \Lambda_n \neq \emptyset}} \sigma_B(X_0, X_1)\Bigg].$$

Furthermore

$$\sigma_A(X_0 + X_1, X_1) = \sigma_{A_0}(X_0)$$

and therefore performing the change of variables

$$(X_0, X_1) \mapsto (X_0 + X_1, X_1)$$

we obtain

$$\widetilde{\varrho_n^+}(\sigma_A) = \sum_{X_0} \sigma_{A_0}(X_0) \exp \beta \sum_{\substack{B \in \tilde{\mathscr{B}} \\ B \cap \Lambda_n \neq \emptyset}} \sigma_{B_0}(X_0) \Bigg/ \sum_{X_0} \exp \beta \sum_{\substack{B \in \tilde{\mathscr{B}} \\ B \cap \Lambda_n \neq \emptyset}} \sigma_{B_0}(X_0).$$

The right hand side here is equal to the expectation value of σ_{A_0} in the "+" state of the two-dimensional Ising model in Λ_n^0. Putting $A = \{0, e_3\}$ and passing to the limit as $n \to \infty$ we see that $\varrho^+(\sigma_{\{0, e_3\}})$ is bounded from below by the spontaneous magnetization of the two-dimensional Ising model, and therefore is not zero at low enough temperatures.

For the model under consideration the family of ergodic equilibrium states at low enough temperatures is non-denumerable.

According to the theorem of the preceeding section it is enough to show that the family of ergodic (under the action of $\mathbb{Z}^3$) measures on $\mathscr{S}/\mathscr{S}^+$ is non-denumerable. Since $\mathscr{S} \supset \mathscr{F}$ and $\mathscr{S}^+ \subset \{E, F\}$ it is enough to find a non-denumerable family of $\mathbb{Z}$ and F-invariant measures on $\mathscr{F}$. Such family of measures is provided by the one-dimensional Ising model if the temperature is varied. Another (denumerable) family of examples is obtained from invariant measures concentrated on the $\mathbb{Z}$-orbits of the periodic elements of $\mathscr{F}$. In these examples μ has finite support.

Whereas it is not hard to vary and to multiply examples like the one above, even for the models discussed in [7] we do not have a description of $\mathscr{S}^+$, or $\mathscr{B}^+$, at low temperatures. By the GKS inequalities $\mathscr{B}^+$ increases with the inverse temperature. For high temperatures $\mathscr{B}^+ = \bar{\mathscr{B}}$ since the state $\overline{\varrho^+}$ of Section 3 vanishes on σ_A, $A \notin \bar{\mathscr{B}}$, and the equilibrium state is for high temperatures unique. We conjecture that $\mathscr{S}^+$ stabilizes at low temperatures, and that the limits as $T \to O$ of translation invariant equilibrium states are given by invariant measures on $\mathscr{S}/\mathscr{S}^+$ where $\mathscr{S}^+$ is the (temperature independent) group corresponding to low temperatures.

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