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The Motion of Black Holes

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Abstract. The motion of axisymmetric, non rotating black holes is discussed using the properties of Weyl solutions. It is shown that there are no such solutions representing more than one black hole or black holes and ordinary massive bodies apart from the exceptional case of a massive body which surrounds or partially surrounds a black hole. A new exact solution is obtained representing a black hole chased by a negative mass particle, both objects being uniformly accelerated and all solutions representing a single black hole tidally distorted by an external static, axisymmetric gravitational field are obtained.

1. Introduction

Most of the work that has been done so far on black holes has concentrated on their stationary states and how these are attained. Apart from two results of Hawking – that black holes may not bifurcate and that the area of the event horizon is an increasing function of time, we have compa'ratively few exact results on the motion of black holes. Of course it is widely believed that they more very much as ordinary bodies do, and in current models for some X-ray sources it is assumed that a black hole can orbit an ordinary star in a binary system but it would be nice to have rigorous justification for this assumption.

Such a justification must await a detailed examination of the equations of motion for black holes in the manner of E.I.H. However before that programme is carried out there are things that can be said on the basis of some exact solutions for black holes and curiously enough a certain amount can be said using some of the simplest solutions to Einstein's equations – the Weyl solutions – which are static.

It has been known for some time [1] that there are no such non-singular, solutions representing more than one isolated body whose energy momentum tensor obeys the weak energy condition [2]. This reflects the fact that two non-rotating bodies always attract one another. It has also been known that there exist solutions consisting of two bodies, one of positive mass and which is chased by another of negative mass, the pair both undergoing a uniform acceleration. Similar results hold for "particles" or particles and massive bodies [1–5]. However the physical significance of these latter results are not so obvious since these isolated particles are generalizations of the Curzon particle [6] which is an example of a naked singularity (in this case the curvature scalars diverge). That is a singularity which is not surrounded by an event horizon. Further it has a more complicated structure than a simple particle [7]. Current views about gravitational collapse hold that such objects could not arise in nature. These results tend to reassure us that bodies move according to General Relativity in very much the same way as they do in Newton's theory even though one may not believe that negative mass can occur in the real world.

It is therefore of interest to ask whether these results may be extended to the case of black holes and this is the concern of the present paper. The method of attack consists in showing that if one uses Carter's [8] boundary conditions for black holes in Weyl coordinates one is led uniquely not to configurations looking like "particles" but "rods" with positive mass density. It is well known that the Schwarzschild solution has the appearance of a rod when written in Weyl coordinates. Here a reason is provided for this. Having derived the result one can then derive a further condition from the requirement of "elementary flatness" from which the main results will follow.

The fact that one is led uniquely to the rod configuration provides a simple proof of Israel's theorem in the axisymmetric case. That is a static, axisymmetric, asymptotically flat vacuum solution of Einstein's equations containing a compact event horizon which is non singular on or outside the event horizon must be Schwarzschild. If one drops the asymptotic flatness conditions one obtains the entire class of static axisymmetric black holes immersed in an external gravitational field. That is all Weyl solutions with compact Killing horizons. These could be used to calculate tidal effects due to static exterior sources such as a rigid, non rotating frame. They could also be used to calculate tidal effects due to stars in a cluster in which the rotational effects averaged to zero and there was no total angular momentum. Until now the only such known solution has been that of Erez and Rosen [9].

Having established the condition for a Weyl solution to contain a black hole one may then demonstrate that there are no non singular Weyl solutions containing two or more isolated black holes, in equilibrium, a result which has been obtained independently by Muller zum Hagen and Seifert [10]. Similar results hold for black holes and massive bodies, provided they do not surround or partially surround the black hole. It is also possible to obtain a solution of Einstein's equations containing a black hole being chased by a particle or body of negative mass both undergoing acceleration or indeed by a "Schwarzschild particle" with negative mass. These solutions are similar to the so-called "C metric" discovered by Levi-Civita and discussed at length by Kinnersley and Walker [11] but differ from the C metric which contains "nodal two surface" which intersects the black hole surface and upon which the manifold ceases to be differentiable, in that the singularities are found only at the site of the particles. It is thus rather analogous to the solution first found by Bondi [1] and discussed in more detail by Bonnor and Swammarayan [5] and Bicak [12].

The plan of this paper is as follows: Section 2 contains a discussion of the coordinates used and the boundary conditions imposed; Section 3 contains a discussion of the field equations and uses them to convert the condition of "elementary flatness" to a more useful form and derives some of the results. Section 4 discusses the uniformly accelerating solutions.

2. Coordinates and Boundary Conditions

A static axisymmetric vacuum spacetime, that is a vacuum spacetime containing two commuting hypersurface orthogonal Killing vectors $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$ which are timelike and spacelike respectively possesses a metric which may be cast in the form:

$$ds^2 = -e^{2u}\,dt^2 + e^{-2u}[e^{2k}(d\varrho^2 + dz^2) + \varrho^2\,d\phi^2] \tag{1}$$

where u and k are functions of ϱ and z only. If matter is present this will not be the most general metric possible but since we are mainly concerned with the vacuum case this need not bother us too much. t is to be regarded as a time coordinate and ϕ as an angular coordinate. This means that $t \in (-\infty, +\infty)$ whilst points with identical t, ϱ and z coordinates but ϕ coordinates differing by 2π are to be identified. The condition that the spacetime be asymptotically flat will be fulfilled if u and k tend to zero for large ϱ and z. For spacetimes containing one or more black holes on the axis, Carter has shown that (1) will be valid everywhere outside the black holes.

The metric becomes degenerate at $\varrho = 0$. If $\exp 2u \neq 0$ there we wish to interpret this as the axis. This requires that k tends to zero there (elementary flatness). If $\exp 2u = 0$ we wish to interpret this as being due to the presence of a black hole. This requires $\varrho = 0$ as well. This however is not sufficient; in order that there exists a non-singular extension Carter, in a careful examination of the boundary conditions for black holes has shown [8] that we must have

$$\exp 2u = \varrho^2 V_1 , \tag{2}$$

$$\exp 2k - 2u = (V_1 + \varrho^2 \Sigma_1)\kappa^{-2} \tag{3}$$

in a neighbourhood of the black holes surface where V_1 and Σ_1 are bounded positive functions of ϱ and z and κ is a constant independent of ϱ and z – the "surface gravity" of the black hole. In addition we must take care of the junction of the symmetry axis with the horizon. To do this it is convenient to introduce around each black hole a set of prolate spheriodal coordinates. This is possible because we may assume with no loss of generality that the black hole surface is topologically a sphere [13]. If the ith black hole has a horizon given by:

$$\varrho = 0; \quad z \in [z_i, z_i + 2c_i]$$

we write $z - z_i - c_i = \lambda_i \mu_i$ (no summation).

$$\varrho^2 = (\lambda_i^2 - c_i^2)(1 - \mu_i^2).$$

The boundary conditions become [8]

$$\exp 2u = (\lambda_i^2 - c_i^2)\, \hat{V}(\mu_i, \lambda_i), \tag{4}$$

$$\exp(2k - 2u) = (1 - \mu_i^2)\, \hat{X}(\mu_i, \lambda_i) \tag{5}$$

near $\lambda_i = c_i$ with $\hat{V}$ and $\hat{X}$ positive bounded functions of μ_i and λ_i. The remaining boundary conditions are those near a different sort of horizon which occurs in the uniformly accelerating solutions. We shall postpone discussion of that until Section 4.

3. The Fields Equations

The non-trivial field equations for a metric of the form (1) are:

$$\exp(2k - 4u) R_{ab} K^a K^b = \nabla^2 u \tag{6}$$

$$R_{\varrho z} = 2 \frac{\partial u}{\partial \varrho} \frac{\partial u}{\partial z} - \frac{1}{\varrho} \frac{\partial k}{\partial z}, \tag{7}$$

$$R_{zz} - R_{\varrho\varrho} = \frac{2}{\varrho} \frac{\partial k}{\partial \varrho} + 2\left(\frac{\partial u}{\partial z}\right)^2 - 2\left(\frac{\partial u}{\partial \varrho}\right)^2, \tag{8}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial \varrho^2} + \frac{1}{\varrho} \frac{\partial}{\partial \varrho} - \frac{\partial^2}{\partial z^2},$$

$$\frac{\partial}{\partial t} = K^a \frac{\partial}{\partial x^a}.$$

Note that (6) is just Poisson's equation in cylindrical coordinates. The structure of these equations in the vacuum case is well known: (6) is the local integrability condition for (7) and (8); so given a solution of (6) we may obtain k by quadrature provided that the resultant k is zero everywhere on the axis. If some of the region D of the ϱz plane is occupied by matter then this condition is equivalent to:

$$\int_D \left(\frac{\partial u}{\partial z} \nabla^2 u\right) \varrho \, d\varrho \, dz = 0 . \tag{9}$$

Bondi [1] has used Eq. (9) to show that no two massive bodies may be in equilibrium. Due to the linearity of Eq. (6) the integral in (9) is made up of contributions of two forms

1. The self force on a Newtonian distribution of density which must vanish by a straightforward use of the divergence theorem or by Newton's Third Law.

2. The classical attraction of one body on another. If $R_{ab}K^a K^b$ is negative (the weak energy condition [2]) this cannot vanish unless one body encloses or partially encloses another (see [14] for a more recent discussion of these results).

We shall now consider the case of a black hole. It is clear that because of (4) the Newtonian potential must diverge as $\lambda_i \to c_i$. This we shall now show happens in such a way that the black hole generates the same potential as a uniform rod with positive density.

Introducing prolate spheriodal coordinates around the ith hole Laplace's equation can be separated and the general solution with the correct angular dependence is

$$u = \sum_{n=0}^{n=\infty} \{A_n P_n(x) + B_n Q_n(x)\} \, P_n(\mu_i) \tag{10}$$

with $x = \lambda_i c_i^{-1}$ and where P_n and Q_n are Legendre functions of the first and second kind respectively.

If

$$\exp 2u = (\lambda_i^2 - c_i^2) \, \hat{V}(\lambda_i, \mu_i) \tag{11}$$

near $\lambda_i = c_i$

$$u = \tfrac{1}{2} \log(\lambda_i - c_i) + \tfrac{1}{2} \log(\lambda_i + c_i) + \tfrac{1}{2} \log \hat{V} \tag{12}$$

the only permissible solutions are:

$$u = \sum_{n=0}^{n=\infty} A_n P_n(x) \, P_n(\mu_i) + \tfrac{1}{2} \log\left(\frac{\lambda_i - c_i}{\lambda_i + c_i}\right) . \tag{13}$$

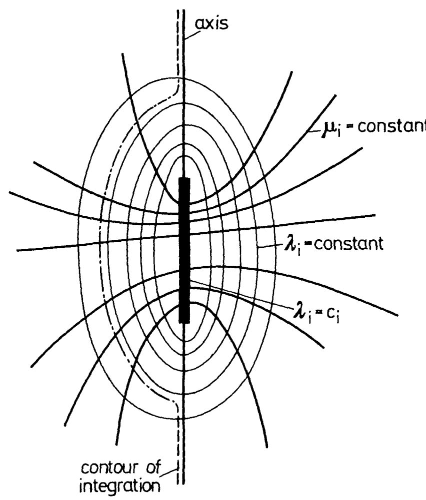

Fig. 1

If we require the solution to the asymptotically flat with no sources outside the black hole, we must have $A_n = 0$. k may now be obtained by quadratures to yield the Schwarzschild solution. If we drop the asymptotic flatness condition we obtain the generalized Erez-Rosen solutions mentioned in the introduction. However we must still check that k is zero on the axis. Eqs. (6), (7), and (8) will ensure that if k is zero on any point on the axis it will remain so on any other point on the axis which can be joined by a curve lying in the axis and such that any other curve joining the points can be shrunk onto the axis. If a black hole is present however k must be found by a line integral which must leave the axis (see Fig. 1) and returns to the axis. If this is zero in the limit that the contour is shrunk onto the line $z_i - c_i \leqq z \leqq z_i + c_i$ then it will be zero for any other such contour by Eqs. (6), (7), and (8) and a simple use of Stokes theorem.

Thus a necessary condition for k to be zero on the axis is that the integral of dk along a line of constant λ should vanish in the limit that $\lambda \to c_i$.

We may write

$$u = W + \tfrac{1}{2} \log \frac{(\lambda_i - c_i)}{(\lambda_i + c_i)}, \tag{14}$$

and set

$$\lambda_i = c_i \cosh \beta, \tag{15}$$

$$\mu_i = \cos \alpha. \tag{16}$$

Then a simple manipulation yields:

$$\frac{\partial k}{\partial \alpha} = -\frac{\cos\alpha \sin\alpha}{4(\cosh^2\beta - \cos^2\alpha)} + \frac{\partial W}{\partial \alpha} \frac{\sin^2\alpha \cosh\beta}{\cosh^2\beta - \cos^2\alpha} + O(\sinh\beta) \quad (17)$$

and

$$\underset{\beta\to 0}{Lt} \int_0^\pi \frac{\partial k}{\partial \alpha} d\alpha = \underset{\beta\to 0}{Lt} \int_0^\pi \frac{\partial W}{\partial \alpha} d\alpha . \quad (18)$$

Thus the necessary condition on W is that it should be the same on the north and south poles of the black hole. This is equivalent to requiring that the "rod" should experience no net force due to the "external" potential W. The limiting process is necessary because k diverges on the surface of the black hole.

It is well known and may easily be checked by similar methods to the above that if u contains a contributions due to a "point particle" (which by itself would generate a Curzon solution) then the part of u not due to the particle must have zero derivative at the particle. That is if

$$u = W - \frac{m}{\sqrt{(\varrho^2 + z^2)}}$$

then

$$\frac{\partial W}{\partial z} = 0 \quad \text{at} \quad \varrho = z = 0 .$$

Thus the question of whether there exist asymptotically flat configurations in which black holes, massive bodies and "particles" in equilibrium depend on whether there exist corresponding configurations of rods, bodies with a different but positive density and particles in Newtonian theory. Thus two black holes cannot be in equilibrium since two rods cannot remain in equilibrium. In the case of massive bodies we require that one body does not surround or partially surround another body or black holes (as for instance a ring of matter encircling the equator of a black hole). However this condition which is clear enough in Newtonian theory is rather obscure in General Relativity, fortunately it does not occur in the case of two or more black holes.

4. Uniform Acceleration

Bondi [1] has pointed out that

$$u = u_1 = \tfrac{1}{2} \log\{[\varrho^2 + z^2]^{\frac{1}{2}} + z\} , \quad (19)$$

$$k = k_1 = \tfrac{1}{2} \log\{\tfrac{1}{2} + \tfrac{1}{2} z[\varrho^2 + z^2]^{-\frac{1}{2}}\} \quad (20)$$

is a solution of (6), (7), and (8) everywhere except on the negative part of the axis, and that the transformation

$$\begin{aligned}
T &= \{[z^2+\varrho^2]^{\frac{1}{2}}+z\} \sinh t\,, \\
Z &= \{[z^2+\varrho^2]^{\frac{1}{2}}+z\} \cosh t\,, \\
X &= \varrho\{[z^2+\varrho^2]^{\frac{1}{2}}+z\}^{-\frac{1}{2}} \cos\phi\,, \\
Y &= \varrho\{[z^2+\varrho^2]^{\frac{1}{2}}+z\}^{-\frac{1}{2}} \sin\phi\,; \\
t &= \tanh^{-1}\frac{T}{Z}\,, \\
z &= \tfrac{1}{2}\{Z^2-T^2-X^2-Y^2\}\,, \\
\varrho &= (Z^2-T^2)^{\frac{1}{2}}(X^2+Y^2)^{\frac{1}{2}}\,, \\
\phi &= \tan^{-1}\frac{Y}{X}
\end{aligned}$$

maps this isometrically onto the portion of Minkowskian space given by $Z \geqq |T|$. The negative z axis mapping onto the null surface $Z=|T|$ where (T, Z, X, Y) are standard coordinates in Minkowski space. In effect one has chosen to adapt a Weyl coordinate system in flat space to the Killing vector

$$\frac{\partial}{\partial t} = Z\frac{\partial}{\partial T} + T\frac{\partial}{\partial Z}$$

which generates Lorentz transformations in the Z, T plane. The curves $z, \varrho, \phi = \text{const}$ move in Minkowski space with uniform acceleration

$$\{[z^2+\varrho^2]^{\frac{1}{2}}+z\}^{-\frac{1}{2}} = (Z^2-T^2)^{-\frac{1}{2}}\,.$$

Bondi pointed out that if to u_1 are added other solutions of (6) corresponding to particles or bodies with both positive and negative mass one would be able to satisfy all the previous boundary conditions and he further showed that the apparent singularities on the negative axis were just coordinate singularities. Subsequently Bonnor and Swammarayan [5] displayed a number of such solutions explicitly and Bicak [12] has examined their radiative properties. It is fairly clear that very much the same thing can be done for black holes. For definiteness we chose to construct (up to a line integral which we are unable to evaluate explicitly) a solution containing a black hole chased by a negative mass particle.

We are trying to find a solution of the form

$$u = u_1 + u_2 + u_3$$

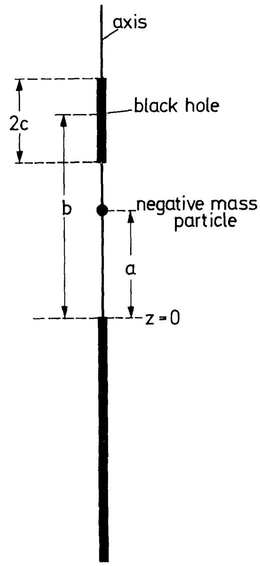

Fig. 2

where u_1 is given by (18)

$$u_2 = + m\{\varrho^2 + (z-a)^2\}^{-\frac{1}{2}},$$

$$u_3 = \tfrac{1}{2}\log\left(\frac{\lambda - c}{\lambda + c}\right)$$

with

$$z - b = \lambda\mu\,;\ \varrho^2 = (\lambda^2 - c^2)(1-\mu^2)\,.$$

That is the acceleration field; a particle of negative mass centred at $\varrho = 0$; $z = a$, and a rod of length $2c$ centred at $z = b$ (see Fig. 2). The acceleration field u_1 contributes a field which is attractive towards $z = 0$. $u_1 + u_2$, $u_1 + u_3$ depend on z along the positive z axis as

$$u_1 + u_2 = \tfrac{1}{2}\log 2z + \frac{m}{|z-a|}, \tag{21}$$

$$u_1 + u_3 = \tfrac{1}{2}\log 2z + \tfrac{1}{2}\log\left(\frac{z-b-c}{z-b+c}\right) \tag{22}$$

now $u_1 + u_2 \to \infty$ as $z \to a$ and $z \to \infty$ and is finite in between therefore it is always possible to find two points (and in fact no more than two) with z differing by $2c$ at which $u_1 + u_2$ is identical. This determines b and leaves m, c, a arbitrary.

Further $u_1 + u_3 \to -\infty$ at $z=0$ and at $z=b-c$ and is finite between, thus there exists at least one (in fact only one) maximum between these points. This determines a. Thus we have a 2 parameter family of solutions, the parameters may be chosen as the mass of the black hole and the mass of the particle. Furthermore it is easy to prove that the conditions cannot be fulfilled if the black hole and the negative mass particle have their positions reversed because there it is necessary to find points between $z=0$ and $z=a$ such that u_1+u_2 takes on equal values between them. However u_1+u_2 tends to $-\infty$ at $z=0$ and $+\infty$ at $z=a$ and is monotonic increasing in between. It remains to consider the extension through the negative z axis.

The conditions for there to exist an extension through this null surface have been given by Bondi [1] and amount to requiring that u_2+u_3 be bounded there. If they are analytic an analytic extension will exist. This is the case being considered. Thus the solution contains two Killing horizons, that due to the black hole and another one which arises from the use of accelerating coordinates – it will not be an event horizon however. The same behaviour occurs in the case of the "C metric" [14].

5. Discussion

What this paper has sought to show is that within the limitation imposed, black holes in General Relativity behave in much the same way as ordinary bodies in gravitational fields and very much as one expects on the basis of Newtonian Theory. It would be of interest to extend these results to the stationary case and thus include the spin-spin effects that result. However although it is possible to repeat the calculations which lead to the equilibrium conditions, because Eq. (6) becomes non-linear it has not been possible to provide a proof that these cannot be fulfilled in the case say of two spinning black holes aligned along an axis.

It is also interesting to note that if the condition that there exists a non-singular axis between the black holes is relaxed solutions are possible, the resulting singularities are often thought of as struts. This interpretation would not be feasible in the case of black holes since the calculation in [15], replacing "rope" by "strut" shows that the stress in a strut keeping two black holes apart would need to be infinite.

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References

1. Bondi, H.: Rev. Mod. Phys. **29**, 423 (1957)
2. Hawking, S. W., Penrose, R.: Proc. Roy. Soc. A **314**, 529 (1970)
3. Siberstein, L.: Phys. Rev. **49**, 268 (1936)

4. Einstein, A., Rosen, N.: Phys. Rev. **49**, 404 (1936)
5. Bonnor, W. B., Swammarayan, N. S.: Z. Physik **177**, 240 (1964)
6. Curzon, H. E.: J. Proc. London Math. Soc. **23**, 29 (1924)
   Curzon, H. E.: J. Proc. London Math. Soc. **23**, 477 (1924)
7. Gautreau, R.: Phys. Letters **28 A**, 606 (1969)
8. Carter, B.: In Black holes, Ed. DeWitt, B. S., DeWitt, C. M. New York: Gordon and Breach, to be published 1973
9. Erez, G., Rosen, N.: Bull. Res. Council of Israel, **8 E** 47, (1959)
10. Müller zum Hagen, H., Seifert, H. J.: preprint (1973)
11. Kinnersley, W., Walker, M.: Phys. Rev. D **2**, 1359 (1970)
12. Bicak, J.: Proc. Roy. Soc. **A 302**, 201) (1968)
13. Hawking, S. W.: Commun. math. Phys. **25**, 152 (1972)
14. Müller zum Hagen, H.: preprint (1973) The Static Two Body Problem
15. Gibbons, G. W.: Nat. Phys. Sci. **240**, 77 (1972)

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