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On a Question of André Verbeure

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Abstract. We show that a non discrete state of van Hove's Universal Receptacle in the fermion case is not unitarily equivalent to any quasi-free state.

We wish to consider the following question of A. Verbeure which arose during an informal seminar given by J. Manuceau and myself concerning local gauge's implementation.

The question was: Are all the states constructed in Van Hove's Universal Receptacle unitarily equivalent to the quasi-free states? We shall show that the answer to the question is negative.

I. Introduction

Let $\mathscr{A} \equiv \overline{\mathscr{A}(H, s)}$ denote the C.A.R.-algebra. (H, s) is a real separable Hilbert space and $\mathscr{A}$ is generated by the elements $B(\psi)$, $\psi \in H$ with the property that:

$$[B(\psi), B(\varphi)]_+ = 2s(\psi, \varphi)\, I\,,$$

see [1].

We shall call the states (representations) of Van Hove's Universal Receptacle (V.H.U.R.), the states (representations) of $\mathscr{A}$ constructed as follows ([2, 3]).

Let

$$H = \bigoplus_{k \in \mathbb{N}} H_k\,, \qquad H_k = [\psi_k^1, \psi_k^2]$$

$$\pi_k(B(\psi_k^j)) = \sigma^j\,, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

π_k is a $*$-representation of $\mathscr{A}_k \equiv \overline{\mathscr{A}(H_k, s)}$ into $\mathscr{H}_k = \mathbb{C}^2$. Let us define π, a representation of $\mathscr{A}$ into $\mathscr{H} = \bigotimes_{k \in \mathbb{N}} \mathscr{H}_k$

$$\pi(B(\psi_k^j)) = \bigotimes_{l}{}_1^{k-1} \sigma_l^3 \otimes \sigma_k^j \otimes \bigotimes_{l}{}_{k+1}^{\infty} I_l\,,$$

$$\sigma_l^3 = \sigma^3 = -i\sigma^1\sigma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad I_l = I_{\mathbb{C}^2}\,.$$

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We shall write $\pi = \dot{\bigotimes}_{k\in\mathbb{N}} \pi_k$. See [4].

Let Ω_k be a unitary vector in $\mathcal{H}_k$, $\Omega_k = \begin{pmatrix}\alpha_k\\ \beta_k\end{pmatrix} \in \mathbb{C}^2$ and $\Omega = \bigotimes_{k\in\mathbb{N}} \Omega_k$, $\mathcal{H}^\Omega = \bigotimes_{k\in\mathbb{N}}^{\mathscr{C}(\Omega)} \mathcal{H}_k$, $\pi_\Omega(x) = \pi(x)|\mathcal{H}^\Omega$ $\forall x \in \mathscr{A}$. For any decomposable Ω such that Ω_k's are unitary the state $\omega_\Omega = (\Omega | \pi_\Omega(\cdot)\,\Omega)$ will be called V.H.U.R.-state relating to decomposition $(H_k)_{k\in\mathbb{N}}$.

Definition 1.1. A representation π_Ω (a state ω_Ω) is discrete if and only if $\sum_{k\in\mathbb{N}} x_k(1-x_k) < +\infty$ with $x_k = |\alpha_k|^2$.

Definition 1.2. A quasi-free state of $\mathscr{A}$ is a state ω_A fully defined by the operator A on H, such that

$$\omega_A(B(\psi)\,B(\varphi)) = s(\psi, \varphi) + is(A\psi, \varphi)$$

(see [1, 5] ...).

Moreover, if A obeys $A^2 = -1$, then ω_A is a pure state (= Fock state) [1].

II. Proposition

Statement. For any non discrete V.H.U.R.-state ω_Ω, there does not exist any quasi-free state of $\mathscr{A}$ which is unitarily equivalent to ω_Ω.

Proof. Let ω_K be a pure quasi-free state of $\mathscr{A}$, $(\mathcal{H}_K, \pi_K, \Xi_K)$ the corresponding Gelfand troika. $\forall x \in \mathscr{A}$, $\omega_K(x) = (\Xi_K | \pi_K(x)\,\Xi_K)$.

We may construct $(\mathcal{H}_K, \pi_K, \Xi_K)$ as follows:

A collection of two-dimensional spaces F_j exists with $KF_j = F_j$ such that $H = \bigoplus_{j\in\mathbb{N}} F_j$. Let $F_j = [\varphi_j^1, \varphi_j^2]$ and $\pi_j(B(\varphi_j^l)) = \sigma^l$, $l = 1, 2$, $\mathcal{H}_K = \bigotimes_{j\in\mathbb{N}}^{\mathscr{C}(\Xi)} \mathcal{H}_j$, $\mathcal{H}_j = \mathbb{C}^2$ and $\Xi_K = \bigotimes_{j\in\mathbb{N}} \Xi_j$, $\Xi_j = \begin{pmatrix}1\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$ for ω_K is quasi-free [6].

From [7, 8], $\omega_K = \int_\Gamma E_\gamma(\cdot)\,d\mu_K(\gamma)$ where μ_K is a Borel measure of mass one on $\Gamma = \{0, 1\}^{\mathbb{N}}$ equipped with product topology. States E_γ are extremal states constructed from vectors $\Omega = \bigotimes_{j\in\mathbb{N}} \begin{pmatrix}\alpha_j\\ \beta_j\end{pmatrix}$ where $\gamma = (\gamma_j)_j$, $\gamma_j = |\alpha_j|$ and $\gamma_j = 0$ or 1. Here μ_K is concentrated on γ where $\Xi_j = \begin{pmatrix}\Xi_j^1\\ \Xi_j^2\end{pmatrix}$ $\gamma_j = \Xi_j^1 = 0$ or 1.

Let $\hat{\mu}_K$ denote the measure on Γ which corresponds with π_K in the Gårding-Wightman classification [9]. If X is a Borel set in Γ

$$\hat{\mu}_K(X) = \sum_{m_1}^{\infty} 2^{-m} \mu_K(X+\delta^m), \quad \text{where} \quad \{\delta^m\}_{m\in\mathbb{N}} = \Delta$$
$$= \{\gamma \in \Gamma \mid \gamma_j = 0 \text{ but for a finite number of } j\text{'s}\}.$$

$\hat{\mu}_K$ is concentrated on $\gamma + \Delta$.

Now let us consider the representation π_Ω as defined in the introduction, the measure μ_Ω is corresponding to it in the Gårding-Wightman classification. $\mu_\Omega = \bigotimes_{k\in\mathbb{N}} \mu_k$ on $\Gamma = \prod_{k'}^{\infty} \{0,1\}$ with

$$\mu_k(0) = |\beta'_k|^2, \quad \mu_k(1) = |\alpha'_k|^2, \quad \alpha'_k\beta'_k \neq 0 \ \forall k\in\mathbb{N} \quad \text{and} \quad \bigotimes_{k\in\mathbb{N}} \begin{pmatrix}\alpha'_k\\ \beta'_k\end{pmatrix} \sim \bigotimes_{k\in\mathbb{N}} \begin{pmatrix}\alpha_k\\ \beta_k\end{pmatrix}.^{1}$$

Suppose μ_Ω is discrete. Then $\exists\, \gamma' \in \Gamma$ such that $\mu_\Omega(\gamma') > 0$ i.e.

$$\prod_{k\in\mathbb{N}} |\alpha'_k|^{2\gamma_k} |\beta'_k|^{2(1-\gamma_k)} > 0 \quad \text{or} \quad \sum_{k\in\mathbb{N}} x_k(1-x_k) < +\infty,$$

therefore π_Ω is discrete.

Conversely, suppose $\sum_{k\in\mathbb{N}} x_k(1-x_k) < +\infty$.

Then $\exists\, M, N \subset \mathbb{N}$, $M\cap N = \emptyset$, $M \cup N = \mathbb{N}$ such that

$$\sum_{k\in M} x_k < +\infty \quad \text{and} \quad \sum_{k\in N} (1-x_k) < +\infty.$$

Let $\gamma \in \Gamma$ such that $\gamma_k = 0$ if $k\in M$ and $\gamma_k = 1$ if $k \in N$. Then

$$\mu_\Omega(\gamma) = \prod_{k\in\mathbb{N}} x_k^{\gamma_k}(1-x_k)^{(1-\gamma_k)} > 0,$$

hence [9] μ_Ω is discrete.

More, if this condition occurs, μ_Ω is concentrated on $\gamma + \Delta$. In the contrary μ_Ω is a non-atomic measure. So, a non discrete state ω_Ω to which corresponds (via G.N.S.) a Gårding-Wightman measure μ_Ω is unable to be unitarily equivalent to any quasi-free state ω_K, the measure $\hat{\mu}_K$ which corresponds to it being a discrete one, therefore inequivalent to μ_Ω ([10], Proposition 3.6, quoted by [9]).

III. Conclusion

We have exhibited a class of states useful in external field problems ([2, 3]) which are not unitarily equivalent to any quasi-free state.

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[1] $\sim$ is the weak equivalence of C_0-vectors defined by von Neumann [11].

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