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Equilibrium of Charged, Spinning, Magnetic Particles

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Abstract. An unsymmetrical, stationary solution of the Einstein-Maxwell equations is given. The solution corresponds to the exterior field of two massive, charged, magnetised, spinning particles. In general line singularities are present in the solution. The field of N such particles is then considered and necessary and sufficient conditions for the equilibrium of the system are given.

§ 1. Introduction

A method of constructing stationary solutions of the Einstein-Maxwell field equations has recently been given by Perjés [1] and independently by Israel and Wilson [2]. These solutions describe the exterior field of massive, charged, magnetised, spinning particles.

Every particle satisfies (in relativistic units $c=G=1$)

$$m=e, \qquad \boldsymbol{h}=\pm\boldsymbol{\mu} \tag{1}$$

m, e, $\boldsymbol{h}$ and $\boldsymbol{\mu}$ being the mass, charge, three dimensional magnetic moment and angular momentum respectively.

Particular solutions of this class were then given by Bonnor and Ward [3] and by Hartle and Hawking [4]. The solution in [3] gave the exterior field due to two Perjeons (to be defined later). It was found that in general there exist singularities on the line joining the particles.

In this paper I give in § 2 the generalisation, to the case when there is no symmetry, of the Bonnor-Ward solution [3]. The condition for the occurrence of singularities in the more general solution is found to be exactly the same as in the axially-symmetric case. In § 3 I use methods first given by Hartle and Hawking [4] and recently discussed by Israel and Spanos [5] to derive necessary and sufficient conditions for a system composed of N Perjeons (with spins in arbitrary directions) to be in equilibrium. There is also an Appendix.

The field equations used in this work are:

$$\begin{aligned} R_{\mu\nu} &= -8\pi E_{\mu\nu} \\ 4\pi E_{\mu\nu} &= -F_{\mu}{}^{\alpha} F_{\nu\alpha} + \tfrac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \\ F_{\mu\nu;\sigma} + F_{\nu\sigma;\mu} &+ F_{\sigma\mu;\nu} = 0 \\ F^{\mu\nu}{}_{;\nu} &= J^{\mu} \end{aligned} \tag{2}$$

where E^{μ}_{ν} is the electromagnetic energy tensor, $F_{\mu\nu}$ the electromagnetic field tensor and J^{μ} the four-current which vanishes because I consider the exterior field only.

§ 2. Generalisation of the Bonnor-Ward Solution

The prescription for generating a PIW metric has been given in a fairly concise form by Bonnor and Ward, and I reproduce this prescription here.

Latin indices run from 1–3 and Greek from 1–4. All functions are independent of X^4. A comma denotes partial differentiation and a stroke | denotes covariant differentiation with respect to the metric γ_{mn} to be defined now. The metric is

$$ds^2 = -f^{-1} \gamma_{mn}\, dx^m\, dx^n + f(dx^4 + \omega_m\, dx^m)^2 \tag{3}$$

the three dimensional positive definite metric γ_{mn} having zero Ricci-Tensor; the electromagnetic field is given in terms of two scalar potentials:

$$F_{4n} = \phi_{,n}, \qquad F^{ab} = \eta^{abm} f \psi_{,m} \tag{4}$$

η^{abm} being the Levi-civita symbol formed from γ_{mn}. The entire solution is generated by two functions L, M harmonic with respect to γ_{mn}.

$$\gamma^{ab} L_{|ab} = 0, \qquad \gamma^{ab} M_{|ab} = 0 \tag{5}$$

through the equations

$$f = \tfrac{1}{4}(L^2 + M^2)^{-1}, \tag{6}$$

$$\omega_{a,b} - \omega_{b,a} = 8\eta_{abm} \gamma^{mt} (ML_{,t} - LM_{,t}), \tag{7}$$

$$\phi = \frac{-\frac{1}{2} L\varepsilon}{L^2 + M^2}, \qquad \psi = \frac{\frac{1}{2} M\varepsilon}{L^2 + M^2}, \qquad \varepsilon = \pm 1. \tag{8}$$

Define $U = 2L + i2M$ and consider the solutions generated by taking

$$U = \sum_{j=1}^{N} \left\{1 + \frac{m_j}{r_j} + i\left(\frac{\boldsymbol{\mu}_j \cdot \hat{\boldsymbol{r}}_j}{r_j^2}\right)\right\} \tag{9}$$

where

$$\begin{aligned} \boldsymbol{\mu}_j &= (\mu_{j1}, \mu_{j2}, \mu_{j3}) \\ \boldsymbol{r}_j &= (x - x_j, y - y_j, z - z_j), \qquad r_j = |\boldsymbol{r}_j| \end{aligned} \tag{10}$$

and (x, y, z) is the field point and (x_j, y_j, z_j) are the positions of the sources.

Perjés gave the metric corresponding to taking $N = 1$ (Bonnor and Ward call this a Perjeon). Bonnor and Ward gave the metric corresponding to

$$\begin{aligned} N &= 2 \qquad & \boldsymbol{\mu}_1 &= (0, 0, \mu_{13}) \\ & & \boldsymbol{\mu}_2 &= (0, 0, \mu_{23}) \\ x_1 &= y_1 = 0 & z_1 &= -a \\ x_2 &= y_2 = 0 & z_2 &= a \qquad a > 0 \end{aligned} \tag{11}$$

so this solution represents the exterior field of two Perjeons with their spins parallel or anti-parallel to each other and also parallel or anti-parallel to their vector separation. Bonnor and Ward concluded that singularities will occur along the line joining the particles unless

$$m_1 \mu_{23} + m_2 \mu_{13} = 0. \tag{12}$$

Here I give the solution corresponding to

$$\begin{aligned} N &= 2 \qquad & \boldsymbol{\mu}_1 &= (\mu_{11}, \mu_{12}, \mu_{13}) \\ & & \boldsymbol{\mu}_2 &= (\mu_{21}, \mu_{22}, \mu_{23}) \\ x_1 &= y_1 = 0 & z_1 &= -a \\ x_2 &= y_2 = 0 & z_2 &= a \qquad a > 0 \end{aligned} \tag{13}$$

This solution will therefore correspond to the exterior field of two Perjeons with their spins in arbitrary directions. The solution is given by (10) with (3) to (8) and ω is found to be

$$
\begin{aligned}
\omega_1 = &- \frac{m_1 \mu_{12}(z+a)}{r_1^4} - \frac{m_2 \mu_{22}(z-a)}{r_2^4} - \frac{2\mu_{12}(z+a)}{r_1^3} - \frac{2\mu_{22}(z-a)}{r_2^3} \\
&+ \frac{m_1 \mu_{13} y}{r_1^4} + \frac{m_2 \mu_{23} y}{r_2^4} + \frac{2\mu_{13} y}{r_1^3} + \frac{2\mu_{23} y}{r_2^4} \\
&+ \frac{m_2 \mu_{12}}{a r_1^3 r_2}(-x^2+y^2+z^2-a^2) + \frac{2m_2 \mu_{11} xy}{a r_1^3 r_2} \\
&- \frac{m_1 \mu_{22}}{a r_2^3 r_1}(-x^2+y^2+z^2-a^2) - \frac{2m_1 \mu_{21} xy}{a r_2^3 r_1} \\
&+ \frac{m_2 \mu_{13} y}{x^2+y^2}\left\{\frac{r_2}{r_1}\left(\frac{1}{2a^2}\right) + \frac{(z+a)}{a r_1^3 r_2}(x^2+y^2+z^2-a^2)\right\} \\
&+ \frac{m_1 \mu_{23} y}{x^2+y^2}\left\{\frac{r_1}{r_2}\left(\frac{1}{2a^2}\right) - \frac{(z-a)}{a r_2^3 r_1}(x^2+y^2+z^2-a^2)\right\},
\end{aligned}
$$

$$
\begin{aligned}
\omega_2 = &\frac{m_1 \mu_{11}(z+a)}{r_1^4} + \frac{m_2 \mu_{21}(z-a)}{r_2^4} + \frac{2\mu_{11}(z+a)}{r_1^3} + \frac{2\mu_{21}(z-a)}{r_2^3} \\
&- \frac{m_1 \mu_{13} x}{r_1^4} - \frac{m_2 \mu_{23} x}{r_2^4} - \frac{2\mu_{13} x}{r_1^3} - \frac{2\mu_{23} x}{r_2^3} \\
&- \frac{m_2 \mu_{11}}{a r_1^3 r_2}(x^2-y^2+z^2-a^2) - \frac{2m_2 \mu_{12} xy}{a r_1^3 r_2} \\
&+ \frac{m_2 \mu_{21}}{a r_2^3 r_1}(x^2-y^2+z^2-a^2) - \frac{2m_1 \mu_{22} xy}{a r_2^3 r_1} \\
&- \frac{m_2 \mu_{13} x}{x^2+y^2}\left\{\frac{r_2}{r_1}\left(\frac{1}{2a^2}\right) + \frac{(z+a)}{a r_1^3 r_2}(x^2+y^2+z^2-a^2)\right\} \\
&- \frac{m_1 \mu_{23} x}{x^2+y^2}\left\{\frac{r_1}{r_2}\left(\frac{1}{2a^2}\right) - \frac{(z-a)}{a r_2^3 r_1}(x^2+y^2+z^2-a^2)\right\},
\end{aligned}
$$

$$
\begin{aligned}
\omega_3 = &- \frac{m_1 \mu_{11} y}{r_1^4} - \frac{m_2 \mu_{21} y}{r_2^4} - \frac{2\mu_{11} y}{r_1^3} - \frac{2\mu_{21} y}{r_2^3} \\
&+ \frac{m_1 \mu_{12} x}{r_1^4} + \frac{m_2 \mu_{22} x}{r_2^4} + \frac{2\mu_{12} x}{r_1^3} + \frac{2\mu_{22} x}{r_2^3} \\
&- \frac{2m_2 \mu_{12} xz}{a r_1^3 r_2} + \frac{2m_1 \mu_{22} xz}{a r_2^3 r_1} \\
&+ \frac{2m_2 \mu_{11} yz}{a r_1^3 r_2} - \frac{2m_1 \mu_{21} yz}{a r_2^3 r_1}.
\end{aligned} \tag{14}
$$

A look at the solution will easily convince the reader that the condition which excludes singularities on the line joining the particles is exactly the same as in the axially symmetric case, i.e. $m_2 \mu_{13} + m_1 \mu_{23} = 0$, or in vector notation (since this result must be true for arbitrary orientation of the line joining the particles)

$$\boldsymbol{r}_{12} \cdot (m_2 \boldsymbol{\mu}_1 + m_1 \boldsymbol{\mu}_2) = 0 . \tag{15}$$

The result is that if the angular momenta per unit mass ($\boldsymbol{\mu}_1/m_1, \boldsymbol{\mu}_2/m_2$ respectively) are equal and opposite, or if the vector $m_2 \boldsymbol{\mu}_1 + m_1 \boldsymbol{\mu}_2$ is perpendicular to the vector separation of the particles then the singularity disappears [1].

§ 3. Equilibrium Conditions for *N* Perjeons

Hartle and Hawking showed that a necessary condition for regularity of the exterior metric is that [2]:

$$\int_S (U^* \nabla U - U \nabla U^*) \cdot \boldsymbol{n} \, dS = 0 \tag{16}$$

where S is any closed two-surface in the background Euclidean 3-space. More recently Israel and Spanos [5] have argued that (16) is also sufficient for regularity in the exterior geometry (assuming U is regular and non-vanishing in the exterior).

It is more convenient for our purposes to amend (16) to the equivalent statement (via Gauss' divergence theorem)

$$\int_V (U^* \nabla^2 U - U \nabla^2 U^*) \, dV = 0 \tag{17}$$

where V is the volume enclosed by S. We shall consider the case in which

$$U = \sum_{j=1}^{N} \left\{ 1 + \frac{m_j}{r_j} + i \left(\frac{\boldsymbol{\mu}_j \cdot \hat{\boldsymbol{r}}_j}{r_j^2} \right) \right\}$$

---

[1] The solution given in (14) is unique only up to the addition of the gradient of a scalar. In (14) the line singularity shows up in the ω_1, ω_2, terms and occupies the whole of the line $x = y = 0$ in the background 3-space. By choosing the scalar carefully the singularity can be chosen to lie only on the section of z-axis between the particles. If $\omega_1, \omega_2, \omega_3$ are expanded in powers of $R^{-1} = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$ and if terms of order "a" are ignored then the solution describes the field of a single Perjeon of mass $m_1 + m_2$ and angular momentum $\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2$ (cf. Bonnor and Ward [3], Eq. (3.4)).

[2] U^* is the complex conjugate of U. ∇ is the usual gradient operator and ∇^2 is the Laplacian.

representing the field of N Perjeons with spins in arbitrary directions. It is then fairly easily shown (outline given in Appendix) that the condition (17) implies:

$$\sum_{\substack{j=1 \\ i \neq j}}^{N} \left\{ \frac{(m_i \boldsymbol{\mu}_j + m_j \boldsymbol{\mu}_i) \cdot \hat{\boldsymbol{r}}_{ij}}{|\boldsymbol{r}_{ij}|^2} \right\} = 0 \tag{18}$$

where the volume V was chosen to enclose the i'th particle. We also note that choosing $N=2$ and $i=1$ then (18) reduces to

$$(m_1 \boldsymbol{\mu}_2 + m_2 \boldsymbol{\mu}_1) \cdot \hat{\boldsymbol{r}}_{12} = 0$$

as of course it should for the two particle case. For equilibrium of the whole system (18) should be satisfied for each particle.

Appendix

For convenience I will consider the case of two Perjeons only (the N-particle case is evaluated in exactly the same way but the notation is clumsy).

Thus in (10) take $N=2$, i.e.

$$U = 1 + \frac{m_1}{r_1} + \frac{m_2}{r_2} + i\left(\frac{\boldsymbol{\mu}_1 \cdot \hat{\boldsymbol{r}}_1}{r_1^2} + \frac{\boldsymbol{\mu}_2 \cdot \hat{\boldsymbol{r}}_2}{r_2^2}\right).$$

The following statements about delta-functions will be required:

$$\nabla^2 \left(\frac{1}{r_1}\right) = -4\pi\, \delta(\boldsymbol{r}_1) = -4\pi\, \delta(x-x_1)\, \delta(y-y_1)\, \delta(z-z_1) \tag{A.1}$$

$$\nabla^2 \left(\frac{x-x_1}{r_1^3}\right) = 4\pi\, \delta'(x-x_1)\, \delta(y-y_1)\, \delta(z-z_1) \stackrel{\text{def}}{=} 4\pi\, \delta_x(r_1) \tag{A.2}$$

$$\frac{\delta'(x)}{x} = -\frac{\delta(x)}{x^2}, \qquad \delta'(-x) = -\delta'(x) \tag{A.3}$$

$$\int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx = f(0), \qquad \int_{-\infty}^{\infty} f(x)\, \delta'(x)\, dx = -f'(0) \tag{A.4}$$

where

$$f'(x) \equiv \frac{df}{dx}.$$

(A.1) to (A.3) are a sub-section of the usual statements made about the delta-function. It should be remembered that these statements are only valid when the appropriate integral is taken. A very full account of the delta-function and its properties can be found in Ref. [6].

Thus using (A.1) and (A.2), Eq. (17) implies directly:

$$\int_V \left(1+\frac{m_1}{r_1}+\frac{m_2}{r_2}\right)(\mu_{1\,1}\,\delta_x(r_1)+\mu_{1\,2}\,\delta_y(r_1)+\mu_{1\,3}\,\delta_z(r_1))\,dV$$

$$+\int_V m_1\,\delta(r_1)\left(\frac{\mu_{1\,1}(x-x_1)+\mu_{1\,2}(y-y_1)+\mu_{1\,3}(z-z_1)}{r_1^3}\right)dV \quad \text{(A.5)}$$

$$+\int_V m_1\,\delta(r_1)\left(\frac{\mu_{2\,1}(x-x_2)+\mu_{2\,2}(y-y_2)+\mu_{2\,3}(z-z_2)}{r_2^3}\right)dV=0$$

where I have chosen V to include only the particle at $(0, 0, -a)$ now

$$\int_V \frac{m_1\,\mu_{1\,1}\,\delta_x(r_1)}{r_1}\,dV=\int_V \frac{m_1\,\mu_{1\,1}\,\delta'(x-x_1)\,\delta(y-y_1)\,\delta(z-z_1)}{r_1}\,dx\,dy\,dz$$

the "y" and "z" dependence can be taken out immediately and using (A.4) we have

$$\int_V \frac{m_1\,\mu_{1\,1}\,\delta_x(r_1)}{r_1}\,dV=\int_x \frac{m_1\,\mu_{1\,1}\,\delta'(x-x_1)}{|x-x_1|}\,dx$$

thus (A.5) implies

$$\int_X \frac{m_1\,\mu_{1\,1}\,\delta'(x-x_1)\,dx}{|x-x_1|}+\text{“}y\text{”, “}z\text{” terms}$$

$$+\int_X \frac{m_2\,\mu_{1\,1}\,\delta'(x-x_1)\,dx}{\{(x-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2\}^{\frac{1}{2}}}+\text{“}y\text{”, “}z\text{” terms}$$

$$+\int_X \frac{m_1\,\mu_{1\,1}(x-x_1)\,\delta(x-x_1)\,dx}{|x-x_1|^3}+\text{“}y\text{”, “}z\text{” terms} \quad \text{(A.6)}$$

$$+\frac{m_1\,\mu_{2\,1}(x_1-x_2)}{r_{1\,2}^3}+\text{“}y\text{”, “}z\text{” terms}\quad =0$$

where

$$r_{1\,2}=+\{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2\}^{\frac{1}{2}}$$

from (A.3) it follows that

$$\frac{\delta'(x-x_1)}{|x-x_1|}=\frac{-(x-x_1)\,\delta(x-x_1)}{|x-x_1|^3} \quad \text{(A.7)}$$

thus we finally get using (A.4) and (A.7) in (A.6) (after some re-arrangement)

$$\hat{\boldsymbol{r}}_{1\,2}\cdot(m_2\,\boldsymbol{\mu}_1+m_1\,\boldsymbol{\mu}_2)=0\,.$$

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