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Gibbs Processes and Generalized Bernoulli Flows for Hard-core One-dimensional Systems

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Abstract. We consider a class of infinite-range potentials for which phase transitions are absent, and prove by the Ornstein-Friedman theorem, that they generate dynamical systems that are Bernoulli flows in a generalized sense.

1. Introduction

A very interesting problem in Statistical Mechanics is the study of the ergodic properties of measures which define the equilibrium states of a gas in the thermodynamic limit. These measures are called infinite volume Gibbs processes following a terminology due to Dobrushin [2]. It is well known that the set of these measures associated to an interaction forms a convex set in the space $M(K)$ (i.e. the translationally invariant probability measures on the space K of configurations of the system) and that the extremal elements of this set are ergodic [7], with respect to the group of spatial translations.

For one-dimensional lattice systems Ruelle proved in [3] that for interactions Φ such that

$$\|\Phi\|_1 = \sum_{S \ni 0} |\Phi(S)| (\operatorname{diam} S) < +\infty \tag{1.1}$$

the thermodynamic limit of finite volume Gibbs processes exists, is an infinite volume Gibbs process and is the only state in the above mentioned convex set. From a physical point of view this means that phase transitions are absent. Furthermore Ruelle proved that the Gibbs process is a K-system [11]. The ergodic properties of the process have been subsequently studied by Gallavotti in [1]. He has shown that these processes are Bernoulli schemes ([4, 10]). Gallavotti uses Ruelle's results to prove

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they are weak Bernoulli shifts and therefore, by the Ornstein-Friedman theorem [4], they are Bernoulli shifts.

We shall use these methods to prove similar results for some one-dimensional continuous Gibbs process with long range hard-core interactions. This will prove the conjecture contained in [10].

2. Results

Let K be the family of subsets $X \subset \mathbb{R}$ such that

$$|x-x'| \geqq a > 0 \qquad x, x' \in X, x \neq x'. \tag{2.1}$$

It is easy to see that K is a compact space in the weakest topology for which a net $\{X_\alpha\}$ of configurations converges to X_0 if for any bounded interval (α, β) with $\alpha, \beta \notin X_0$ the configurations $X_\alpha \cap (\alpha, \beta)$ converge pointwise to $X_0 \cap (\alpha, \beta)$. Let $C(K)$ be the space of continuous functions over K with the uniform norm. If Δ is a bounded measurable region we can integrate these functions with the measure

$$\int_\Delta dX = \sum_{n \geqq 0} \frac{1}{n!} \int_\Delta dx_1 \dots dx_n \tag{2.2}$$

that is to say,

$$\int_\Delta dX\, f(X) = \sum_{n \geqq 0} \frac{1}{n!} \int_\Delta dx_1 \dots dx_n\, f(\{x_1, \dots, x_n\}) \tag{2.3}$$

where the integral is taken over allowed configurations. A state μ of the system is a probability measure on K, that is a state on the algebra $C(K)$. We write for $A \in C(K)$

$$\mu(A) = \int_\Delta dX \int A(X \cup Y)\, \mu_\Delta(X, dY) \tag{2.4}$$

and define μ_Δ by this formula.

A potential Φ is a real function on K such that

1. $\Phi(\phi) = 0$.
2. $\Phi(X + x) = \Phi(X)$ (translation invariance).

We shall denote by T_t the translation $T_t X = X + t$, $t \in \mathbb{R}$.

Let $\mathfrak{B}$ be the Banach space of continuous potentials Φ such that

$$\|\Phi\| = \sup_{\substack{X \in K \\ X \ni 0}} \sum_{\substack{Y \cap \mathbb{R}^+ \neq \phi \\ Y \cap \mathbb{R}^- \neq \phi \\ Y \subset X}} |\Phi(Y)| < +\infty \tag{2.5}$$

where $\mathbb{R}^+$ and $\mathbb{R}^-$ are non-negative and non-positive real numbers.

Gallavotti and Miracle-Sole [6], studied interactions

$$\Phi = \Phi_0 + \tilde{\Phi} \tag{2.6}$$

where $\tilde{\Phi} \in \mathfrak{B}$ and Φ_0 is a non-negative measurable finite-range pair potential (i.e. $\Phi_0(X) = 0$ if $|X| \neq 2$ and $\Phi_0(\{x_1, x_2\}) = 0$ if $|x_1 - x_2| > c_0 > 0$). They proved that only one equilibrium state exists in the thermodynamic limit and that it is continuously differentiable w.r.t. the thermodynamic potentials. So was reached the conclusion that for the above systems phase transitions are absent.

This infinite equilibrium state can be characterized as follows, [7]. It is the only translationally invariant state μ such that for all bounded regions Δ and all $X \subset \Delta$

$$\mu_\Delta(X, dY) = f_\Delta(X, Y)\, \mu_\Delta(\phi, dY) \tag{2.7}$$

where f_Δ is a conditional probability defined as

$$f_\Delta = \begin{cases} 0 & \text{if} \quad X \cup Y \notin K \\ \exp\left[- \sum\limits_{\substack{S \subset X \cup Y \\ S \cap X \neq \phi}} \Phi(S) \right] & \text{if} \quad X \cup Y \in K . \end{cases} \tag{2.8}$$

This state is the Gibbs process we study in this paper. Thus we have defined a dynamical system, as the triple (K, μ, T_t) and our main result is

Theorem 1. *The set of transformations $\{T_t\}$ over K is a generalized Bernoulli flow.*

We give the definitions of the mentioned objects. A one-to-one invertible measure-preserving transformation T is a generalized Bernoulli shift [8], if there is a subalgebra $\mathscr{A}$ of the full measure algebra such that $T^i \mathscr{A}$ are independent and generate it. If $\mathscr{A}$ has a continuous part the entropy of T is infinite. Then by a generalized Bernoulli flow we mean a flow $\{T_t\}$ in which every transformation T_t, $t \neq 0$, is a generalized Bernoulli shift.

Now we fix $t \in \mathbb{R}$ and $T = T_t$. In conformity with this choice of t, we define

$$\Gamma_t(r) = \{x \in \mathbb{R} : tr \leqq x < t(r+1\} \qquad r \in \mathbb{Z} \tag{2.9}$$

and for

$$\Lambda' \subset \Lambda, \qquad \text{with} \qquad \Lambda = (i_1, \ldots, i_{|\Lambda|}), \qquad i_j \in \mathbb{Z} \quad \text{for} \quad j = 1, \ldots, |\Lambda|$$

$$\begin{aligned} A_{\Lambda', \Lambda}(t) = \{X \in K : X \cap \Gamma_{t|\Lambda|^{-1}}(r) = \phi \quad &\text{if} \quad r \in \Lambda \backslash \Lambda' \\ \text{and} \qquad X \cap \Gamma_{t|\Lambda|^{-1}}(r) \neq \phi \quad &\text{if} \quad r \in \Lambda'\} \end{aligned} \tag{2.10}$$

where $|\Lambda|$ is the number of elements of Λ. For every Λ the sets $\{A_{\Lambda, \Lambda'}, \Lambda' \subset \Lambda\}$ form a partition of K and we shall denote it by P^Λ.

In particular by P^n we mean the partition P^Λ with $\Lambda = \{0, \ldots, n-1\}$. Each partition P^Λ with the translation T generate an algebra $\mathscr{A}_\Lambda$

$$\mathscr{A}_\Lambda = \overline{\bigvee_{-\infty}^{+\infty}{}_i\, T^i P^\Lambda} \tag{2.11}$$

that is the completion of the smallest algebra containing all the sets in $T^i P^\Lambda$, $i \in \mathbb{Z}$. In particular we have

$$\mathscr{A}_n = \overline{\bigvee_{-\infty}^{+\infty} T^i P^{(n)}}\,. \tag{2.12}$$

The key result is the

Theorem 2. *The dynamical system $(K, \mathscr{A}_n, \mu, T)$ is a Bernoulli scheme.*

The proof of this result is an adaptation of the Ornstein-Friedman proof for Markov processes [4]. If we assume Theorem 2 and denote by $(K, \mathscr{A}, \mu, T)$ our original dynamical system, it is easy to realize that

$$\mathscr{A} = \overline{\cup_\Lambda \mathscr{A}_\Lambda} = \overline{\cup_n \mathscr{A}_n}\,. \tag{2.13}$$

Then to prove that

Theorem 3. *The dynamical system $(K, \mathscr{A}, \mu, T)$ is a generalized Bernoulli scheme with an infinite entropy.*

We need a "limit theorem" due to Ornstein and contained in [8].

Theorem 4. *If T is a $1-1$ invertible measure preserving transformation on a field $\mathscr{A}$ that is increasing union of T-invariant subfields $\mathscr{A}_i$ and if T is a Bernoulli shift on each $\mathscr{A}_i$ then it is a generalized Bernoulli shift on $\mathscr{A}$.*

In our case we have by Eq. (2.12) that the $\mathscr{A}_i$ are invariant and by Theorem 2 that T is a Bernoulli shift on $\mathscr{A}_i$. Now Eq. (2.13) and the "limit theorem" tell us that T is a generalized Bernoulli shift. Further one can explicitely check that its entropy is actually infinite and so we have Theorem 3. Since the above arguments are independent of the particular choice of t, also Theorem 1 is proved.

An extension of the results of this paper which exclude the hard-core condition can probably be carried out for the class of Gibbs processes associated with superstable potentials [9]. This is based on the conjectured possibility of the extension to this class of the results of Refs. [3] and [6].

3. Proofs

In this section we give the proof of Theorem 2 of Section 1, and show that the entropy of the abstract dynamical system $(K, \mathscr{A}, \mu, T_t)$ is actually infinite for every t.

A. Proof of Theorem 2

Our proof is analogous to the one for the lattice case [1]. Particular care is to be paid to the presence of the finite range, positive and possibly unbounded potential Φ_0, see Eq. (2.6). We fix $t>0$ and the partition of $\mathbb{R}$, $\{\Gamma_t(r)\}$. Let $\Lambda=(0,\ldots,m-1)$, then the corresponding partition P^m is by definition a generator for $(K,\mathscr{A}_m,\mu,T)$, where μ is our Gibbs process associated with Φ. We show that P^m is also weakly Bernoulli and therefore the thesis is proved. We remember the generating partition P^m is weakly Bernoulli if for every $\varepsilon>0$ there exists $N_\varepsilon>0$ such that for $m_1, m_2>0$ the partitions $\bigvee_{i\,0}^{m_1} T^i P^m$ and $\bigvee_{i\,n}^{n-m_2} T^i P^m$ are ε-independent, as soon as $n-m_1>N_\varepsilon$. To write down this condition we use the following notations

$$\Lambda_1=(0,\ldots,mm_1-1)\,;\qquad \Lambda_2=(0,\ldots,mm_2-1)\,, \tag{3.1}$$

$$f_{\Lambda_i}(\Lambda')=\mu(A_{\Lambda_i,\Lambda'}(mt)),\qquad \Lambda'\subset\Lambda_i,\qquad i=1,2\,, \tag{3.2}$$

$$f_{\Lambda_1\cup(\Lambda_2+nm)}(\Lambda')=\mu(A_{\Lambda_1\cup(\Lambda_2+nm),\Lambda'}(m_1+m_2)t)\,. \tag{3.3}$$

Therefore $f_{\Lambda_1}(\Lambda')$, $f_{\Lambda_2}(\Lambda')$, $f_{\Lambda_1\cup(\Lambda_2+nm)}(\Lambda')$ are the Gibbs measures of the corresponding atoms of the partitions

$$\bigvee_{i\,0}^{m_1} T^i P^m,\ \bigvee_{i\,0}^{m_2} T^i P^m,\ \left(\bigvee_{i\,0}^{m_1} T^i P^m\right)\vee\left(\bigvee_{n}^{n+m_2} T^i P^m\right)$$

and the weak Bernoulli condition can be written as follows

$$\sum_{\substack{\Lambda_1'\subset\Lambda_1\\ \Lambda_2'\subset\Lambda_2+nm}} |f_{\Lambda_1\cup(\Lambda_2+nm)}(\Lambda_1'\cup\Lambda_2')-f_{\Lambda_1}(\Lambda_1')\,f_{\Lambda_2}(\Lambda_2')|<\varepsilon\,. \tag{3.4}$$

Eq. (3.4) must hold as $n-m_1>N_\varepsilon$ independently of m_1 and m_2. The proof of Eq. (3.4) is completely analogous to that of Eq. (1.8) of [1], for the lattice case. Instead of the Ruelle's theorem [3], we use its extension to the hard-core case as carried out in [6]. The only modification occurs in Lemma 1 and 2 and Theorem 2 of [1], which in fact hold with the additional requirement that n is sufficiently large so that $nt>c_0$ the range of the potential.

B. Proof of Theorem 3

There is only to prove that the entropy[1] of T is infinite. As in the previous section we consider the partitions P^m. Then we find a lower

[1] We recall some well known definitions. If P is a partition and T a measure preserving one-to-one invertible transformation

$$H(P)=-\sum_i \mu(P_i)\log\mu(P_i),\qquad P_i\in P\,,$$

$$H(T,P)=\lim_{n\to\infty} n^{-1} H\left(\bigvee_0^n T^i P\right).$$

bound for $H(T, P^m)$, the entropy of the transformation T relative to the partition P^m [4]. This bound involves the classical entropy S^c [12], which is known to be finite [5], and another term depending on the basic interval length t/m as $\log t/m$. Therefore the entropy $H(T, P^m)$ is an unbounded function of m and so the Kolmogorov-Sinai entropy $H(T) = \sup_{\Lambda} H(T, P^{\Lambda})$ is infinite. We first recall the definition of the classical entropy for the interval $(0, L)$

$$S_L = - \int_{K(0,L)} dX\, \Omega_L(X) \log \Omega_L(X) \tag{3.5}$$

and $\Omega_L(X)$ is related to $\mu_L(X, dY)$ as

$$\Omega_L(X) = \int \mu_L(X, dY)\,.$$

We fix m, put $L = mnt$ and compare S_L with $H\left(\bigvee_{0}^{n}{}_i\, T^i P^m\right)$. We consider the atoms of $\bigvee_{0}^{n}{}_i\, T^i P^m$,

$$A_{\Lambda,\Lambda'}(nt), \qquad \Lambda = (0, \ldots, nm-1), \qquad \Lambda' \subset \Lambda \tag{3.6}$$

and the corresponding averages of $\Omega_L(X)$, namely

$$\Omega_L(\Lambda') = [C_\Lambda(\Lambda')]^{-1} \int_{A_{\Lambda,\Lambda'} \cap K(0,L)} dX\, \Omega_L(X)\,, \tag{3.7}$$

$$C_\Lambda(\Lambda') = \int_{A_{\Lambda,\Lambda'} \cap K(0,L)} dX\,. \tag{3.8}$$

Then we have

$$f_\Lambda(\Lambda') = C_\Lambda(\Lambda')\, \Omega_L(\Lambda')\,. \tag{3.9}$$

The basic inequality we use in this proof is then the following,

$$\int_{K(0,L)} dX\, \Omega_L(X) \log \Omega_L(X) \geqq \sum_{\Lambda' \subset \Lambda} C_\Lambda(\Lambda')\, \Omega_L(\Lambda') \log \Omega_L(\Lambda')\,. \tag{3.10}$$

Eq. (3.10) is a consequence of

$$\cdot \int_a^b dx\, f(x) \log f(x) \geqq \bar{f} \log \bar{f}$$

where $\bar{f} = (b-a)^{-1} \int_a^b dx\, f(x)$ which follows directly from the convexity of the function $t \to t \log t$.

Now we rewrite Eq. (3.10) introducing the classical entropy S_L via Eq. (3.5) and the Kolmogorov-Sinai entropy by Eq. (3.9) [4]. Then we have

$$L^{-1} S_L \leqq (tn)^{-1} H\left(\bigvee_0^n{}_i\, T^i P^m\right) + L^{-1} \sum_{\Lambda' \subset \Lambda} f_\Lambda(\Lambda') \log C_\Lambda(\Lambda') \tag{3.11}$$

where

$$H\left(\bigvee_0^n{}_i\, T^i P^m\right) = -\sum_{\Lambda' \subset \Lambda} f_\Lambda(\Lambda') \log f_\Lambda(\Lambda').$$

Now we point out the dependence on m of $C_\Lambda(\Lambda')$, with the following upper bound

$$C_\Lambda(\Lambda') \leqq (t/m)^{|\Lambda'|/2}, \qquad m > t. \tag{3.12}$$

We perform the limit for $n \to \infty$ in Eq. (3.11) and obtain

$$S^c = \lim_{L\to\infty} L^{-1} S_L \leqq t^{-1} H(T, P^m) + b \log m^{-1} t, \tag{3.13}$$

$$2b = \limsup_{n\to\infty} \sum_{\Lambda' \subset \Lambda} f_\Lambda(\Lambda')(nt)^{-1} |\Lambda'|. \tag{3.14}$$

It remains to prove that $b > 0$. We make extensive use of some results contained in [6] and we collect them in the

Theorem. *There exists a function $h \in C(K^+)$ and a measure ν on $C(K^+)$ such that*

- $\int d\nu(Y) = 1$
- $\int d\nu(Y) A(Y) = e^{-lP} \int d\nu(Y) \int_{K(0,l)} dX\, f_{(0,l)}(X, Y) A(X \cup Y + l)$ $\qquad A \in C(K^+)$
- $\int d\nu(Y) h(Y) = 1$
- $\int d\nu(Y) h(Y) A(Y) = \int d\mu(Y) A(Y) \qquad A \in C(K^+)$

where μ is our Gibbs process, and P is the corresponding pressure. The function $f_{(0,L)}$ was defined in Eq. (2.8).

Using these results we have

$$\begin{aligned} b &= (nt)^{-1} \int d\nu(Y) h(Y) \sum_{\Lambda' \subset \Lambda} \chi_{\Lambda\Lambda'}(Y) |\Lambda'| \\ &= (nt)^{-1} \int d\nu(Y) \int_{K(0,L)} dX \exp(-LP) h(X \cup Y) f_{(0,L)}(X, Y) \\ &= (nt)^{-1} \exp(-LP) \int d\nu(Y) \int_{K(0,L)} dX\, h(X \cup Y) \frac{d}{d\lambda} f_{(0,L)}(X, Y) \end{aligned} \tag{3.15}$$

where $-\lambda$ is the coefficient of the one-body potential. So we have

$$b = L^{-1} \exp(-LP) \frac{d}{d\lambda} \exp(LP) = \frac{d}{d\lambda} P. \tag{3.16}$$

The positivity of b can then be checked using Eqs. (37), (34) of Ref. [6].

Finally we mention that it can be proved a stronger result about the asymptotic behaviour of $H(T, P^m)$, actually

$$t^{-1} H(T, P^m) - \{b \log|mt^{-1}| + S^c\} \xrightarrow[m \to \infty]{} 0 .$$

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