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Analyticity and Uniqueness for Spin $\frac{1}{2}$ Classical Ferromagnetic Lattice Systems at Low Temperatures

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Abstract. Analyticity and uniqueness of correlation functions is investigated for a number of systems by application of Ruelle's theorem on zeros of Asano contracted polynomials to the partition function. To answer the question when the partition function of a system is the Asano contraction of those of subsystems the groups appearing in the low and high temperature expansions are employed.

Introduction

In a series of papers that appeared recently ([7, 11, 12], and references given there) a technique generalizing that of Lee-Yang was applied to find regions free of zeros of the partition function, and to deduce analyticity properties of the pressure. The results are obtained mainly for systems with two-body interactions.

The technique consists in decomposing the system in a finite volume into subsystems with uniformly bounded size, when the volume tends to infinity, and in using theorems relating the zeros of the system to those of subsystems. Systematic development of these ideas allows us here to prove analyticity and uniqueness properties of the pressure and the correlation functions of quite general systems.

In applying this method one has to solve two problems. First, one has to show that the partition function of the system can be obtained from those of subsystems by the operation called Asano contraction. Second, suitable information about the zeros of the partition functions of subsystems should be available.

We give a solution of the first problem in terms of groups that appear in the low and high temperature expansions [3, 4, 6, 14]; these groups were studied recently in connection with the duality [6, 14]. In the low temperature region, in which we are interested here, and for ferro-

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magnetic interactions, a solution of the second problem comes from the following observation:

If the partition function for a system with energy

$$U = - \sum_{B \in \mathscr{B}} J(B) \sigma_B, \qquad J(B) > 0$$

is expressed in terms of variables

$$z_B = e^{-2\beta J(B)}$$

then, up to a harmless factor, one obtains a polynomial

$$M(z_{\mathscr{B}})$$

with free term equal to 1. Therefore, if z_B are small enough, and this is the case for positive $J(B)$ and low temperature, $M(z_{\mathscr{B}})$ is not zero.

Uniqueness and analyticity of the (magnetic) correlation functions $\varrho(\sigma_A)$ are obtained by showing that the modified pressure with $\beta \sum_{B \in \mathscr{B}} J(B) \sigma_B$ replaced by $\beta \sum_{B \in \mathscr{B}} J(B) \sigma_{\mathscr{B}} + \lambda \sum_{A' \in \mathscr{A}} \sigma_{A'}$, where $\mathscr{A}$ is the set of translates of A, is a function of λ analytic at 0. This is true only if σ_A is a product of the σ_B's from U, and follows from the fact that in $M(z_{\mathscr{B} \cup \mathscr{A}})$ the variables $z_{A'}$ appear only in product with z_B and therefore M does not vanish if $z_{A'}$ are close to 1 and z_B are small. We now describe the content of the paper in more details.

Section 1 introduces the notation and the groups of the low temperature expansion. In Sections 1.1 to 1.4 we give the "magnetic" formulation of results proved usually in the lattice gas language. The low temperature expansion and the groups are introduced in Section 1.5; we follow here [6]. In Sections 1.6–1.8 we give additional information about these groups. In Section 1.9 we show that the pressure is, up to a non interesting term, equal to its reduced (with respect to ground state for ferromagnetic interaction) value. The polynomial M corresponds to the partition function for the reduced pressure, and the proof that it is non-vanishing for small arguments can be viewed as a perturbation argument, around the ground state.

Section 2.1 contains the theorem by Ruelle [11] on zeros of polynomials contracted according to Asano. In Section 2.2, we show that the polynomial M of a system is the Asano contraction of those of subsystems if a group of Section 1.5 corresponding to the system is generated by the subgroups corresponding to subsystems. This allows easily (Section 2.3) take into account the modifications of the partition function needed in discussion of the correlation functions, and to prove (Theorem 2.4) that the absence of zeros of M in suitable region implies the uniqueness and analyticity of the correlation functions. That the assumptions of Theorem 2.4 are satisfied for quite a general class of interactions, cf. Section 2.5 for a list, is shown later. It seems that a theorem of this type should hold for all finite range interactions.

Theorem 2.4 yields a result about the entropy. Namely, it is shown (Proposition 2.5) that for ferromagnetic systems and for low enough temperatures, the entropy is the same for all translation invariant equilibrium states and tends to zero with the temperature (the proof depends in fact only on the analyticity properties of the pressure). This can be compared with the situation for antiferromagnetic interactions where examples are known, [16], with non zero residual entropy. One of the differences between ferromagnetic and antiferromagnetic interactions is in the number of ground states: in the first case it is given by the cardinality of the symmetry group $\mathscr{S}_\Lambda$ (Section 1.5) and, as proved in Section 1.8,

$$\frac{1}{|\Lambda|}\log|\mathscr{S}_\Lambda|_{\Lambda\to\infty}\to 0$$

whereas in the antiferromagnetic case ground states are not related to the symmetry group. In the quantum case even for ferromagnetic interactions the residual entropy is in general non vanishing (cf. [15]).

In Section 3 we consider even interactions with Ising type bonds contained in the set of all bonds. This is a special case of essentially regular systems considered in later sections but because of the presence of the Ising type bonds considerable simplifications occur. We show that Theorem 2.4 applies here, and that for such systems at most two ergodic equilibrium states are possible for low temperatures, i.e. each translation invariant equilibrium state is a combination of the state ϱ^+ defined by the + boundary conditions and the state obtained from ϱ^+ by flipping all spins. The last statement admits a generalization to a description of all the translation invariant equilibrium states of systems to which Theorem 2.4 applies [13]. As a special case we reobtain here by a different method a result of [2]. In Section 3.4, we discuss a generalization to partially antiferromagnetic systems which is close to but weaker for two-body interaction than [1]. Section 3.5 deals with systems called here trivial. We show in [13] that for such systems the invariant equilibrium state is unique.

In Sections 4.1–4.5 we show that the assumptions of Theorem 2.4 are fulfilled if the bonds satisfy a regularity condition (Definition 4.2). This condition is weakened in Section 4.6 and in the next sections. The last section contains a series of remarks.

1. Translation Invariant Equilibrium States

1.1. For *spin* $\frac{1}{2}$ *classical systems* on the ν-dimensional lattice $\mathbb{Z}^\nu$ a configuration is defined by specifying for each lattice site one of two possible configurations which are identified here with 1 and -1 [1]. Therefore in what follows a *configuration* is a function $X : \mathbb{Z}^\nu \to \{1, -1\}$. The set of all configurations

$$\mathscr{X} = \{1, -1\}^{\mathbb{Z}^\nu}$$

is made into a compact separable space by the product topology. A state of the algebra $\mathscr{C}(\mathscr{X})$ of continuous functions on $\mathscr{X}$ or, what is the same, a probability measure on $\mathscr{X}$ will be called a *state* of our system. For $i \in \mathbb{Z}^\nu$ we let σ_i denote the function on $\mathscr{X}$ which to each configuration assigns its value at the lattice site i, and for a finite subset A of the lattice we put

$$\sigma_A = \prod_{i \in A} \sigma_i .$$

The linear span of $\{\sigma_A\}_{A \in \mathscr{P}_f(\mathbb{Z}^\nu)}$ [2] is dense in $\mathscr{C}(\mathscr{X})$. Therefore in order to specify the state ϱ it is enough to give $\{\varrho(\sigma_A)\}_{A \in \mathscr{P}_f(\mathbb{Z}^\nu)}$.

For a subset Λ of the lattice we let $\mathscr{X}_\Lambda$ denote the configurations in Λ:

$$\mathscr{X}_\Lambda = \{1, -1\}^\Lambda$$

and we do not distinguish between σ_A, $A \subset \Lambda$, and the corresponding function on $\mathscr{X}_\Lambda$.

1.2. An *interaction* will be identified with a real valued function on the set of finite subsets of the lattice. The elements of the support of an interaction are called *bonds*, i.e. for the interaction J the set of bonds is

$$\mathscr{B} = \{B \in \mathscr{P}_f(\mathbb{Z}^\nu) : J(B) \neq 0\} .$$

The *energy of the subsystem* in a finite volume Λ is

$$U_\Lambda = - \sum_{B \in \mathscr{B}_\Lambda} J(B)\, \sigma_B ;$$

here $\mathscr{B}_\Lambda = \{B \in \mathscr{B} : B \subset \Lambda\}$. The *partition function* for the subsystem in Λ is given by:

$$Z_\Lambda = \sum_{X \in \mathscr{X}_\Lambda} e^{-\beta U_\Lambda(X)}$$

and the *Gibbs state* in Λ ascribes to the configuration $X \in \mathscr{X}_\Lambda$ probability

$$\varrho_\Lambda(X) = Z_\Lambda^{-1}\, e^{-\beta U_\Lambda(X)} .$$

Putting, according to tradition,

$$K(B) = \beta J(B)$$

[1] The configurations are also called "up" and "down", or "+" and "−", or "0" and "1".

[2] For a set Λ, $\mathscr{P}_f(\Lambda)$ denotes the family of all finite subsets of Λ.

we rewrite:

$$Z_\Lambda = \sum_{X \in \mathscr{X}_\Lambda} e^{\sum_{B \in \mathscr{B}_\Lambda} K(B)\,\sigma_B(X)},$$
$$\varrho_\Lambda(X) = Z_\Lambda^{-1}\, e^{\sum_{B \in \mathscr{B}_\Lambda} K(B)\,\sigma_B(X)} \tag{1.1}$$

In what follows we consider only *translation invariant* systems with a *finite range* interaction (see, however, Section 4.8 d). This means that for a bond B and $a \in \mathbb{Z}^\nu$ the translate $\tau_a(B)$, denoted also by $B + a$, is again a bond and $K(\tau_a(B)) = K(B)$. Moreover, each bond is congruent with an element of a certain finite family of bonds. Minimal families with the property that any bond is congruent to an element of the family will be called *fundamental.*

Translation invariant finite range interactions form a linear subspace of the space of all interactions.

1.3. For a translation invariant finite range interaction the *pressure*

$$p(K) = \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \log Z_\Lambda$$

exists and is a convex function of K. Here Λ is allowed to tend to infinity in the sense of Van Hove (cf. [8], p. 14); for instance through a sequence of parallelepipeds with the length of all the edges tending to infinity.

For any translation invariant state the *mean energy*

$$\lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \varrho(U_\Lambda)$$

is defined, and, as is easy to see, is equal to

$$-\varrho\Big(\sum_{B \in \mathscr{B}_0} K(B)\,\sigma_B\Big)$$

where $\mathscr{B}_0$ is any fundamental family of bonds. We adopt here the definition that a *(translation invariant) equilibrium state* is a translation invariant state maximizing the difference between the entropy and the mean energy. This maximum is equal to the pressure. In other words, ϱ is an equilibrium state if

$$p(K) = s(\varrho) + \varrho\Big(\sum_{B \in \mathscr{B}_0} K(B)\,\sigma_B\Big). \tag{1.2}$$ [3]

We let Δ_K denote the set of all equilibrium states corresponding to an interaction K.

1.4. **Proposition.** *Let for a finite subset A of the lattice the interaction K_A be defined by*

$$K_A(B) = \begin{cases} 1 & \text{if } B \text{ is a translate of } A \\ 0 & \text{otherwise}. \end{cases}$$

[3] We refer to [5, 9, 18] for a discussion of the notion of an equilibrium state.

If the function $\lambda \mapsto p(K+\lambda K_A)$ *is differentiable at zero then* $\varrho(\sigma_A)$ *has the same value for all* $\varrho \in \Delta_K$ *and*

$$\varrho(\sigma_A) = \frac{d}{d\lambda} p(K+\lambda K_A)|_{\lambda=0}\,. \tag{1.3}$$

[In fact also the converse statement is true: in general $\varrho(\sigma_A)$, $\varrho \in \Delta_K$, fill out the interval between the left and right derivatives of $\lambda \mapsto p(K+\lambda K_A)$ at zero.]

The proposition follows directly from formula (4) of [10] and well known properties of convex functions.

1.5. To introduce the low temperature expansion we start with a finite set Λ, an interaction K on Λ with the family $\mathscr{B}$ of bonds, and

$$Z = \sum_{X \in \mathscr{X}} e^{\sum_{B \in \mathscr{B}} K(B)\,\sigma_B(X)} \quad \text{where} \quad \mathscr{X} = \{1, -1\}^{\Lambda}\,.$$

We introduce now following [4, 3, 14, 6] several abelian groups.

The *group structure in* $\mathscr{X}$ is defined by considering it as the product

$$\{1, -1\} \times \cdots \times \{1, -1\} \qquad (|\Lambda|\text{-times})\,,$$

$\{1, -1\}$ being equipped with the group operation for which 1 is the unity. For any set, say Λ, we consider the family of all its subsets, $\mathscr{P}(\Lambda)$, as an abelian group, the group operation being the symmetric difference

$$A, B \mapsto (A \setminus B) \cup (B \setminus A)\,.$$

The elements of $\mathscr{P}(\mathscr{B})$ will be denoted by $\alpha, \beta, \ldots$, and the elements of $\mathscr{P}(\Lambda)$ by $A, B, \ldots$. The group operation in $\mathscr{X}$ and $\mathscr{P}(\mathscr{B})$ will be written additively, and in $\mathscr{P}(\Lambda)$ it will be denoted by juxtaposition.

The mapping

$$A \mapsto \sigma_A$$

defines an *isomorphism of* $\mathscr{P}(\Lambda)$ *onto the group dual to* $\mathscr{X}$.

The inclusion mapping $\mathscr{B} \to \mathscr{P}(\Lambda)$ *extends uniquely to a homomorphism* $\mathscr{P}(\mathscr{B}) \to \mathscr{P}(\Lambda)$ denoted by

$$\beta \mapsto \bar{\beta}\,.$$

For $X \in \mathscr{X}$ let

$$\gamma(X) = \{B \in \mathscr{B} : \sigma_B(X) = -1\}\,;$$

$X \mapsto \gamma(X)$ *is a homomorphism of* $\mathscr{X}$ *into* $\mathscr{P}(\mathscr{B})$. *The image of* $\mathscr{P}(\mathscr{B})$ *in* $\mathscr{P}(\Lambda)$ *is denoted by* $\bar{\mathscr{B}}$, *and the image of* γ *by* Γ. That $\beta \mapsto \bar{\beta}$ is a homomorphism is obvious; we will show that the same is true about γ. $B \in \gamma(X_1 + X_2)$ means that $\sigma_B(X_1+X_2) = -1$, that is $\sigma_B(X_1)\,\sigma_B(X_2) = -1$. The last equality means that one of the numbers $\sigma_B(X_1)$, $\sigma_B(X_2)$, but not both, is equal to -1 which is equivalent to $B \in \gamma(X_1) + \gamma(X_2)$.

In the case of the Ising model $\gamma(X)$ can be identified with the broken lines that are drawn in discussions of the spontaneous magnetization. For more examples we refer to [6].

Two further groups, $\mathcal{K}$ and $\mathcal{S}$, are defined as the kernels of the homomorphisms $\beta \mapsto \bar{\beta}$ and $X \mapsto \gamma(X)$, respectively. $\mathcal{S}$ is called the (internal) *symmetry group* of the system since it has the property:

$$U(X+Y)=U(X), \qquad X\in\mathcal{X}, \ Y\in\mathcal{S}.$$

It can also be defined as $\{X\in\mathcal{X} : \sigma_B(X)=1, \text{ all } B\in\mathcal{B}\}$. For the Ising model $\mathcal{S}$ contains just two elements: the zero of $\mathcal{X}$ and the configuration equal everywhere to -1.

We can now write down the *low temperature expansion* of the partition function:

$$Z=|\mathcal{S}| \prod_{B\in\mathcal{B}} e^{K(B)} \sum_{\beta\in\Gamma} \prod_{B\in\beta} e^{-2K(B)}. \tag{1.4}$$

It is obtained by the following transformations.

$$\sum_{X\in\mathcal{X}} e^{\sum_{B\in\mathcal{B}} K(B)\sigma_B(X)} = \sum_{X\in\mathcal{X}} e^{\sum_{B\in\mathcal{B}} K(B)(\sigma_B(X)-1)+\sum_{B\in\mathcal{B}} K(B)}$$

$$= \prod_{B\in\mathcal{B}} e^{K(B)} \sum_{X\in\mathcal{X}} e^{\sum_{B\in\mathcal{B}} K(B)(\sigma_B(X)-1)} = \prod_{B\in\mathcal{B}} e^{K(B)} \left[|\mathcal{S}| \sum_{\beta\in\Gamma} e^{-2\sum_{B\in\beta} K(B)}\right].$$

The last equality is a consequence of the fact that $\sum_{B\in\mathcal{B}} K(B)(\sigma_B(X)-1) = -2\sum_{B\in\gamma(X)} K(B)$ [by the definition of $\gamma(X)$].

We introduce the pressure

$$p=\frac{1}{|\Lambda|}\log Z$$

and the *reduced pressure*

$$p^0=\frac{1}{|\Lambda|}\log \sum_{\beta\in\Gamma} \prod_{B\in\beta} e^{-2K(B)}$$

so that

$$p=\frac{1}{|\Lambda|}\log|\mathcal{S}|+\frac{1}{|\Lambda|}\sum_{B\in\mathcal{B}} K(B)+p^0. \tag{1.5}$$

The inequality (2.15) of [8] gives

$$|p(K_1)-p(K_2)| \leqq \frac{1}{|\Lambda|}\sum_{B\subset\Lambda} |K_1(B)-K_2(B)|$$

and this yields

$$|p^0(K_1)-p^0(K_2)| \leqq \frac{1}{|\Lambda|}\big|\log|\mathcal{S}_1|-\log|\mathcal{S}_2|\big| + \frac{2}{|\Lambda|}\sum_{B\subset\Lambda} |K_1(B)-K_2(B)|. \tag{1.6}$$

Putting here $K_2=0$ we get

$$|p^0(K)| \leqq \frac{1}{|\Lambda|}\big|\log|\mathcal{S}|-\log 2^{|\Lambda|}\big| + \frac{2}{|\Lambda|}\sum_{B\in\mathcal{B}} |K(B)|. \tag{1.7}$$

1.6. We proceed with a discussion of the structure introduced in the preceding section. For $\alpha, \beta \in \mathscr{P}(\mathscr{B})$ we define

$$\langle \alpha, \beta \rangle = (-1)^{|\alpha \cap \beta|}$$

$\alpha, \beta \mapsto \langle \alpha, \beta \rangle$ is a bicharacter of $\mathscr{P}(\mathscr{B})$, i.e. when one of the arguments is fixed it defines a character. It is non-degenerate in the sense that $\langle \alpha, \beta \rangle = 1$ for all $\alpha \in \mathscr{P}(\mathscr{B})$ implies that β is zero.

Lemma. *Let for a subgroup G of* $\mathscr{P}(\mathscr{B})$

$$G^{\perp} = \{\alpha \in \mathscr{P}(\mathscr{B}) : \langle \alpha, \beta \rangle = 1 \quad \textit{for all} \quad \beta \in G\} .$$

Then $\Gamma = \mathscr{K}^{\perp}$ *and* $\mathscr{K} = \Gamma^{\perp}$.

For a proof we remark first that

$$\langle \beta, \gamma(X) \rangle = \sigma_{\bar{\beta}}(X), \qquad \beta \in \mathscr{P}(\mathscr{B}), \ X \in \mathscr{X} .$$

For $\beta = \{B\}$ this follows directly from the definitions of γ and $\langle , \rangle$, and then for general β from the fact that when X is fixed both sides of the equality define a character of $\mathscr{P}(\mathscr{B})$. It follows from this formula that $\mathscr{K} = \Gamma^{\perp}$. For, $\langle \beta, \gamma(X) \rangle = 1$, all $X \in \mathscr{X}$, is equivalent to $\sigma_{\bar{\beta}}(X) = 1$, all X, and this implies that the set $\bar{\beta}$ is empty.

That $\Gamma = \mathscr{K}^{\perp}$ follows from $\mathscr{K} = \Gamma^{\perp}$ and the nondegenerateness of $\langle , \rangle$ by the duality theory of finite abelian groups, or, if the language of vector-spaces over $\mathbb{Z}_2$ is introduced, by the reflexivity of the orthogonality relation for subspaces, with respect to non-degenerate symmetric form.

1.7. We will later need information on the change of $\mathscr{K}$ when $\mathscr{B}$ is enlarged by adding elements of $\bar{\mathscr{B}}$.

Suppose that $A_1, \ldots, A_n \in \bar{\mathscr{B}}$ *and put* $\mathscr{B}' = \mathscr{B} \cup \{A_1, \ldots, A_n\}$, $\mathscr{K}'$ *the kernel of* $\mathscr{P}(\mathscr{B}') \to \mathscr{P}(\Lambda)$. *Let* $\alpha_1, \ldots, \alpha_n \in \mathscr{P}(\mathscr{B})$ *be such that* $\bar{\alpha}_i = A_i$ *and denote:* $\alpha'_i = \alpha_i \cup \{A_i\}$. *Then* $\mathscr{K}'$ *is generated by* $\mathscr{K} \cup \{\alpha'_1, \ldots, \alpha'_n\}$.

For $\beta \in \mathscr{K}'$ let $\{A_{i_1}, \ldots, A_{i_r}\} = \beta \cap \{A_1, \ldots, A_n\}$. Then $\beta + \alpha'_{i_1} + \cdots + \alpha'_{i_r} \in \mathscr{K}$ and $\beta = (\beta + \alpha'_{i_1} + \cdots + \alpha'_{i_r}) + (\alpha'_{i_1} + \cdots + \alpha'_{i_r})$ is the needed decomposition.

1.8. Systems for which $\mathscr{K}$ has only one element: the empty subset of $\mathscr{B}$, will be called here *trivial.*

Let B_0 be a finite subset of $\mathbb{Z}^{\nu}$, let Λ be a parallelepiped and let $\mathscr{B}$ be the set of all the translates of B_0 that are contained in Λ. We will show now that the system so defined is trivial.

Let us introduce in $\mathbb{Z}^{\nu}$ the lexicographic order, i.e. $a < b$ if the first nonzero component of $b - a$ is positive. For $B \in \mathscr{B}$ we denote by $b(B)$ the earliest point of B: if $B_1 \neq B_2$ then $b(B_1) \neq b(B_2)$. If $\beta = \{B_1, \ldots, B_n\} \in \mathscr{P}(\mathscr{B})$ then the earliest point in $\{b(B_1), \ldots, b(B_n)\}$ belongs to only one element of β, and therefore belongs also to $\bar{\beta}$. This shows the triviality of $\mathscr{K}$.

1.9. Returning to infinite systems we let Λ to be a finite subset of the lattice and we identify $\mathscr{B}$ of Section 1.5 with the set $\mathscr{B}_\Lambda$ of bonds contained in Λ. We will now show that $\lim_\Lambda p^0_\Lambda$, *denoted further by* p^0, *and called: reduced pressure, exists and*

$$p(K) = \sum_{B \in \mathscr{B}_0} K(B) + p^0(K)\,; \tag{1.8}$$

here $\mathscr{B}_0$ *is any fundamental family of bonds.*

From (1.5)

$$\frac{1}{|\Lambda|}\log Z_\Lambda = \frac{1}{|\Lambda|}\log|\mathscr{S}_\Lambda| + \frac{1}{|\Lambda|}\sum_{B \in \mathscr{B}_\Lambda} K(B) + p^0_\Lambda\,.$$

$\sum_{B \in \mathscr{B}_\Lambda} K(B)$ can be computed by summing first over the translates of a fixed element B_0 of $\mathscr{B}_0$, and then performing the summation over $\mathscr{B}_0$. It is not hard to see that first sum differs from $|\Lambda|\,K(B_0)$ by no more than $K(B_0)\cdot|\partial\Lambda|\cdot$(diameter of B_0). Hence

$$\lim_\Lambda \frac{1}{|\Lambda|}\sum_{B \in \mathscr{B}_\Lambda} K(B) = \sum_{B \in \mathscr{B}_0} K(B)\,.$$

To prove that $\lim_\Lambda \frac{1}{|\Lambda|}\log|\mathscr{S}_\Lambda| = 0$ we remark first that enlarging of $\mathscr{B}$ leads to smaller $\mathscr{S}_\Lambda$ as $\mathscr{S}_\Lambda = \{X \in \mathscr{X}_\Lambda : \sigma_B(X) = 1,\ \text{all}\ B \in \mathscr{B}_\Lambda\}$. Therefore it is enough to show that $\lim_\Lambda \frac{1}{|\Lambda|}\log|\mathscr{S}_\Lambda| = 0$ for systems whose bonds are all translates of one. For such systems $\mathscr{K}_\Lambda$ are trivial (Section 1.8) and therefore (Lemma 1.6) $\Gamma_\Lambda = \mathscr{P}(\mathscr{B}_\Lambda)$. Hence $\mathscr{S}_\Lambda$, being the kernel of a homomorphism of $\mathscr{X}_\Lambda$ onto $\mathscr{P}(\mathscr{B}_\Lambda)$, has the number of elements equal to $|\mathscr{X}_\Lambda|/\mathscr{P}(\mathscr{B}_\Lambda)$, i.e.

$$|\mathscr{S}_\Lambda| = 2^{|\Lambda| - |\mathscr{B}_\Lambda|}$$

and hence

$$\frac{1}{|\Lambda|}\log|\mathscr{S}_\Lambda| = \frac{|\Lambda| - |\mathscr{B}_\Lambda|}{|\Lambda|}\log 2\,.$$

As was remarked in discussing $\lim \frac{1}{|\Lambda|}\sum_{B \in \mathscr{B}_\Lambda} K(B)$, $\big||\Lambda| - |\mathscr{B}_\Lambda|\big|$ is majorized by $|\partial\Lambda|\cdot$(diameter of B). This shows that $\frac{1}{|\Lambda|}\log|\mathscr{S}_\Lambda| \to 0$ and finishes the proof of (1.8).

2.6. Analyticity and Uniqueness

2.1. Let P be a complex polynomial in several variables, which is of degree 1 with respect to each. That is there is a finite set $\mathscr{B}$ and

$$P(z_{\mathscr{B}}) = \sum_{\beta \subset \mathscr{B}} c_\beta z^\beta$$

where

$$z_{\mathscr{B}} = \{z_B\}_{B \in \mathscr{B}} \quad \text{and} \quad z^\beta = \prod_{B \in \beta} z_B .$$

Let $\mathscr{B} = \bigcup_i \mathscr{B}_i$ be a finite covering of $\mathscr{B}$ and let $P_i(z_{\mathscr{B}_i}) = \sum_{\beta \subset \mathscr{B}_i} c_{i,\beta} z^\beta$ be a family of polynomials. We say that $P(z_{\mathscr{B}})$ is the *Asano contraction* of $\{P_i(z_{\mathscr{B}_i})\}$ if

$$c_\beta = \prod_i c_{i,\beta \cap \mathscr{B}_i} . \tag{2.1}$$

We will say that the *variable* z_B *is undergoing contraction* if B belongs to more than one of $\{B_i\}$.

Theorem. (Ruelle [11]). *Let $P(z_{\mathscr{B}})$ be the Asano contraction of $\{P_i(z_{\mathscr{B}_i})\}$ and let for each $B \in \mathscr{B}_i$ a subset $R_{i,B}$ of the complex plane be given which is closed and does not contain* 0 *if z_B is undergoing contraction.*

Suppose that $P_i(z_{\mathscr{B}_i})$ is nonzero if $z_B \notin R_{i,B}$, all $B \in \mathscr{B}_i$. Then $P(z_{\mathscr{B}})$ does not vanish when $z_B \notin -\prod_i (-R_{i,B})$ for all $B \in \mathscr{B}$, here for a finite family $\{R_i\}_{i=1}^n$ of subsets of $\mathbb{C}$:

$$\prod_{i=1}^n R_i = \{z_1 \cdot \dots \cdot z_n : z_i \in R_i,\ i = 1, \dots, n\} .$$

(In [11] it is assumed that all $R_{i,B}$ are closed and do not contain zero but an examination of the proof shows that the theorem holds also in this form.)

2.2. We now place ourself in the situation of Section 1.5. We define a polynomial M by

$$M(z_{\mathscr{B}}) = \sum_{\beta \in \Gamma} z^\beta ; \tag{2.2}$$

$M(z_{\mathscr{B}})$ is determined by the family $\mathscr{B}$ of sets unambiguously.

For a finite covering $\{\mathscr{B}_i\}$ of $\mathscr{B}$ we consider the corresponding groups Γ_i, $\mathscr{K}_i$ defined by $\mathscr{B}_i$ and we introduce the polynomials $M(z_{\mathscr{B}_i})$. Obviously, $\Gamma_i = \{\beta \cap \mathscr{B}_i\}_{\beta \in \Gamma}$ and $\mathscr{K}_i = \mathscr{K} \cap \mathscr{P}(\mathscr{B}_i)$.

Proposition. *$M(z_{\mathscr{B}})$ is the Asano contraction of $\{M(z_{\mathscr{B}_i})\}$ if and only if the subgroup of $\mathscr{P}(\mathscr{B})$ generated by $\bigcup_i \mathscr{K}_i$ coincides with $\mathscr{K}$ (we write: $\left[\bigcup_i \mathscr{K}_i\right] = \mathscr{K}$).*

By (2.1) and (2.2) it has to be proved that the implication:

$$\beta \cap \mathscr{B}_i \in \Gamma_i, \quad \text{all} \quad i \Rightarrow \beta \in \Gamma$$

is equivalent to: $\mathscr{K} = \left[\bigcup_i \mathscr{K}_i\right]$. This equivalence follows from Lemma 1.6. For if $\mathscr{K} \neq \left[\bigcup_i \mathscr{K}_i\right]$ then there is an element of $\mathscr{P}(\mathscr{B})$, say β, which is $\langle , \rangle$-orthogonal to $\left[\bigcup_i \mathscr{K}_i\right]$ and not orthogonal to $\mathscr{K}$.

Then $\beta \cap \mathscr{B}_i \in \Gamma_i$, all i, by Lemma 1.6 as $\langle \beta \cap \mathscr{B}_i, \alpha \rangle = \langle \beta, \alpha \rangle$ for $\alpha \subset \mathscr{B}_i$, and $\beta \notin \Gamma$, again by Lemma 1.6. If, on the other hand, $\mathscr{K} = \left[\bigcup_i \mathscr{K}_i \right]$ then the fact that $\beta \cap \mathscr{B}_i \in \Gamma_i$ for all i, i.e. (Lemma 1.6) orthogonality of $\beta \cap \mathscr{B}_i$ to $\mathscr{K}_i$, implies orthogonality of β to $\mathscr{K}$ and therefore (Lemma 1.6) $\beta \in \Gamma$.

2.3. We consider the effect of an enlargement of $\mathscr{B}$ on the zeros of M, as in Section 1.7. We refer to that section for the notation.

Let us assume that $A_i \notin \mathscr{B}$, all i, and that α_i are minimal, i.e. no proper subset of α_i yields A_i.

If $M(z_{\mathscr{B}}) \neq 0$ when $|z_B| < r_B$, $B \in \bigcup_i \alpha_i$, and $z_B \notin R_B$ for $B \in \mathscr{B} \setminus \bigcup_i \alpha_i$ then for any $\eta \in [0, 1]$ $M(z_{\mathscr{B}'})$ does not vanish when $|z_{A_i} - 1| < \left(\frac{1-\eta}{1+\eta} \right)^{|\alpha_i|}$, $|z_B| < \eta^{n_B} r_B$ for $B \in \bigcup_i \alpha_i$ and $z_B \notin R_B$ for $B \in \mathscr{B} \setminus \bigcup_i \alpha_i$. Here n_B is the number of α_i's containing the bond B.

By Section 1.7, and Proposition 2.4 $M(z_{\mathscr{B}'})$ is the Asano contraction of $\{M(z_{\mathscr{B}}), M(z_{\alpha_i})\}$. By Theorem 2.1 it is enough to show that $M(z_{\alpha_i}) \neq 0$ if $|z_{A_i} - 1| < \left(\frac{1-\eta}{1+\eta} \right)^{|\alpha_i|}$, $|z_B| < \eta$, all $B \in \alpha_i$. We omit the index i at α_i in the following.

By the minimality of α, $\mathscr{K}(\alpha')$ consists of only two elements: the empty subset of α' and α'. It follows from Lemma 1.6 that $\Gamma(\alpha')$ is the family of all even subsets of α'. Hence

$$M(z_{\alpha'}) = \sum_{|\beta| \text{ even}} z^\beta .$$

The right-hand side is equal to

$$\frac{1}{2} \left[\prod_{B \in \alpha'} (1 + z_B) + \prod_{B \in \alpha'} (1 - z_B) \right].$$

Therefore if $z_A \neq -1$ and $z_B \neq 1$, $B \in \alpha$, $M(z_{\alpha'}) = 0$ implies

$$\prod_{B \in \alpha} \frac{1 + z_B}{1 - z_B} = - \frac{1 - z_A}{1 + z_A}.$$

If $|z_A - 1| < \theta$, $0 \leqq \theta \leqq 1$, then $\left| \frac{1 - z_A}{1 + z_A} \right| < \theta$. On the other hand, if $|z_B| < \eta$, all $B \in \alpha$, then $\left| \prod_{B \in \alpha} \frac{1 + z_B}{1 - z_B} \right| > \left(\frac{1-\eta}{1+\eta} \right)^{|\alpha|}$. Hence $M(z_{\alpha'}) \neq 0$ if $|z_A - 1| < \theta$, $|z_B| < \eta$, $B \in \alpha$, and $\left(\frac{1-\eta}{1+\eta} \right)^{|\alpha|} > \theta$. Putting $\theta = \left(\frac{1-\eta}{1+\eta} \right)^{|\alpha|}$ finishes the proof.

2.4. Let $\Lambda_n \to \infty$ in the sense of Van Hove. Let δ be a positive number, and let $\mathscr{B}_n \subset \mathscr{P}(\Lambda_n)$ be a modification of $\mathscr{B}_{\Lambda_n}$ in the δ-layer around the boundary $\partial \Lambda_n$ of Λ_n [4]. That is, the subset $\mathscr{B}_{n,\delta}$ of the elements that are distant from $\partial \Lambda_n$ by more than δ is the same for $\mathscr{B}_n$ and $\mathscr{B}_{\Lambda_n}$.

Let for $B \in \mathscr{B}_n \setminus \mathscr{B}_{n,\delta}$, $r_n(B)$ be a positive number, and let $M_n(z_{\mathscr{B}_{n,\delta}})$ be the polynomial obtained from $M(z_{\mathscr{B}_n})$ by substitution $r_n(B)$ for z_B, $B \in \mathscr{B}_n \setminus \mathscr{B}_{n,\delta}$. We will say that $\{M_n(z_{\mathscr{B}_{n\delta}})\}$ is a *boundary modification* of $\{M(z_{\mathscr{B}_{\Lambda_n}})\}$ if

$$\lim_n \frac{1}{|\Lambda_n|} \left[\sum_{B \in \mathscr{B}_n \setminus \mathscr{B}_{n,\delta}} |\log r_n(B)| \right] = 0 .$$

Theorem. *Let for $B \in \mathscr{B}$, r_B be a positive number and let $r_{B'} = r_B$ if B' is congruent with B. Suppose that $\Lambda_n \to \infty$ in the sense of Van Hove and that there exists a boundary modification $M_n(z_{\mathscr{B}_{n,\delta}})$ of $M(z_{\mathscr{B}_{\Lambda_n}})$ such that*

$$M_n(z_{\mathscr{B}_{n,\delta}}) \neq 0 \quad \text{for} \quad |z_B| < r_B .$$

Then

a) *there exists a function f of $\{z_B\}_{B \in \mathscr{B}_0}$ analytic for $|z_B| < r_B$ such that*

$$f(z_B)|_{z_B = e^{-2K(B)}} = p^0(K) ,$$

b) *if $e^{-2K(B)} < r_B$ then for each $A \in \bar{\mathscr{B}}$, $\varrho(\sigma_A)$ has the same value for all $\varrho \in \Delta_K$, and $\varrho(\sigma_A)$ extends to an analytic function of $\{z_B\}_{B \in \mathscr{B}_0}$ as in* a).

We will show that a stronger version of a) holds, in which the regions $|z_B| < r_B$ are replaced by open simply connected domains D_B that intersect $\mathbb{R}^+$ and $D_{B'} = D_B$ if B' is a translate of B. We follow the proof of the analyticity of the pressure with respect to external magnetic field in [8], p. 111.

Let $M^n(z_{\mathscr{B}_0})$ be the polynomial in variables $\{z_B\}_{B \in \mathscr{B}_0}$ obtained from $M_n(z_{\mathscr{B}_{n,\delta}})$ by substitution z_B, $B \in \mathscr{B}_0$, for each variable $z_{B'}$, $B' \in \mathscr{B}_{n,\delta}$, for which B' is congruent with B. By assumption, the domain

$$D_{\mathscr{B}_0} = \prod_{B \in \mathscr{B}_0} D_B$$

is free of zeros of M^n. We first show that if $r(B)$, $B \in \mathscr{B}_0$, are positive numbers, then

$$\lim_n \frac{1}{|\Lambda_n|} \log M^n(r(\mathscr{B}_0)) = p^0(K) \tag{2.3}$$

where $M^n(r(\mathscr{B}_0)) = M^n(z_{\mathscr{B}_0})|_{z_B = r_B}$ and K is the translation invariant interaction with $\mathscr{B}$ as the set of bonds and such that

$$e^{-2K(B)} = r(B) , \quad B \in \mathscr{B}_0 .$$

Let K_n be the interaction in Λ_n with bonds $\mathscr{B}_n$ and

$$K_n(B) = \begin{cases} -\frac{1}{2} \log r(B) & \text{for} \quad B \in \mathscr{B}_{n,\delta} \\ -\frac{1}{2} \log r_n(B) & \text{for} \quad B \in \mathscr{B}_n \setminus \mathscr{B}_{n,\delta} \end{cases}$$

[4] For a subset Λ of $\mathbb{Z}^\nu$ its boundary $\partial\Lambda$ is the set of points in Λ that have nearest neighbors in $\mathbb{Z}^\nu \setminus \Lambda$. $\Lambda(\delta) = \{x \in \Lambda : \operatorname{dist}(x, \partial\Lambda) \leq \delta\}$.

and let $p_n^0(K_n)$ be the corresponding reduced pressure. Then

$$\frac{1}{|\Lambda_n|} \log M^n(r(\mathscr{B}_0)) = p_n^0(K_n).$$

By (1.6)

$$\begin{aligned} |p_{\Lambda_n}^0(K) - p_n^0(K_n)| \leqq & \frac{1}{|\Lambda_n|} \big|\log|\mathscr{S}_{\Lambda_n}| - \log|\mathscr{S}_n|\big| \\ & + \frac{2}{|\Lambda_n|} \sum_{B \subset \Lambda_n} |K(B) - K_n(B)|. \end{aligned} \tag{2.4}$$

It was proved in Section 1.9 that $\lim_n \frac{1}{|\Lambda_n|} \log|\mathscr{S}_{\Lambda_n}| = 0$, and it is not hard to see that $|\mathscr{S}_n|$ is majorized by $|\mathscr{S}_{\Lambda_n(\delta)}|$ multiplied by the number of subsets of $\Lambda_n \setminus \Lambda_n(\delta)$. Therefore

$$\frac{1}{|\Lambda_n|} \log|\mathscr{S}_n| \leqq \frac{1}{|\Lambda_n|} \log|\mathscr{S}_{\Lambda_n(\delta)}| + \frac{\delta \cdot |\partial \Lambda_n|}{|\Lambda_n|} \log 2.$$

This shows that the first term on the right-hand side of (2.4) tends to zero as $\Lambda_n \to \infty$. On the other hand

$$\begin{aligned} \frac{2}{|\Lambda_n|} \sum_{B \subset \Lambda_n} |K(B) - K_n(B)| \leqq & \frac{2}{|\Lambda_n|} \sum_{B \in \mathscr{B}_{\Lambda_n} \setminus \mathscr{B}_{n,\delta}} |K(B)| \\ & + \frac{2}{|\Lambda_n|} \sum_{B \in \mathscr{B}_n \setminus \mathscr{B}_{\Lambda_n(\delta)}} |K_n(B)|. \end{aligned}$$

The first term on the right-hand side tends to zero since it is majorized by

$$\frac{2}{|\Lambda_n|} \cdot |\mathscr{B}_0| \cdot \delta \cdot \max_{B \in \mathscr{B}_0} |K(B)|;$$

the second term tends to zero by the definition of boundary modification. Thus (2.3) is proved.

The functions $M^n(z_{\mathscr{B}_0})^{\frac{1}{|\Lambda_n|}}$ and $\frac{1}{|\Lambda_n|} \log M^n(z_{\mathscr{B}_0})$ are analytic in $D_{\mathscr{B}_0}$, and

$$\left| M^n(z_{\mathscr{B}_0})^{\frac{1}{|\Lambda_n|}} \right| \leqq M^n(r(\mathscr{B}_0))^{\frac{1}{|\Lambda_n|}} \quad \text{if} \quad |z_B| < r(B). \tag{2.5}$$

Here and in the rest of the proof by log and $z \mapsto z^\alpha$ we mean those branches of these functions that are assuming positive values for real argument greater than 1. In particular $\log z^\alpha = \alpha \log z$.

(2.5) and the convergence of $\frac{1}{|\Lambda_n|} \log M^n(r(\mathscr{B}_0))$ show that the functions $\left\{ M^n(z_{\mathscr{B}_0})^{\frac{1}{|\Lambda_n|}} \right\}_n$ are uniformly bounded on every compact subset of $D_{\mathscr{B}_0}$. The same convergence and the fact that each component of $D_{\mathscr{B}_0}$

contains an interval of $\mathbb{R}^+$ allow for a use of the Vitali theorem to finish the proof of a).

If $A \in \mathscr{B}$, we may assume that $A \in \mathscr{B}_0$, then b) follows directly from a) and Proposition 1.4: the analytic function of b) is equal to

$$1 - 2z_A \frac{\partial}{\partial z_A} f(z_{\mathscr{B}_0}). \tag{2.6}$$

Let now $A \in \bar{\mathscr{B}}$, $A \notin \mathscr{B}$, and let α be a minimal element of $\mathscr{P}_f(\mathscr{B})$ that yields A. We define:

$$\mathscr{B}' = \mathscr{B} \cup \{A_x\}_{x \in \mathbb{Z}^\nu}, \qquad \mathscr{B}'_0 = \mathscr{B}_0 \cup \{A\}, \qquad \alpha' = \alpha \cup \{A\},$$

$$\mathscr{B}'_n = \mathscr{B}_n \cup \{A_x : \alpha_x \subset \mathscr{B}_{n,\delta}\}, \qquad \mathscr{B}'_{n,\delta} = \mathscr{B}'_n \cap \mathscr{P}(A_n(\delta))$$

here $A_x = A + x$, and the elements of α_x are the translates by x of the bonds of α. $\mathscr{B}'_n \setminus \mathscr{B}'_{n,\delta} = \mathscr{B}_n \setminus \mathscr{B}_{n,\delta}$, and the polynomials $M'_n(z_{\mathscr{B}'_{n,\delta}})$ obtained from $M(z_{\mathscr{B}'_n})$ by substitution for z_B, $B \in \mathscr{B}'_n \setminus \mathscr{B}'_{n,\delta}$, the same numbers as previously form a boundary modification of $\{M(z_{\mathscr{B}'_{A_n}})\}$. Applying Lemma 2.3 to $M(z_{\mathscr{B}'_n})$ we conclude that for each $\eta \in [0, 1]$ $M'_n(z_{\mathscr{B}'_{n,\delta}}) \neq 0$ when $|z_B| < \eta^n r_B$ for $B \in \bigcup_x \alpha_x$ and $|z_{A_x} - 1| < \left(\frac{1-\eta}{1+\eta}\right)^{|\alpha|}$ for all x; n_B of Lemma 2.3 is replaced here by a natural number $\bar{n}$ independent of B since, by translation invariance, there exists $\bar{n}$ such that each $B \in \mathscr{B}$ belongs to no more than $\bar{n}$ of $\{\alpha_x\}_{x \in \mathbb{Z}^\nu}$.

The strengthened version of a) proved above yields now a function $f'_\eta(z_{\mathscr{B}'_0})$ analytic for

$$|z_B| < \eta^{\bar{n}} r_B, \qquad B \in \mathscr{B}_0, \qquad |z_A - 1| < \left(\frac{1-\eta}{1+\eta}\right)^{|\alpha|}$$

and such that

$$f_\eta'(z_{\mathscr{B}'_0})\big|_{\substack{z_B = e^{-2K(B)} \\ z_A = e^{-2\lambda}}} = p^0(K + \lambda K_A).$$

By Proposition 1.4, b) holds with the analytic function given for $|z_B| < \eta^{\bar{n}} r_B$ by

$$1 - 2z_A \frac{\partial}{\partial z_A} f'_\eta(z_{\mathscr{B}'_0})\big|_{z_A = 1}.$$

Letting here $\eta \to 1$ we finish the proof of b) and of the theorem.

Remark. The theorem modified suitably still holds if the regions $|z_B| < r_B$ are replaced by domains each component of which is simply connected and intersects $\mathbb{R}^+$. This follows from the following property of the polynomials $M(z_{\alpha'})$ of Section 2.3: there exists a function $\delta \mapsto \varepsilon(\delta)$ such that $\varepsilon(\delta) \to 0$ when $\delta \to 0$, and $M(z_{\alpha'}) \neq 0$ if $|z_A - 1| < \delta^{|\alpha|}$ and $|z_B + 1| \geqq \varepsilon(\delta)$ (ε can be taken proportional to δ).

2.5. In the next sections we prove that the assumptions of Theorem 2.4 are satisfied for the following systems:

a) Systems with even bonds and such that $\mathscr{B}$ contains the Ising bonds, Sections 3.1–3.2.

b) Trivial systems, Section 3.5.

c) Regular and essentially regular systems, Sections 4.1 to 4.7.

d) Systems with general two-body, finite range interactions, Section 4.9.

The r_B's of Theorem 2.4 are in general small and therefore we obtain analyticity and uniqueness of $\varrho(\sigma_A)$, $A \in \overline{\mathscr{B}}$, for ferromagnetic interactions and low temperatures:

$$J(B) > 0 \quad \text{and} \quad \beta \text{ large}.$$

2.6 **Proposition.** *For ferromagnetic systems to which Theorem 2.4 applies (hence for the systems of Section 2.5) the entropy is the same for all the translation invariant equilibrium states at low enough temperatures, and tends to zero with temperature.*

From (1.2) and (1.8)

$$s(\varrho) = p^0(K) + \sum_{B \in \mathscr{B}_0} K(B)\,[1 - \varrho(\sigma_B)]\,.$$

By Theorem 2.4 $\varrho(\sigma_B)$ has the same value for all $\varrho \in \Delta_K$ at low temperatures. This proves the first part of Proposition.

Making explicit the dependence on the temperature, we get

$$s(\varrho) = p^0(\beta J) + \sum_{B \in \mathscr{B}_0} J\beta\,[1 - \varrho(\sigma_B)]\,.$$

From formula (2.6):

$$\beta[1 - \varrho(\sigma_B)] = 2\beta e^{-2\beta J(B)} \frac{\partial}{\partial z_B} f(z_{\mathscr{B}_0})\big|_{z_{\mathscr{B}_0} = e^{-2\beta J(\mathscr{B}_0)}}.$$

Since f is analytic in a neighborhood of the origin and $J(B) > 0$ for all $B \in \mathscr{B}_0$ this shows that $\beta[1 - \varrho(\sigma_B)]$ tends to zero (exponentially) as $\beta \to \infty$. It remains to discuss the behavior of $p^0(\beta J)$.

$$p^0(\beta J) = f(z_{\mathscr{B}_0})\big|_{z_B = e^{-2\beta J(B)}}$$

and

$$f(z_{\mathscr{B}_0}) = \lim_{\Lambda_n \to \infty} \frac{1}{|\Lambda_n|} \log M^n(z_{\mathscr{B}_0})\,.$$

Therefore

$$\lim_{\beta \to \infty} p^0(\beta J) = \lim_{\Lambda_n} \frac{1}{|\Lambda_n|} \log M^n(0)$$

$M^n(0)$ can be thought as the reduced partition function for a system in $\Lambda_n \setminus \Lambda_n(\delta)$ with bonds $\mathscr{B}_n \setminus \mathscr{B}_{n,\delta}$ and interaction $K_n(B) = -\frac{1}{2}\log r_n(B)$. By (1.7)

$$|\log M^n(0)| \leqq \left|\log|\mathscr{S}_{n,\delta}| - \log 2^{|\Lambda_n \setminus \Lambda_n(\delta)|}\right| + 2 \sum_{B \in \mathscr{B}_n \setminus \mathscr{B}_{n,\delta}} K(B)\,.$$

Since $|\mathscr{S}_{n,\delta}| \leqq 2^{|\Lambda_n \setminus \Lambda_n(\delta)|}$ and $|\Lambda_n \setminus \Lambda_n(\delta)| \leqq \delta \cdot |\partial \Lambda_n|$ this shows that $\frac{1}{|\Lambda_n|} \log M^n(0) \to 0$. Thus $s(\varrho) \underset{\beta \to \infty}{\to} 0$.

3.4. Generalized Ising Models and Trivial Systems

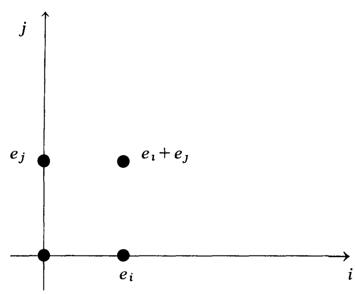
3.1. An *Ising interaction* in $\mathbb{Z}^\nu$ has $\mathscr{I}_0 = \{\{0, e_i\}\}_{i=1}^{\nu}$ as a fundamental set of bonds. The set of bonds of an Ising interaction will be denoted by $\mathscr{I}$.

The group $\mathscr{K}_\Lambda$ for an Ising interaction in a parallelepiped Λ is generated by all the translates of

$$\beta_{ij} = \{\{0, e_i\}, \{0, e_j\}, \{e_i, e_i + e_j\}, \{e_j, e_i + e_j\}\} \quad i, j = 1, \ldots, \nu; \ \nu \geqq 2$$

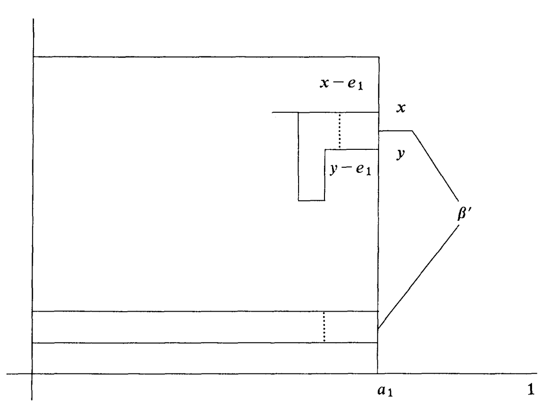
that are contained in Λ; this is not a minimal family of generators.

β_{ij}:



For $\nu = 1$ the groups are trivial (Section 1.8).

We may assume that $\Lambda = \Lambda(a)$ $(= \{x \in \mathbb{Z}^\nu, 0 \leqq x_i \leqq a_i\})$. Let $\beta \in \mathscr{K}_\Lambda$ and let β' be the set of all the bonds of β that are contained in the hyperplane $x_1 = a_1$. For $B = \{x, y\} \in \beta'$ let $\beta_B = \{\{x, y\}, \{x, x - e_1\}, \{y, y - e_1\}, \{x - e_1, y - e_1\}\}$; β_B belongs to our



family of generators, and $\beta + \sum_{B \in \beta'} \beta_B$ has no bonds on the hyperplane $x_1 = a_1$. It follows that it has also no bonds of the type: $\{x, x - e_1\}$, $x_1 = a_1$, and therefore it is contained in $\Lambda(a_1 - 1, a_2, \ldots, a_\nu\}$. Repeating this construction a_1-times, we arrive at a system on the $\nu - 1$-dimensional hyperplane $x_1 = 0$. This shows that our statement can be proved by induction with respect to ν.

3.2 **Theorem.** *Let $\mathscr{B}$ be the set of bonds of a ferromagnetic interaction J. Suppose that the bonds are even and that $\mathscr{B} \supset \mathscr{I}$. Then at low enough temperatures the even order correlation functions (of translation invariant equilibrium states) are unique and have the analyticity properties of Theorem 2.4.*

For a parallelepiped Λ the subgroup $\overline{\mathscr{B}}_\Lambda$ of $\mathscr{P}(\Lambda)$ generated by $\mathscr{B}_\Lambda$ consists of all the even subsets of Λ: for a two-point subset $\{x, y\}$ suitable element of $\mathscr{P}(\mathscr{B}_\Lambda)$ is obtained by taking bonds lying on any path joining x with y that is contained in Λ. General even subset can be decomposed into a sum of two-point subsets.

Therefore, for a finite even subset A of $\mathbb{Z}^\nu$ there exists $\alpha_A \in \mathscr{P}_f(\mathscr{I})$ such that $\overline{\alpha}_A = A$ and α_A is contained in any parallelepiped containing A. Let us choose such α_B for each $B \in \mathscr{B} \setminus \mathscr{I}$ in a translation invariant way.

According to Section 1.7, for any parallelepiped Λ the group $\mathscr{K}_\Lambda$ is generated by the translates of β_{ij} and by $\{\alpha_B \cup \{B\}\}_{B \in \mathscr{B}_\Lambda \setminus \mathscr{I}}$. Therefore (Proposition 2.2) $M(z_{\mathscr{B}_\Lambda})$ is the Asano contraction of the polynomials corresponding to the translates of β_{ij} and of $\{M(z_{\alpha_B \cup \{B\}}\}_{B \in \mathscr{B}_\Lambda \setminus \mathscr{I}}$. The number of variables of these polynomials has a bound independent of Λ:

$$\max\{4, \max_{B \in \mathscr{B}_0} |\alpha_B| + 1\}.$$

Each of these polynomials has the free term equal to 1. Therefore there exists $r' > 0$ such that the polynomials do not vanish when the modulus of their arguments is majorized by r'. Since there is a bound independent of Λ on the number of the polynomials containing given variable there exists $r > 0$ such that

$$M(z_{\mathscr{B}_\Lambda}) \neq 0 \quad \text{if} \quad |z_B| < r, \quad \text{all} \quad B \in \mathscr{B}_\Lambda.$$

Thus Theorem 2.4 applies (without boundary modifications) and yields the required analyticity. Since the interaction is ferromagnetic it yields also the uniqueness properties at low temperature.

3.3 **Corollary.** *The systems of Theorem 2.3 admit at most two ergodic equilibrium states at low temperatures.*

This is a special case of a theorem of [13] but we give here a more direct proof.

If all the odd order correlation functions vanish, there is only one ergodic equilibrium state. Suppose therefore that $\tilde{\varrho}$ is an ergodic element of Δ and that $\tilde{\varrho}(\sigma_{\tilde{A}}) \neq 0$, $|\tilde{A}|$-odd. Let ϱ be another ergodic element of Δ and let A be an odd subset of $\mathbb{Z}^\nu$. Then $\tilde{A} \cdot \tau_x(A) \in \overline{\mathscr{B}}$ and therefore

$$\tilde{\varrho}(\sigma_{\tilde{A}\tau_x(A)}) = \varrho(\sigma_{\tilde{A}\cdot\tau_x(A)}).$$

Passing to the limit $a \to \infty$ in the equality:

$$\sum_{x \in \Lambda(a)} \frac{1}{|\Lambda(a)|} \tilde{\varrho}(\sigma_{\tilde{A}} \tau_x \sigma_A) = \sum_{x \in \Lambda(a)} \frac{1}{|\Lambda(a)|} \varrho(\sigma_{\tilde{A}} \tau_x \sigma_A)$$

and taking into account the ergodicity of $\tilde{\varrho}$ and ϱ, we get

$$\tilde{\varrho}(\sigma_{\tilde{A}})\, \tilde{\varrho}(\sigma_A) = \varrho(\sigma_{\tilde{A}})\, \varrho(\sigma_A).$$

The corollary follows.

3.4. We now give a variation of the preceding theorem and corollary which covers the situation when the non-Ising part of the interaction is not necessarily ferromagnetic.

We choose α_B for $B \in \mathscr{B} \setminus \mathscr{I}$ as in Section 3.2, and we let $|\alpha_B|_i$, $1 \leqq i \leqq \nu$, to denote the number of the translates of the bond $\{0, e_i\}$ that are contained in α_B. We denote $J(0, e_i\})$ by J_i.

If

$$J_i - \sum_{B \in \mathscr{B}_0 \setminus \mathscr{I}} |\alpha_B|_i \, |J(B)| > 0, \qquad \text{all } i, \tag{3.1}$$

then for low enough temperatures there are only two ergodic equilibrium states. Some analyticity properties can be also deduced.

In the polynomials $M(z_{\alpha_B \cup \{B\}})$ the variable z_B appears only in product with variables z_A, $A \in \alpha_B$. Therefore they are not zero if $|z_A z_B|$ is small enough for all $A \in \alpha_B$. This includes low temperature region if $J_A - |J_B| > 0$. The stronger condition (3.1) ensures that the free of zeros region of the polynomial obtained by contractions also includes low temperature domain. To finish the proof, it is enough to apply an obvious modification of Theorem 2.4.

When only two-body interaction is present and $\alpha_{\{x,y\}}$ is the set of Ising bonds lying on a shortest broken line joining x with y the condition (3.1) is stronger than the corresponding condition of [1]: in [1] before the sum appears factor $\frac{1}{2}$.

3.5. We will say about a system that it is *trivial* if the groups $\mathscr{K}_\Lambda$ are trivial.

If Λ is a finite subset of the lattice then by Proposition 2.2 $M(z_{\mathscr{B}_\Lambda})$ is the Asano contraction of $\{M(z_{\mathscr{B}_i})\}$ for any covering $\{\mathscr{B}_i\}$ of $\{\mathscr{B}_\Lambda\}$. Choosing the covering by the one-element subsets of $\mathscr{B}_\Lambda$ we get

$$M(z_{\mathscr{B}_\Lambda}) = \prod_{B \in \mathscr{B}_\Lambda} (1 + z_B). \tag{3.2}$$

This follows from the fact that for disjoint covering no variable is contracted and from:

$$M(z_{\{B\}}) = 1 + z_B .$$

Let $M^{|\Lambda|}(z_{\mathscr{B}_0})$ be the polynomial in variables $\{z_B\}_{B \in \mathscr{B}_0}$ obtained from $M(z_{\mathscr{B}_\Lambda})$ by substitution z_B, $B \in \mathscr{B}_0$ for $z_{\mathscr{B}'}$, $B' \in \mathscr{B}_\Lambda$ if B' is a translate of B. Then for $z_B \neq 1$, $B \in \mathscr{B}_0$

$$\lim_{\Lambda} \frac{1}{|\Lambda|} \log M^{|\Lambda|}(z_{\mathscr{B}_0}) = \sum_{B \in \mathscr{B}_0} \log(1 + z_B) .$$

In fact for trivial systems $\mathscr{B}_0$ can have no more than one element: by Section 1.9 $|\mathscr{S}_\Lambda| = 2^{|\Lambda| - |\mathscr{B}_\Lambda|}$ and since $|\mathscr{B}_\Lambda|$ differs from $|\mathscr{B}_0|\,|\Lambda|$ by a number of order $|\partial\Lambda|$, $\Lambda \to \infty$, $|\mathscr{B}_0| > 1$ would imply $|\mathscr{S}_\Lambda| < 1$ for Λ large enough which is impossible.

Formula (3.2) implies, by Theorem 2.4, that the correlation functions $\varrho(\sigma_A)$, $A \in \overline{\mathscr{B}}$, are unique for ferromagnetic interactions at all temperatures. It is not hard to see that the same holds for antiferromagnetic interaction. We showed that for trivial ferromagnetic systems the translation invariant equilibrium state is unique at all temperatures.

In some cases we proved that the equilibrium state, not necessarily translation invariant is unique. This holds presumably for all trivial systems.

4. Regular and Essentially Regular Systems

4.1. The theorems on $\mathscr{K}_\Lambda$ proved in the next sections show that the assumptions of Theorem 2.4 are satisfied for a large family of systems. We now give the argument for regular systems.

By Theorem 4.4 for each regular system there exists a parallelepiped Λ_0 such that

$$\mathscr{K}_\Lambda = \left[\bigcup_i \mathscr{K}_{\Lambda_i}\right]$$

where $\{\Lambda_i\}$ are the translates of Λ_0 contained in the parallelepiped Λ. By Proposition 2.2, this implies that $M(z_{\mathscr{B}_\Lambda})$ is the Asano contraction of $\{M(z_{\mathscr{B}_{\Lambda_i}})\}$. Since

$$M(z_{\mathscr{B}_{\Lambda_i}}) = \sum_{\beta \in \Gamma_i} z^\beta$$

and $z^\emptyset = 1$, $M(z_{\mathscr{B}_{\Lambda_i}}) \neq 0$ if all the arguments are small enough, say $|z_B| < r_0$. On the other hand, each $B \in \mathscr{B}$ is contained in no more than $|\Lambda_0|$ of $\{\mathscr{B}_{\Lambda_i}\}$. Therefore, by Theorem 2.1,

$$M(z_{\mathscr{B}_\Lambda}) \neq 0 \quad \text{if} \quad |z_B| < r^{|\Lambda_0|}, \qquad \text{all } B \in \mathscr{B}_\Lambda .$$

This allows to apply Theorem 2.4 and gives a rough estimation of the analyticity domain.

4.2. A *μ-plane*, or simply a *plane*, in $\mathbb{Z}^\nu$ is a set of the type

$$\{x \in \mathbb{Z}^\nu : x_i = a_i,\ i \in I\}, \qquad I \subset \{1, \ldots, \nu\}, \qquad |I| = \nu - \mu.$$

A μ-plane is determined by the set I and the function a from I to $\mathbb{Z}$; it is denoted by $P(a)$ or $P(I, a)$. We also write $P(i, a)$ instead of $P(I, a)$ if $I = \{i\}$, and $P(I)$ if $a_i = 0$.

μ-plane is identified with $\mathbb{Z}^\mu$ by the mapping

$$(x_1, \ldots, x_\mu) \mapsto \sum_{i=1}^{\mu} x_i e_{j_i} + \sum_{i \in I} a_i e_i.$$

Here $j_1 < \cdots < j_\mu$, $\{j_1, \ldots, j_\mu\} = \{1, \ldots, \nu\} \setminus I$, and $e_j = (\delta_{ij})_{i=1}^{\nu}$, $j = 1, \ldots, \nu$ are the canonical generators of $\mathbb{Z}^\nu$. We note the transitivity of our identification of planes with $\mathbb{Z}^\mu$'s.

For $a, b \in \mathbb{Z}^\nu$ we put

$$\Lambda(a) = \{x \in \mathbb{Z}^\nu : 0 \leqq x_i \leqq a_i, i = 1, \ldots, \nu\}, \qquad \Lambda(a; b) = \{x : a_i \leqq x \leqq b_i\},$$
$$\Lambda_a = \{x : |x_i| \leqq a_i\}, \quad \Lambda_r = \{x \in \mathbb{Z}^\nu : |x_i| \leqq r\} \quad \text{for} \quad r \in \mathbb{Z}.$$

Then $\Lambda(a) = \Lambda(0; a)$, $\Lambda(a + x; b + x) = \tau_x \Lambda(a; b)$ and $\Lambda_r = \Lambda_{\bar{r}}$ where $\bar{r} = (r, \ldots, r)$.

4.3 *Definition.* The subsets of $\mathbb{Z}^\nu$ that are obtained from $\{\Lambda(a; b)\}_{a, b \in \mathbb{Z}^\nu}$ by substraction $\nu - 1$-planes will be called *regular*. A family $\mathscr{B} \subset \mathscr{P}_f(\mathbb{Z}^\nu)$ is regular if each element of $\mathscr{B}$ is regular.

For a family $\mathscr{B}$ of subsets of $\mathbb{Z}^\nu$ and a plane P we define

$$\mathscr{B}(P) = \{B \cap P\}_{B \in \mathscr{B}}.$$

If $P \supset Q$ then $\mathscr{B}(P)(Q) = \mathscr{B}(Q)$. If $\mathscr{B}$ is regular so is $\mathscr{B}(P)$.

For $\nu - 1$-plane $P = P(i, a)$ we introduce the halfspaces

$$P_\pm = \bigcup_{n \geqq 0} \tau_{\pm n e_i} P.$$

The main reason for dealing with regular systems is that they enjoy the following *extension property:*

Let $\mathscr{B}$ be a translation invariant family of regular subsets of $\mathbb{Z}^\nu$ and let P be a $\nu - 1$-plane. Then for each $A \in \mathscr{B}(P)$ there exist A_+ and A_- in $\mathscr{B}$ such that $A_+ \subset P_+$, $A_- \subset P_-$, $A_+ \cap P = A_- \cap P = A$, and both A_+ and A_- projected on P yield A. A_+ and A_- will be called *the extensions of* A to P_+ and P_-, respectively.

We define: $|a| = \max_{1 \leqq i \leqq \nu} |a_i|$, for $a \in \mathbb{Z}^\nu$; $\operatorname{diam}(A) = \sup_{a, b \in A} |a - b|$, for a finite subset A of $\mathbb{Z}^\nu$; for a family β of subsets of $\mathbb{Z}^\nu$ the support

$$\operatorname{supp}(\beta) = \bigcup_{B \in \beta} B.$$

For $a, b \in \mathbb{Z}^\nu$: $\max(a, b)_i = \max(a_i, b_i)$, and $a \geqq b$ if $a_i \geqq b_i$.

If $\mathscr{B}$ is a translation invariant family of bonds with a finite fundamental family we put $d_0 = \max_{B \in \mathscr{B}} \operatorname{diam}(B)$

and for $1 \leqq \mu \leqq \nu$ inductively:

$$d_\mu = \max_{\substack{I \subset \{1, \dots, \nu\} \\ |I| = \nu - \mu}} \max_{\substack{A \in \overline{\mathscr{B}(P(I))} \\ A \subset \Lambda_{d_{\mu-1}}}} \min_{\substack{\beta \subset \mathscr{B}(P(I)) \\ \bar{\beta} = A}} \operatorname{diam}(\operatorname{supp} \beta)$$

$d_0 \leqq d_1 \leqq \cdots \leqq d_\nu$, and all d_μ are finite numbers as follows by an easy induction with respect to μ.

4.4 **Theorem.** *Let $\mathscr{B}$ be translation invariant, regular and with a finite fundamental family. Then there exists a parallelepiped Λ_0 such that for each large enough parallelepiped Λ*

$$\mathscr{K}_\Lambda = \left[\bigcup_i \mathscr{K}_{\Lambda_i} \right]$$

where Λ_i are the translates of Λ_0 that are contained in Λ.

4.5 **Lemma.** *Let Q be a μ-plane, and let $A \in \overline{\mathscr{B}(Q)}$, $A \subset \Lambda_a(Q)$. Then there exists $\alpha \subset \mathscr{B}(Q)$ such that $\bar{\alpha} = A$ and supp $(\alpha) \subset \Lambda_{\max(a, \bar{d}_{\mu-1})}(Q)$.*

We add to the lemma the following:

(*) If Q is a μ-plane and $a \geqq \bar{d}_{\mu-1}$, $a_i > d_{\mu-1}$ then there exists $\alpha \subset \mathscr{B}(Q)$ such that $\operatorname{supp}(\alpha) \subset \Lambda_a(Q)$ and $A \cdot \bar{\alpha} \subset \Lambda_{(a_1, \dots, a_i - 1, \dots, a_\nu)}(Q)$, and prove both statements simultaneously by induction with respect to μ.

Assuming (*) holds for $\mu - 1$-planes, we prove that the lemma is also true for $\dim Q = \mu - 1$. In case $a \leqq d_{\mu-2}$ existence of such α follows from the definition of $d_{\mu-1}$. The general case can be reduced to this one. For, if some of a_i's, $1 \leqq i \leqq \mu - 1$ are larger than $d_{\mu-2}$, we can by repeated use of (*) (for $\dim Q = \mu - 1$) prove the existence of $\alpha' \subset \mathscr{B}(Q)$ such that $A \cdot \bar{\alpha}' \subset \Lambda_{d_{\mu-1}}(Q)$ and $\operatorname{supp}(\alpha') \subset \Lambda_{\max(a, \bar{d}_{\mu-2})}(Q)$.

Assuming now the Lemma for $\mu - 1$ we prove (*) in dimension μ. We identify Q with $\mathbb{Z}^\mu$ as in Section 4.2 and we write $\mathscr{B}, \Lambda_a$ instead of $\mathscr{B}(Q)$, $\Lambda_a(Q)$. The lemma applied to $A \cap P(i, a_i)$, $\Lambda_{(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_\mu)} \cdot (P(i, a_i))$ yields the existence of $\beta \in \mathscr{B}(P(i, a_i))$ such that $\bar{\beta} = A \cap P(i, a_i)$ and $\operatorname{supp}(\beta) \subset \Lambda_{(a_1, \dots, a_{i-1}, a_i, \dots, a_\mu)}(P(i, a_i))$.

By the extension property of Section 4.3, there exists an extension α' of β to $P(i, a_i)_-$. Moreover, $\operatorname{supp}(\alpha') \subset P(i)_+ \cap \Lambda_a$. Repeating the same for $A \cap P(i, -a_i)$ we arrive at $\alpha'' \subset \mathscr{B}$ such that $\bar{\alpha}'' \cap P(i, -a_i) = A \cap P(i, -a_i)$ and with support in $P(i)_- \cap \Lambda_a$.

It is now easy to see that $\alpha = \alpha' + \alpha''$ satisfies the conditions of (*).

It remains to show that (*) holds if $\dim Q = 1$. This can be done by repeating the construction of the preceding paragraph.

4.6 **Proposition.** *If an element A of $\overline{\mathscr{B}}$ is contained in a parallelepiped Λ with edges not shorter than $2d_\nu$ then there exists $\alpha \subset \mathscr{B}_\Lambda$ such that $\bar{\alpha} = A$.*

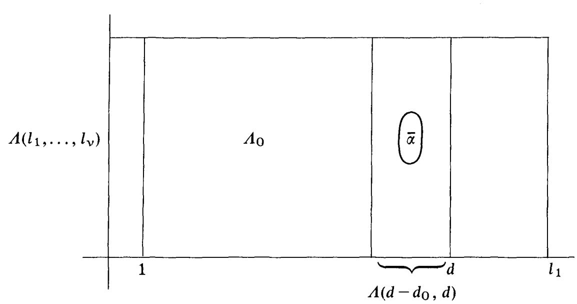
For such a parallelepiped Λ there exists $x \in \mathbb{Z}^\nu$ such that $\tau_x(\Lambda) \supset \Lambda_{d_\nu}$. Therefore the proposition can be proved by applying Lemma 4.5 to $\tau_x(\Lambda)$, $\tau_x(\Lambda)$ and by translating thus obtained α back to Λ.

We turn to the proof of Theorem 4.3. We denote $2d_\nu + 1$ by d and, for notational convenience, we assume that $\Lambda = \Lambda(l)$. "Large enough" of the theorem means that $l_i \geqq d$.

Let $\beta \in \mathcal{K}_\Lambda$ and let α be the part of β that is contained in

$$\Lambda_0 = \Lambda(d, l_2, \ldots, l_\nu); \ \alpha = \beta \cap \mathcal{B}_{\Lambda_0}.$$

Then $\bar{\alpha}$ is contained in the slice $\Lambda(d-d_0; d) \times \Lambda(l_2, \ldots, l_\nu)$ as $\bar{\alpha} = \overline{\beta \setminus \alpha}$ and elements of $\beta \setminus \alpha$ are contained in $\Lambda(d-d_0; l_1) \times \Lambda(l_2, \ldots, l_\nu)$; for $\nu = 2$ the situation is shown on the picture below.



By Proposition 4.6, there exists γ with support in $\Lambda(1; d) \times \Lambda(l_2, \ldots, l_\nu)$ such that $\overline{\alpha + \gamma} = \emptyset$. If we define $\beta_0 = \alpha + \gamma$, $\beta_0' = (\beta \setminus \alpha) + \gamma$ then $\operatorname{supp}(\beta_0) \subset \Lambda_0$, $\operatorname{supp}(\beta_0') \subset \Lambda(1; l_1) \times \Lambda(l_2, \ldots, l_\nu)$ and $\beta_0 + \beta_0' = \beta$.

If $l_1 - 1 > d$ we repeat the construction with β_0', $\Lambda(1; l_1) \times \Lambda(l_2, \ldots, l_\nu)$ replacing β, Λ. Repeating it $l_1 - d$ times we get the decomposition:

$$\beta = \sum_{i=0}^{l_1 - d} \beta_i, \qquad \beta_i \in \mathcal{K}_\Lambda, \qquad \operatorname{supp}(\beta_i) \subset \Lambda(i; i+d) \times \Lambda(l_2, \ldots, l_\nu).$$

Decomposing in the same way each of β_i's in direction of e_2 we obtain:

$$\beta_i = \sum_{j=1}^{l_2 - d} \beta_{ij}, \qquad \beta_{ij} \in \mathcal{K}_\Lambda,$$
$$\operatorname{supp}(\beta_{ij}) \subset \Lambda(i; i+d) \times \Lambda(j; j+d) \times \Lambda(l_3, \ldots, l_\nu).$$

Repeating it again we arrive at the decomposition:

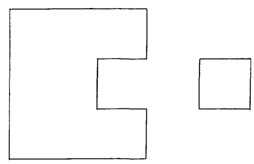
$$\beta = \sum_{i_1, \ldots, i_\nu} \beta_{i_1 \ldots i_\nu}, \qquad \beta_{i_1, \ldots, i_\nu} \in \mathcal{K}_\Lambda,$$
$$\operatorname{supp}(\beta_{i_1 \ldots i_\nu}) \subset \Lambda(i_1; i_1 + d) \times \cdots \times \Lambda(i_\nu; i_\nu + d).$$

This proves the theorem with $\Lambda_0 = \Lambda_{\bar{d}}$.

4.7. A translation invariant subfamily $\mathscr{B}$ of $\mathscr{P}_f(\mathbb{Z}^\nu)$ will be called *essentially regular* if there exists a finite set of regular elements of $\bar{\mathscr{B}}$, say $\mathscr{A}_0$, such that the translates of elements of $\mathscr{A}_0$ generate $\bar{\mathscr{B}}$.

We can assume that the elements of $\mathscr{A}_0$ are pairwise non-congruent, and we let $\mathscr{A}$ denote the set of all translates of the elements of $\mathscr{A}_0$.

A system with fundamental bonds of the shape



is essentially regular but not regular. A system with $\bar{\mathscr{B}} = \mathscr{P}_f(\mathbb{Z}^\nu)$ is essentially regular: $\{\{x\}\}$, $x \in \mathbb{Z}^\nu$, may serve as $\mathscr{A}_0$. Similarly a system with $\bar{\mathscr{B}}$ containing all finite even subsets of $\mathbb{Z}^\nu$ is always essentially regular.

We fix for what follows $\mathscr{A}, \mathscr{A}_0$, and for each $A \in \mathscr{A}$ we choose $\beta_A \in \mathscr{P}_f(\mathbb{Z}^\nu)$ such that $\bar{\beta}_A = A$; we make this choice in a translation invariant way. We fix a natural number δ such that $\operatorname{supp}(\beta_A) \subset \tau_{A_\delta}(A)$ for all A and we define

$$\mathscr{B}'_\Lambda = \mathscr{B}_\Lambda \cup (\mathscr{A}_\Lambda \setminus \mathscr{A}_{\Lambda(\delta)}) ;$$

as $\Lambda \to \infty$ $\mathscr{B}'_\Lambda$ is a boundary modification of $\mathscr{B}_\Lambda$. With this notation, we have the following version of Theorem 4.4.

There exists $r \in \mathbb{Z}$ such that for all large enough $a \in \mathbb{Z}^\nu$

$$\mathscr{K}'_{\Lambda(a)} = \left[\bigcup_i \mathscr{K}'_{\Lambda_i} \right]$$

where $\{\Lambda_i\}$ are the translates of Λ_r contained in $\Lambda(a)$, $\mathscr{K}'_\Lambda = \mathscr{K}(\mathscr{B}'_\Lambda)$ and

$$\mathscr{K}'_{\Lambda_i} = \mathscr{K}'_\Lambda \cap \mathscr{P}(\Lambda_i) .$$

(Thus, for essentially regular systems, the assumptions of Theorem 2.4 are satisfied.)

It follows from the proof of Theorem 4.4 that there exists δ such that for each $\beta \in \mathscr{K}'_\Lambda$

$$\beta = \sum \beta'_i , \qquad \beta'_i \in \mathscr{K}(\mathscr{A}_{\Lambda_i} \cup \mathscr{B}_{\Lambda_i})$$

where $\{\Lambda'_i)$ are the translates of Λ_s that are contained in Λ. This is because $\bar{\alpha}$ of the beginning of the proof of Theorem 4.4 being in $\overline{\mathscr{B}'_\Lambda}$ is by Proposition 4.6, for Λ large enough, also in $\bar{\mathscr{A}}_\Lambda$. We now define β_i by substracting from β'_i these A's that are in $A_{\Lambda(\delta)}$ and adding β_A in place of them. Clearly, $\beta_i \in \mathscr{K}'_\Lambda$ and $\operatorname{supp}(\beta_i) \subset \tau_{A_\delta}(\Lambda'_i) \cap \Lambda$. It follows that we can put $r = s + \delta$.

4.8 *Remark.* One obtains a slight extension of the preceding theorems if the notion of regularity, and essential regularity, with respect to a basis of $\mathbb{Z}^\nu$ is introduced. The modifications are obvious and omitted here. In the next section, regularity of a system means that there exists a basis with respect to which the system is essentially regular.

4.9. In this section, we show that Theorem 2.4 applies to essentially two-body systems, i.e. to systems for which $\bar{\mathscr{B}}$ is generated by its two-body part. This includes all systems with purely two-body (finite range) interaction. We consider first the decomposition into connected parts[5].

Given $\mathscr{B}$, translation invariant but not necessarily regular or finite range, we will say that $x \in \mathbb{Z}^\nu$ is connected by $\mathscr{B}$ with $y \in \mathbb{Z}^\nu$ if there exists a finite subfamily of $\mathscr{B}$, say $(B_1, \ldots, B_n)$, such that $x \in \mathscr{B}_1$, $y \in B_n$ and $B_i \cap B_{i+1} \neq \emptyset$. "$x$ is connected by $\mathscr{B}$ with y" is obviously an equivalence relation. Elements of the corresponding partition

$$\mathbb{Z}^\nu = \bigcup_i \mathbb{L}_i$$

will be called the *$\mathscr{B}$-components of $\mathbb{Z}^\nu$.*

Because of the translation invariance of $\mathscr{B}$ a translate of a $\mathscr{B}$-component is again a $\mathscr{B}$-component. Since $\mathbb{Z}^\nu$ acts on $\{\mathbb{L}_i\}$ in a transitive way the isotropy subgroup, i.e. $\{x \in \mathbb{Z}^\nu : \tau_x \mathbb{L}_i = \mathbb{L}_i\}$, is the same for all $\mathbb{L}_i$. It coincides with the $\mathscr{B}$-component containing 0 and will be denoted by $\mathbb{L}$. We note that $\mathscr{B} = \bigcup_i \mathscr{B}_i$, where $\mathscr{B}_i = \mathscr{B} \cap \mathscr{P}(\mathbb{L}_i)$.

By a theorem on subgroups of $\mathbb{Z}^\nu$ (c.f. [17]) $\mathbb{L}$ is isomorphic to $\mathbb{Z}^\mu$, $1 \leqq \mu \leqq \nu$ and there exists a family $\{f_i\}_{i=1}^\nu$ of generators of $\mathbb{Z}^\nu$ and integers $n_1, \ldots, n_\mu$ such that $\{n_i f_i\}_{i=1}^\mu$ generate $\mathbb{L}$. If $\{\Lambda_n^0\}_{n=1}^\infty$ is a sequence of subsets of $\mathbb{L}$ tending to ∞ in the sense of Van Hove, then

$$\Lambda_n = \bigcup_{\substack{0 \leqq m_i \leqq n_i,\ i=1,\ldots,\mu \\ 0 \leqq m_i < n,\ i=\mu+1,\ldots,\nu}} \left(\Lambda_n^0 + \sum_{m_i} m_i f_i\right)$$

has the same property. From this and from the factorization:

$$M(z_{\mathscr{B}_{\Lambda_n}}) = \prod_m M(z_{\mathscr{B}_{\Lambda_n^m}})$$

where $\Lambda_n^m = \Lambda_n^0 + \sum_i m_i f_i$, follows that if the connected subsystems are essentially regular, then Theorem 2.4 applies.

That essentially two-body systems are essentially regular on the connected components follows from the following:

If $\mathscr{B}$ is connected and purely two-body then $\bar{\mathscr{B}}$ is the family of all the even finite subsets of $\mathbb{Z}^\nu$.

[5] Most of the content of this Section I owe to suggestions of D. Ruelle.

For a proof of the last statement it is enough to show that all the two-element sets are in $\bar{\mathscr{B}}$. To prove that this holds we will show that if $(B_1, \ldots, B_n)$ is a family of bond connecting x with y, $x \neq y$, then there exists a subfamily $(B_{i_1}, \ldots, B_{i_m})$ such that $B_{i_1} \cdot \ldots \cdot B_{i_m} = \{x, y\}$.

We define i_1 as the last element of $(1, \ldots, n)$ for which $x \in \mathscr{B}_{i_1}$. Let $\mathscr{B}_{i_1} = \{x, x_1\}$; if $x_1 = y$ the proof is finished. If not, we remark that, as follows from the definition of i_1, x_1 is connected by $(B_{i_1+1}, \ldots, B_n)$ with y. Now $i_2, \ldots, i_m$ are defined inductively in the same way as i_1.

4.10. *Remarks.* a) No attempt was made here to determine exactly the domains in which the methods of this paper give analyticity and uniqueness. We remark however that any information about absence of zeros of polynomials M yields statements on the uniqueness and analyticity of the correlation functions. For example, calculations of [12] allow to conclude that in the ν-dimensional Ising model there are at most two ergodic equilibrium states for

$$\beta J > (\nu - 1) \log(\sqrt{2} + 1).$$

b) To enlarge the domain of analyticity, or to treat partially antiferromagnetic interactions, one can substitute the variables z_B for only part of $\{e^{-2K(B)}\}$:

Let $\mathscr{A} \subset \mathscr{B}$ and suppose that $\mathscr{B} \subset \bar{\mathscr{A}}$. Then for each $\beta \in \Gamma(\mathscr{A})$ there is only one element $h(\beta)$ of $\Gamma(\mathscr{B})$ such that $h(\beta) \cap \mathscr{A} = \beta$; we set

$$\beta' = h(\beta) \setminus \beta, \qquad c_\beta = \prod_{B \in \beta'} e^{-2K(B)}.$$

Suppose now that $\mathscr{B} = \bigcup_i \mathscr{B}_i$ and that $\mathscr{B}_i \subset \bar{\mathscr{A}}_i$ where $\mathscr{A}_i = \mathscr{A} \cap \mathscr{B}_i$. Then

$$\mathscr{K}(\mathscr{A}) = \left[\bigcup_i \mathscr{K}(\mathscr{A}_i)\right] \quad \text{and} \quad \sum_{B \in \beta'} K(B) = \sum_i \sum_{B \in \beta'_i} K(B)$$

imply that $\sum_{\beta \in \Gamma(\mathscr{A})} c_\beta z^\beta$ is the Asano contraction of $\left\{ \sum_{\beta \in \Gamma(\mathscr{A}_i)} c_{i,\beta} z^\beta \right\}$.

In the case of the Ising model in an external field, one can choose for $\mathscr{A}$ the family of one-point bonds and $\mathscr{B}_i$ of the type $\{\{x\}, \{y\}, \{x, y\}\}$. This leads to the usual treatment that yields uniqueness of the translation invariant equilibrium state at all temperatures.

c) Section 3.4 generalizes in an obvious way to systems for which the interaction is dominated by its ferromagnetic part in the sense that suitable modification of the condition (3.1) in which $\mathscr{I}$ is replaced by $\mathscr{A} \subset \mathscr{B}$, such that $\mathscr{B} \subset \bar{\mathscr{A}}$, holds.

d) One can get the analyticity and uniqueness also for infinite range interactions. Rough estimation based on Section 2.3 needs faster than exponential decrease of J.

e) The notions used in this paper seem to be well suited also for a discussion of the spontaneous magnetization. For instance, the proof of Theorem 5.3.1 of [8] generalizes in a natural way to many body interactions satisfying (3.1).

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Note added in proof: From a recent work by W. Holsztynski (Institute for Advanced Studies) it follows that Theorem 4.4. is true without the assumption of the regularity of $\mathscr{B}$. This extends the results of the present paper to all finite range ferromagnetic interactions.