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Hamiltonian Formalism for Non-invariant Dynamics*

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Abstract. In a completely Hamiltonian dynamical system, there will be a generating function H_Y for each infinitesimal space-time transformation Y. In the non-autonomous case, the H_Y depend on the observer. This dependence is here described by a system of commutation relations. It is also shown that these relations can be made to mirror exactly the commutation relations of the Y's in the Lorentz-invariant case.

1. Introduction

A system involving an external field can be enlarged to a completely autonomous system whenever a transformation-law for the field is specified. By a method presented below we can enlarge any non-autonomous system to an "augmented" system which is completely autonomous. When the original system preserves some Hamiltonian structure, an analysis of the augmented system leads to the results stated about the H_Y. The systems considered are of the classical type in that there are only finitely many degrees of freedom but are more general in that the entire space-time group, and not merely temporal changes as in classical dynamics, are allowed as changes in observes.

For any given infinitesimal change of observer, the generating function H_Y is almost unique: unique up to an additive term independent of the dynamical coordinates. If such terms are improperly adjusted, then the relation $\{H_Y, H_Z\} = H_{[Y,Z]}$, expected in the invariant case, may not hold. In the Lorentz-invariant case we show that these relations can be achieved.

2. Space-time, Coordinators, and Dynamics

For any discussion of dynamical systems, one must have a space-time manifold M, and M must have a space-time structure. The latter includes (usually tacitly) a differentiable-manifold structure (usually 4-dimen-

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sional), and a system of tensor fields on M (cf. [2, (2.2)]). Perhaps even an affine space structure [7, p. 350] may be included. Here we always suppose M to be 4-dimensional, and what is much more serious, we suppose there is at least one coordinate system in M *valid on all of* M and mapping M onto $\mathbb{R}^4$ (Cartesian 4-space). This carries the space-time structure over to $\mathbb{R}^4$ and these tensor fields, etc., constitute the *space-time structure of* $\mathbb{R}^4$. The mappings of M onto itself which preserve its space-time structure constitute the space-time group $\mathscr{G}$. It should be conceived as distinct, although obviously isomorphic, to the group of transformations of $\mathbb{R}^4$ which preserve the structure there. In the Einstein-Minkowski case it is supposed that the latter structure (in $\mathbb{R}^4$) consists just of the metric

$$(dx^4)^2-(dx^1)^2-(dx^2)^2-(dx^3)^2\,.$$

The space-time group of $\mathbb{R}^4$ in this case is usually called the Poincaré group $\mathscr{P}$.

Any map $\boldsymbol{x}: M \to \mathbb{R}^4$ which carries the structure in M onto that of $\mathbb{R}^4$ shall be called a *coordinator*. Evidently a coordinator is a coordinate system for M, but of a special sort, e.g. Lorentz coordinate system. The class of coordinators will be denoted by $\mathscr{C}$. If T belongs to $\mathscr{G}$, S to the space time group G of $\mathbb{R}^4$ and $\boldsymbol{x}$, $\boldsymbol{y}$ are coordinators, then $\boldsymbol{x}\circ T$ and $S\circ\boldsymbol{y}$ belong to $\mathscr{C}$, $\boldsymbol{x}\circ\boldsymbol{y}^{-1}$ to $\mathscr{G}$, and $\boldsymbol{y}^{-1}\circ\boldsymbol{x}$ to G. Thus $\mathscr{G}$ "acts" on the left in $\mathscr{C}$ and G, on the right [6, p. 294].

A dynamical system consists of three things: $\mathscr{C}$, K, Δ. Here $\mathscr{C}$ is the class of coordinators, K is a manifold called the space of states, and Δ is the *dynamics*, namely a family, $\{\Delta_{\boldsymbol{x}}^{\boldsymbol{y}}\}$, indexed by all possible pairs of coordinators of mappings of K onto K, satisfying [2; 2.4, 2.41]

$$\Delta_{\boldsymbol{y}}^{\boldsymbol{z}}\circ\Delta_{\boldsymbol{x}}^{\boldsymbol{y}}=\Delta_{\boldsymbol{x}}^{\boldsymbol{z}}\,, \tag{2.1}$$

$$\Delta_{\boldsymbol{y}}^{\boldsymbol{x}} \textit{ is the inverse of } \Delta_{\boldsymbol{x}}^{\boldsymbol{y}}\,. \tag{2.2}$$

A *motion* κ is a function defined on $\mathscr{C}$ with values in K satisfying the condition $\kappa(\boldsymbol{y})=\Delta_{\boldsymbol{x}}^{\boldsymbol{y}}(\kappa(\boldsymbol{x}))$ for all $\boldsymbol{x}$, $\boldsymbol{y}$ in $\mathscr{C}$. Given $\boldsymbol{x}$ in $\mathscr{C}$ and k in K one can define a motion κ such that $\kappa(\boldsymbol{x})=k$ by setting $\kappa(\boldsymbol{y})=\Delta_{\boldsymbol{x}}^{\boldsymbol{y}}(k)$. This motion will be denoted by $\Delta_{\boldsymbol{x}}(k)$. In an n-particle interaction [2, p. 157] the given k would be the n positions and n velocities while $\Delta_{\boldsymbol{x}}(k)$ would essentially be the set of n world lines (in M) representing that movement of those n particles ensuing from the initial conditions ($x^4=0$) given by k.

We will say that an element T of $\mathscr{G}$ is an *equivalence* for the dynamics Δ if for every two coordinators $\boldsymbol{x}$, $\boldsymbol{y}$

$$\Delta_{\boldsymbol{x}\circ T}^{\boldsymbol{y}\circ T}=\Delta_{\boldsymbol{x}}^{\boldsymbol{y}}\,.$$

If κ is a motion (for Δ) and T belongs to $\mathscr{G}$ we can define $T\kappa$ by $T\kappa(\boldsymbol{x}) = \kappa(\boldsymbol{x} \circ T)$. This is a function from $\mathscr{C}$ to K, but it is not necessarily a *motion* for Δ. However, it will be if T is an equivalence for Δ.

More precisely

(2.3) *if T is an equivalence then* $T^{-1}(\Delta_{\boldsymbol{x}} k) = \Delta_{\boldsymbol{x} \circ T}(k)$.

A dynamical system will be called *completely autonomous* if every T in $\mathscr{G}$ is an equivalence[1]. A simple test is the following.

(2.4) *The dynamics Δ is completely autonomous if and only if $\Delta_{\boldsymbol{x}}^{\boldsymbol{y}}$ depends only on $\boldsymbol{y} \circ \boldsymbol{x}^{-1}$.*

This latter map $\boldsymbol{y} \circ \boldsymbol{x}^{-1}$ belongs to the space-time group G of $\mathbb{R}^4$.

(2.5) **Theorem.** *Suppose Δ is completely autonomous and κ is a motion. Let T and T_1 belong to $\mathscr{G}$. Then $T\kappa$ is also a motion and $T_1(T\kappa) = (T_1 \circ T)(\kappa)$.*

Proof. The assertion about $T\kappa$ is essentially (2.3). The second is proved as follows. For $\boldsymbol{x}$ in $\mathscr{C}$, $T_1(T\kappa)(\boldsymbol{x}) = T\kappa(\boldsymbol{x} \circ T_1) = \kappa(\boldsymbol{x} \circ T_1 \circ T) = (T_1 \circ T)(\kappa)(\boldsymbol{x})$.

Thus $\mathscr{G}$ acts in the space of motions in the completely autonomous case, just as the space-time group of $\mathbb{R}^4$, G, acts in the space K of states [2, p. 158].

These two actions of these two different groups are related. For, select a coordinator $\boldsymbol{x}$. Then $\Delta_{\boldsymbol{x}}$ maps K onto the space of motions, while the map $S \to \boldsymbol{x}^{-1} \circ S \circ \boldsymbol{x}$ establishes an isomorphism of G onto $\mathscr{G}$. These two maps are linked as follows.

(2.6) **Proposition.** $\Delta_{\boldsymbol{x}}(Sk) = (\boldsymbol{x}^{-1} \circ S \circ \boldsymbol{x})(\Delta_{\boldsymbol{x}} k)$.

Proof. $Sk = \Delta_{S^{-1} \circ \boldsymbol{x}}^{\boldsymbol{x}} k$ by [2, p. 158], so the left side is $\Delta_{S^{-1} \circ \boldsymbol{x}}(k)$. This is exactly what 2.3 yields for the right side.

While it takes a coordinator to relate them, neither action requires the help of any coordinator.

When computing, it is much more convenient to deal with this action of G in K rather than that of $\mathscr{G}$ in the space of motions. In fact, working with G is inevitable because the only effective way of introducing coordinates into the space of motions is through the maps $\Delta_{\boldsymbol{x}}$ in (2.6), while for $\mathscr{G}$ coordinates are best provided through the isomorphisms given in 2.6.

These remarks concern only completely autonomous systems.

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[1] In [2], such a system is called *invariant*. We change the terminology to prevent confusion with the concept of *invariant of the dynamics* which is meaningful even when the dynamics is not completely autonomous.

We turn to the concept of *invariants*. We attempt no all-inclusive definition. However, suppose Ω is a tensor field on K. A dynamorphism Δ_x^y is a differentiable map of K onto K and there is presumably (from the definition of tensor field) an appropriate concomitant way for Δ_x^y to transform Ω, say to Ω'. If this Ω' is always the same as Ω, then Ω is a *dynamical invariant*. This can happen even in the non-autonomous case. The main example of such an invariant is that of an invariant closed non-degenerate 2-form. A closed non-degenerate (or non-singular) 2-form is also called a *symplectic structure* or a Hamiltonian structure [6, pp. 144–147]. A system will be called *completely Hamiltonian* if there is on its space of states a symplectic structure which is dynamically invariant.

It was intended in [2, 4.3] to present an example of such a system. Actually, there is a removable defect in the presentation and this is a good occasion to straighten it out. The error consisted failing to observe that time-reversal [2, p. 172] does not preserve the symplectic structure

(2.7) $dp_i \wedge dq^i$,

but changes its sign. So the system is not completely Hamiltonian, although it surely is Poincaré invariant as asserted. This failure occurs even in the most elementary systems, such as one or more free particles [1, p. 273]. The difficulty is essentially terminological and could be avoided by using a term like $\mathscr{P}_0$-Hamiltonian where $\mathscr{P}_0$ (in this case the orthochronous Poincaré group) is the subgroup which does preserve the symplectic structure. Another way out would be to allow changes of sign of the symplectic structure, but that would conflict with established terminology. The simplest thing is to include in the space-time structure a *sense* (or direction) *of time*. Having done that, the coordinators have to take this into account, with the result

(2.8) *G will not contain time-reversal*.

Amended in this sense, [2, 4.3] does present a completely Hamiltonian system, although now it has more invariance (or autonomy) than is being claimed. We will adhere to the sense of time idea leading to 2.8 in this paper.

In the classical case, the space time group is implicitly limited to a one-dimensional group ("time translations") and for a Hamiltonian system, only this group is required to preserve a symplectic structure.

In the classical case, there is a *generating function* for the infinitesimal time translation. This is the *Hamiltonian H*, or sometimes its negative depending on the exact definition of the Poisson bracket. In the situation to be treated here there will be a generating function for every infinitesimal space-time transformation. It is our intent to investigate the Poisson brackets of these generating functions, in the autonomous as well as the general case. Our method is to "imbed" the system in a completely autonomous system. This construction is the topic of the next section.

3. Semidirect Product of Two Systems

We are about to consider a great number of different dynamical systems. The class $\mathscr{C}$ of coordinators will be the same for all.

Suppose we have one system $(\mathscr{C}, F, \Gamma)$ with state space F and dynamics Γ. Let Φ be the class of all motions of this system.

Now suppose that for each motion φ of this system there is given a system $(\mathscr{C}, K, {}_{\varphi}\Delta)$. Here the state-space is the same for all φ, but the dynamics may vary with φ, as indicated. The example we have in mind is that $(\mathscr{C}, K, {}_{\varphi}\Delta)$ is the system provided by a charged particle moving in an electro-magnetic field φ. In order to present this example explicitly one would have to construct a system $(\mathscr{C}, F, \Gamma)$ such that its motions correspond (preferably in some natural fashion) to the collection of electromagnetic fields desired. For example, one can proceed as follows. Let F be the class of solutions to Maxwell's equations in $\mathbb{R}^4$. Given an f from F and a pair of coordinators $\boldsymbol{x}$, $\boldsymbol{y}$ we form $S = \boldsymbol{x} \circ \boldsymbol{y}^{-1}$, which is a map from $\mathbb{R}^4$ to $\mathbb{R}^4$. (In fact, it belongs to G.)

Now f is a tensor field, and hence there is a standard way in which S transforms it (for scalar fields, the new field is $f \circ S$ and this idea is escalated to affine tensor fields). The new field we call $\Delta_{\boldsymbol{x}}^{\boldsymbol{y}}(f)$, for brevity.

In this example we would naturally use the Einstein Minkowski space-time structure to define $\mathscr{C}$. Thus one can set up the system of Maxwell's equations on M, and class Φ of solutions φ. These solutions are practically the same as a motion of the system $(\mathscr{C}, F, \Gamma)$. In fact, given a solution φ, we can define for each $\boldsymbol{x}$ in $\mathscr{C}$, the field $\boldsymbol{x}(\varphi)$ in $\mathbb{R}^4$ obtained by transference from φ. Setting $\bar{\varphi}(\boldsymbol{x}) = \boldsymbol{x}(\varphi)$ defines a motion $\bar{\varphi}$. Thus it makes very little difference whether you say that the motion of the charged particle is conditioned by the field φ or some motion of the external system $(\mathscr{C}, F, \Gamma)$.

We leave this example now and continue the general discussion.

We can form a system with $F \times K$ as its space of states, and the following dynamics Θ:

$$\text{(3.1)} \quad \Theta_{\boldsymbol{x}}^{\boldsymbol{y}}(f, k) = (\Gamma_{\boldsymbol{x}}^{\boldsymbol{y}} f, {}_{\Gamma_{\boldsymbol{x}} f}\Delta_{\boldsymbol{x}}^{\boldsymbol{y}} k)\,.$$

Here $\Gamma_{\boldsymbol{x}} f$ is the motion of Γ which has at $\boldsymbol{x}$ the value f. It is easily verified that $\Theta_{\boldsymbol{y}}^{\boldsymbol{z}} \circ \Theta_{\boldsymbol{x}}^{\boldsymbol{y}} = \Theta_{\boldsymbol{x}}^{\boldsymbol{z}}$. For want of a better name, let us call this the *semi-direct product* system.

When this construction is considered for the charged particle in a field, the semi-direct product system is completely autonomous. This is the reason for making this semi-direct product, and so we will trace the origin of the autonomy. We make an additional assumption.

(3.2) *For each T in the space-time group, for each motion φ of Γ and each pair of coordinators,*

$$_{\varphi}\Delta^{\boldsymbol{y}\circ T}_{\boldsymbol{x}\circ T} = {}_{T\varphi}\Delta^{\boldsymbol{y}}_{\boldsymbol{x}} \,.$$

(3.3) *Remark.* By choosing a coordinator, say $\boldsymbol{w}$, this condition may be expressed using the action of G in the space of states F, with a labeling of the dynamics by the elements f of F rather than by the motions φ. Denoting $_{\varphi}\Delta$ by $_{\boldsymbol{w}^{-1}(\varphi)}E$, the condition reads

$$_{f}E^{\boldsymbol{v}\circ S\circ \boldsymbol{w}}_{\boldsymbol{u}\circ S\circ \boldsymbol{w}} = {}_{S(f)}E^{\boldsymbol{v}\circ \boldsymbol{w}}_{\boldsymbol{u}\circ \boldsymbol{w}} \,.$$

This is very unwieldy, since the special coordinator is still in evidence. This is partly why we decided the dynamics ought to be labeled by the *motions*, rather than the *states* of $(\mathscr{C}, F, \Gamma)$. This decision made it impossible to formulate the theory in terms of only one the group G of space-time transformations in $\mathbb{R}^4$.

Condition 3.2 is designed to produce the following result.

(3.4) **Theorem.** *If the dynamics of Γ is completely autonomous, then so is the semi-direct product.*

(3.5) In greater detail, *if T is an equivalence of Γ then T is also an equivalence of the semi-direct product.*

To prove 3.4, we must show that $\Theta^{\boldsymbol{y}\circ T}_{\boldsymbol{x}\circ T}(f, k) = \Theta^{\boldsymbol{y}}_{\boldsymbol{x}}(f, k)$. The left side is $(\Gamma^{\boldsymbol{y}\circ T}_{\boldsymbol{x}\circ T} f, {}_{\Gamma_{\boldsymbol{x}\circ T}}\Delta^{\boldsymbol{y}\circ T}_{\boldsymbol{x}\circ T} k)$, by definition. T being an equivalence of Γ makes the first component here equal to $\Gamma^{\boldsymbol{y}}_{\boldsymbol{x}} f$, while 2.3 changes the second component to $_{T^{-1}(\Gamma_{\boldsymbol{x}} f)}\Delta^{\boldsymbol{y}\circ T}_{\boldsymbol{x}\circ T} k$. The latter is $_{\Gamma_{\boldsymbol{x}} f}\Delta^{\boldsymbol{y}}_{\boldsymbol{x}} k$ by virtue of (3.3). Thus the left side does not depend on T and (3.4) is established.

If we have a system $(\mathscr{C}, F, \Gamma)$ and a collection of systems $(\mathscr{C}, K, {}_{\varphi}\Delta)$ as above, and if Γ is completely autonomous, and moreover, (3.2) holds, we will say that we have a *field system.*

The concept of a field system enables us to justify precisely the intuitive idea that if a system $(\mathscr{C}, K, \Delta)$ is not completely autonomous, then there must be some external field, and that if we make a larger system in which the field is also transformed, the desired complete autonomy can be attained.

A system $(\mathscr{C}, F, \Gamma)$ which can be used in all cases, is the following. The space of states F is simply $\mathscr{C}$ itself. For $\boldsymbol{x}$, $\boldsymbol{y}$ in $\mathscr{C}$ and the "state" $\boldsymbol{z}$ (also a point of $\mathscr{C}$) we define $\Gamma^{\boldsymbol{y}}_{\boldsymbol{x}}(\boldsymbol{z}) = \boldsymbol{y} \circ \boldsymbol{x}^{-1} \circ \boldsymbol{z}$. Not only (2.1), (2.2) are satisfied, but the dynamics is completely autonomous, by (2.4).

(3.6) **Proposition.** *For* T *in* $\mathscr{G}$ *define* $\varphi(\boldsymbol{x}) = \boldsymbol{x} \circ T$ *for each* $\boldsymbol{x}$ *in* $\mathscr{C}$. *Then* φ *is a motion of* $(\mathscr{C}, \mathscr{C}, \Gamma)$. *Each motion is given in this way, and with a unique* T.

Proof. Let φ be a motion. Then $\varphi(\boldsymbol{y}) = \Gamma_{\boldsymbol{x}}^{\boldsymbol{y}}(\varphi(\boldsymbol{x}))$ or $\boldsymbol{y}^{-1} \circ \varphi(\boldsymbol{y}) = \boldsymbol{x}^{-1} \circ \varphi(\boldsymbol{x})$. Hence there is a unique T (an element of $\mathscr{G}$) such that $\boldsymbol{x}^{-1} \circ \varphi(\boldsymbol{x}) = T$ for all $\boldsymbol{x}$ in $\mathscr{C}$. That is clearly what is desired.

To continue assembling a field system, we must invent a family of dynamics $(\mathscr{C}, K, {}_{\varphi}\Delta)$. K is the given space of states. Using the given dynamics Δ, we define ${}_{\varphi}\Delta_{\boldsymbol{x}}^{\boldsymbol{y}}$ to be $\Delta_{\varphi(\boldsymbol{x})}^{\varphi(\boldsymbol{y})}$. This is meaningful because φ is a function defined on $\mathscr{C}$, with values in $\mathscr{C}$, so that $\varphi(\boldsymbol{x})$, $\varphi(\boldsymbol{y})$ are coordinators.

Now we check (3.2). We know ${}_{\varphi}\Delta_{\boldsymbol{x} \circ T}^{\boldsymbol{y} \circ T} = \Delta_{\varphi(\boldsymbol{x} \circ T)}^{\varphi(\boldsymbol{y} \circ T)}$ and ${}_{T\varphi}\Delta_{\boldsymbol{x}}^{\boldsymbol{y}} = \Delta_{T\varphi(\boldsymbol{x})}^{T\varphi(\boldsymbol{y})}$. According to Section 1, $T\varphi$ is defined by $T\varphi(\boldsymbol{x}) = \varphi(\boldsymbol{x} \circ T)$. Thus (3.2) holds, and we have a field system. In this case, we will call the semi-direct product system the *augmented system*, the idea being that $(\mathscr{C}, K, \Delta)$ has been augmented by including a hypothetical field.

The augmented system being completely autonomous, we have an action of $\mathscr{G}$ in its space of motions. As stated in Section 2, it is more convenient to deal with an action of G in the space of states. This space of states is $\mathscr{C} \times K$, and the action of G, when the notation is untangled, emerges as follows.

(3.7) **Theorem.** *In the augmented system, the effect of* S *(a member of* G*) on a state* $(\boldsymbol{x}, k)$ *is to transform it into* $(S \circ \boldsymbol{x}, \Delta_{\boldsymbol{x}}^{S \circ \boldsymbol{x}} k)$.

This latter we abbreviate by $S(\boldsymbol{x}, k)$.

Obviously, G does not act transitively in $\mathscr{C} \times K$. Indeed the dimension of $\mathscr{C} \times K$ exceeds that of G exactly by the dimension of K. The orbits of an action are the subsets of the space in question on which G acts transitively.

(3.8) **Proposition.** *The orbits for the action of* G *in* $\mathscr{C} \times K$ *are in* 1 : 1 *correspondence with the motions of the original system* $(\mathscr{C}, K, \Delta)$.

Proof. To each point $(\boldsymbol{x}, k)$ of $\mathscr{C} \times K$ we assign the motion whose value at $\boldsymbol{y}$ in $\mathscr{C}$ is $\Delta_{\boldsymbol{x}}^{\boldsymbol{y}}(k)$. This rule assigns to $S(\boldsymbol{x}, k)$ the motion whose value at $\boldsymbol{y}$ is $\Delta_{S \circ \boldsymbol{x}}^{\boldsymbol{y}}(\Delta_{\boldsymbol{x}}^{S \circ \boldsymbol{x}} k)$, i.e., the same motion as before. Thus all points of this orbit determine the same motion. This suffices to establish (3.8).

The action of G here is not to be confused with the action of G in the state-space K (which in fact arises only in the autonomous case). This latter action is the analogue of the action of $\mathscr{G}$ on the motions. In the action of G in $\mathscr{C} \times K$, the motions, being the orbits, are invariant.

Having properly made the distinction between G and $\mathscr{G}$ and having pointed out where each group acts in a coordinator-free way, but wishing next to consider more technical matters, we take M to be $\mathbb{R}^4$ itself.

Then $\mathscr{G}=\mathscr{C}=G$. The action of $\mathscr{G}$ in $\mathscr{C}$ is left multiplication; the action of G in $\mathscr{C}$ is right multiplication. The action of G in $\mathscr{C}\times K$ for the augmented system is given by

$$(3.9)\quad S(T,k)=(S\circ T, \Delta_T^{S\circ T}k).$$

There is no loss of mathematical generality if only systems of this special sort are considered.

4. The Infinitesimal Dynamorphisms

Suppose G is a Lie group, and let its Lie algebra be denoted by $\mathfrak{g}$. To be precise, let $\mathfrak{g}$ be the linear space of right invariant vector fields on G. (These are infinitesimal *left*-multiplications.) For Y in $\mathfrak{g}$ one has the one-parameter subgroup $s\to\exp sY$. For a given T in G and k in K consider the curve

$$f(s)=\Delta_T^{(\exp sY)\circ T}(k)$$

in K. Its tangent $f'(0)$ at k we denote by $\Delta_{Y,T}(k)$, which is a slight change from [2, (2.63)]. Thus $\Delta_{Y,T}$ is a vector field on K. If the dynamics is completely autonomous, this vector field is the same for all T, and one has the commutation relation

$$(4.1)\quad [\Delta_{Y,T},\Delta_{Z,T}]=\Delta_{[Y,Z],T}\,;\ \Delta_{Y+Z,T}=\Delta_{Y,T}+\Delta_{Z,T}\,.$$

Conversely, if vector fields $\Delta_{Y,T}$ independent of T are given on K, satisfying (4.1), then dynamorphisms Δ_S^T can be found, at least for all $T\circ S^{-1}$ sufficiently close to the identity element of G, such that the corresponding infinitesimal dynamorphisms are just these given $\Delta_{Y,T}$.

The question was raised in [2, loc. cit.] what happens to the condition (4.1) in the non-autonomous case. This question we now answer.

Each Y in $\mathfrak{g}$ is a vector field on G. For each k in K, there is a mapping F_k of G into $G\times K$ with $F_k(T)=(T,k)$ and this mapping transforms Y into a vector field on $G\times\{k\}$. Varying k we obtain a vector field $\bar{Y}$ on $G\times K$. Similarly, for each T define $F_T(k)=(T,k)$. This F_T transfers $\Delta_{Y,T}$ to $\{T\}\times K$. Varying T gives a vector field Δ_Y on $G\times K$. In a sense, $\bar{Y}$ is parallel to the G axis and Δ_Y is parallel to the K axis: a vector V in $G\times K$ with base point (T,k) is called "parallel to the K axis" if it is tangent to the subspace $\{T\}\times K$. One might also say that the G component is 0.

Now G acts in $G\times K$ (see (3.9)) and $\bar{Y}+\Delta_Y$ is the infinitesimal action due to Y, in $G\times K$. Because we have an action, we must have

$$(4.2)\quad [\bar{Y}+\Delta_Y,\bar{Z}+\Delta_Z]=\overline{[Y,Z]}+\Delta_{[Y,Z]}$$

for each pair Y, Z in $\mathfrak{g}$. This is essentially the condition replacing (4.1). It can be given a little more useful form. It is obvious that $[\bar{Y}, \bar{Z}] = \overline{[Y, Z]}$, whence

$$(4.3) \quad [\Delta_Y, \bar{Z}] + [\bar{Y}, \Delta_Z] + [\Delta_Y, \Delta_Z] - \Delta_{[Y,Z]} = 0\,.$$

Now the components of $\bar{Y}$, $\bar{Z}$ do not depend on k. Hence the vector field on the left of (4.3) is parallel to the K axis. To see this one must think of vector fields as linear differential operators [6, pp. 90–94]. What we are saying is that if f is a function on $G \times K$ and $f(T, k)$ depends only on T, and if the linear differential operator on the left side of (4.3) is denoted by L, then $Lf = 0$ automatically. Hence the total content of (4.3) is that $Lf = 0$ for all f such that $f(T, k)$ depends only on k. Let f be such a function ("depending only on k"). Then

$$\Delta_Y(\bar{Z}(f)) - \bar{Z}(\Delta_Y(f)) + \bar{Y}(\Delta_Z(f)) - \Delta_Z(\bar{Y}(f)) + [\Delta_Y, \Delta_Z](f) - \Delta_{[Y,Z]}(f) = 0\,.$$

Observe that $\bar{Z}(f)$ and $\bar{Y}(f)$ are 0. Leaving off, as is customary, certain parentheses, we obtain

$$(4.4) \quad \begin{aligned} &[\Delta_Y, \Delta_Z] f = \Delta_{[Y,Z]} f - Y(\Delta_Z f) + Z(\Delta_Y f)\,, \\ &\Delta_{Y+Z} f = \Delta_Y f + \Delta_Z f\,. \end{aligned}$$

(4.5) **Theorem.** *If vector fields Δ_Y on $G \times K$ are given, satisfying* (4.4) *and depending sufficiently smoothely on the coordinates, then dynamorphisms Δ_S^T can be constructed at least for $T \circ S^{-1}$ sufficiently close to the identity of G such that the $\Delta_{Y,T}$ are the infinitesimal dynamorphisms for that (local) dynamics.*

Proof. From (4.4) we can get back to vector fields $\bar{Y} + \Delta_Y$ on $G \times K$. By a "local" existence theorem [5, Theorem 88], we obtain an action of a neighborhood of the identity in G, on $G \times K$. Then we use (3.9) to define the $\Delta_T^{S \circ T}$ for all S sufficiently close to the identity.

In the classical case, (4.4) is always fulfilled because $\mathfrak{g}$ is one-dimensional, so that Y, Z are linearly dependent.

5. Completely Hamiltonian Systems

In this and the following sections the vector field $\bar{Y}$ in $G \times K$ corresponding to Y in $\mathfrak{g}$ will be denoted simply by Y.

On the next theorem we require the state space K to be simply connected. We also require the dynamics to preserve some symplectic structure ω as the definition of complete Hamiltonicity requires. Moreover, we suppose that the symplectic structure ω is *exact*, that is, there is a 1-form μ on K such that $\omega = d\mu$. Using notation of (3.9), we "extend" μ to $G \times K$ in the manner corresponding to the Cartesian projection of $G \times K$ on K, and call that extension μ also. Our statement involves also the Poisson Bracket associated with ω [6, (7.10)].

(5.1) **Theorem.** *Suppose* $(\mathscr{C}, K, \Delta)$ *is completely Hamiltonian. Suppose* K *is simply connected. Then one can find a* 1*-form* λ *on* $G \times K$ *such that*

(5.1.1) λ *is orthogonal to the* K *axis, i.e.* $\langle \lambda, X \rangle = 0$ *for each vector* X *paralled to the* K *axis,*

(5.1.2) $\{\langle \lambda, Y \rangle, f\} = \Delta_Y f$ *for each right invariant vector field on* G, *and function* f *on* $G \times K$,

(5.1.3) *for* Y, Z *right invariant vector fields,*

$$\{\langle \lambda, Y \rangle, \langle \lambda, Z \rangle\} = \langle \lambda, [Y, Z] \rangle$$
$$- Y(\langle \lambda, Z \rangle) + Z(\langle \lambda, Y \rangle)$$

and

(5.1.4) *the vectors in* $G \times K$ *tangent to the orbits under the action of* G *are precisely the singular vectors for* $d(\mu - \lambda)$.

To prove this, we first translate it into the language of coordinates, bases and components.

We suppose there are coordinates $p_1, \dots, p_n$, $q^1, \dots, q^n$ such that $\mu = p_k\, dq^k$ (summation convention used). These coordinates extend to $G \times K$ by defining $p_k(T, k) = p_k(k)$, etc. This defines μ on $G \times K$ as (5.1) says, and the new μ is $p_k\, dq^k$ in terms of the new p, q. We now choose any basis $Y_1, \dots, Y_v$ for the Lie algebra of right invariant vector fields on G. This gives rise to structure constants $r^{\gamma}_{\alpha\beta}$ for which $[Y_\alpha, Y_\beta] = r^{\gamma}_{\alpha\beta}$ (summation here from 1 to v). The constants are called r for "right". They are the negatives of those underlying the table (4.6) in [3], where following most texts the *left* invariant vector fields are involved. The dual linear space of the lie algebra generated by $Y_1, \dots, Y_v$ is spanned by the right invariant Maurer-Cartan forms $\lambda^1, \dots, \lambda^v$ for which $\lambda^\alpha(Y_\beta) = \delta^\alpha_\beta$. We will write $\langle \lambda^\alpha, Y_\beta \rangle$ for $\lambda^\alpha(Y_\beta)$. Now (5.11) says, concerning the λ promised by (5.1) that

(5.2.1) λ *has the form (of a sum)* $H_\alpha \lambda^\alpha$.

Here H_α are certain functions on $G \times K$. Now $\langle \lambda, Y_\alpha \rangle$ is easily seen to be H_α. Thus (5.1.2) says

(5.2.2) H_α *is a generating function for the infinitesimal dynamorphism* $D_\alpha (\equiv \Delta_{Y_\alpha})$.

Now, (5.1.3) says

$$\text{(5.2.3)} \quad \{H_\alpha, H_\beta\} = r^{\gamma}_{\alpha\beta} H_\gamma - Y_\alpha(H_\beta) + Y_\beta(H_\alpha)\,.$$

The Poisson Bracket $\{f, g\}$ is

$$\frac{\partial f}{\partial p_k}\,\frac{\partial g}{\partial q^k} - \frac{\partial f}{\partial q^k}\,\frac{\partial g}{\partial p_k}.$$

We will always be using the summation convention.

Finally, (5.1.4) says the same as

(5.2.4) *The singular directions for* $d(p_i\,dq^i - H_\alpha \lambda^\alpha)$ *are the vectors tangent to the orbits.*

When reformulating (5.1.1)–(5.1.4) in terms of coordinates we are not supposing this coordinate system to be valid on all of K. As a matter of fact, coordinates can always be found giving μ the form $p_k\,dq^k$ [6, p. 140]. This remark shows that (5.1.1)–(5.1.4) implies (5.2.1)–(5.2.4). Conversely, if the latter hold for all such coordinate systems, then (5.1.1)–(5.1.4) hold.

To prove (5.1), we will establish the conclusion in the form (5.2.1)–(5.2.4). The essential element of the proof of (5.1) is the observation that ω is dynamically invariant if and only if $d\mu = dp_k \wedge dq^k$, as a 2-form on $G \times K$, is invariant under the action of G in $G \times K$ (this remark and the succeeding one apply to all covariant tensor fields on K).

Besides the "obvious" way to extend differential forms from K to $G \times K$ already used to define μ, there is another way, based on the map $(T, k) \to T^{-1}(T, k)$ of $G \times K$ onto $\{e\} \times K$. This map Q provides a way of defining a tensor field $Q\Omega$ on $G \times K$, given a covariant tensor field Ω on $\{e\} \times K$ (or on K). Namely, $Q\Omega$ evaluated for vectors in $G \times K$ shall be Ω evaluated on their images under Q. Any vector tangent to an orbit for the action of G is singular for $Q\Omega$, because under Q such a vector projects onto 0. Moreover $Q\Omega$ is invariant under G. $Q\Omega$ is not an extension of r, in the sense of agreeing with Ω on $\{e\} \times K$, but it does agree with Ω for vectors tangent to $\{e\} \times K$.

Both $Q\omega$ and $d\mu$ are invariant under G. They agree for vectors tangent to $\{e\} \times K$. Hence they agree for vectors tangent to $\{T\} \times K$. Let $\beta = Q\mu - \mu$. Then $d\beta = dQ\mu - d\mu = Q(\omega) - d\mu$, and as we just said $\langle d\beta; Z, W\rangle = 0$ if Z and W are tangent to $\{T\} \times K$. We now assert that for each point k_0 of K there is a neighborhood U of k and a function S_U defined on $G \times U$ such that $Q\mu - \mu - dS_U$ is a linear combination of the Maurer-Cartan forms λ^α. To prove this, we choose a simply connected neighborhood U of k_0 on which there is coordinate system $x^1, \ldots, x^m$ where m is the dimension of K. Let y^α be some coordinate system for G. (This latter assumption can be avoided by using the Maurer-Cartan forms and is not necessary, but it simplifies the exposition.) Let $\beta = Q\mu - \mu$. β is of the form $A_i\,dx^i + \beta_\alpha\,dy^\alpha$, whence

$$d\beta = \frac{\partial A_i}{\partial x^i}\,dx^j \wedge dx^i + \frac{\partial A_i}{\partial y^\gamma}\,dy^\gamma \wedge dx^i + \frac{\partial B_\alpha}{\partial x^j}\,dx^j \wedge dy^\alpha + \frac{\partial B_\alpha}{\partial y^\gamma}\,dy^\gamma \wedge dy^\alpha.$$

What we said about $d\beta$ earlier forces $\partial A_i/\partial x^j$ to be 0 (among other things). Thus we can obtain a function S_U on $G \times U$ such that

$$A_i = \frac{\partial}{\partial x^i} S_U .$$

This function can be obtained for each T, k by integrating along some curve $(T, k(s))$ where $k(0) = k_0$ and $k(1) = k$ and hence depends differentiably on T, k.

Now let U and V be two neighborhoods on which such S_U and S_V have been constructed. It follows that $dS_U - dS_V$ is of the form $f_\alpha(x, y)\, dy^\alpha$, so to speak. More precisely, $S_{UV} = S_U - S_V$ depends only on T in G. We can therefore replace S_V by $S_V + S_{UV}$ and obtain a *new* S_V which agrees with S_U on the intersection of U and V. This is a situation of the sort where simple connectivity is always invoked to provide a single function S such that $Q\mu - \mu - dS$ is a linear combination $-H_\alpha \lambda^\alpha$. These functions H_α will turn out to be our generating functions. In any case, $-Q\mu + \mu + dS$ is the desired λ, completing the proof of (5.2.1).

Consider $d(\mu - \lambda)$. This is $dQ\mu = Q(\omega)$. Anything obtained by the Q-process has the property that vectors tangent to orbits are singular for it. Conversely, vectors not tangent to orbits project (under Q) into non-zero vectors in $\{e\} \times K$ which are not singular for (the non degenerate) ω. This proves (5.2.4).

In coordinates p, q, $\mu - \lambda$ is of the form $p_i\, dq^i - H_\alpha \lambda^\alpha$.

To establish (5.2.2) and (5.2.3) we need $d(\mu - \lambda)$. We use the following important formulae. One is for functions H on $G \times K$. It is of interest even for functions independent of the p's and q's, that is functions depending only on the coordinates in G.

$$(5.3) \quad dH = \frac{\partial H}{\partial p_k} dp_k + \frac{\partial H}{\partial q^k} dq^k + \lambda^\alpha Y_\alpha(H) .$$

The other is for Maurer-Cartan forms and can be deduced by comparing formulae (2.1) and (2.3) of [6, pp. 217–219].

$$(5.4) \quad d\lambda^\gamma = -\tfrac{1}{2} r^\gamma_{\alpha\beta} \lambda^\alpha \wedge \lambda^\beta .$$

This result is

$$\begin{aligned} d(\mu - \lambda) &= dp_k \wedge dq^k + (-H^k_\alpha\, dp_k - H_{\alpha k}\, dq^k) \wedge \lambda^\alpha \\ &\quad - Y_\beta(H^\alpha) \lambda^\beta \wedge \lambda^\alpha + \tfrac{1}{2} H_\alpha r^\alpha_{\beta\gamma} \lambda^\beta \wedge \lambda^\gamma . \end{aligned}$$

Here H^k means $\partial H/\partial p_k$ and H_k means $\partial H/\partial q^k$.

Now we express the fact that $Y_\delta + D_\delta$ is singular for $d(\mu - \lambda)$. We make use of the fact that, if φ_i and ψ_i are 1-forms, then Z is singular for

$\Sigma\, \varphi_i \wedge \psi_i$ if and only if $\Sigma\, [\langle \varphi_i, Z \rangle \psi_i - \langle \psi_i, Z \rangle \varphi_i] = 0$. Applied to our problem, this says

$$\begin{aligned} D_\delta(p_k)\, dq^k - D_\delta(q^k)\, dp_k + [-H_\alpha^k D_\delta(p_k) - H_{\alpha k} D(q^k)]\, \lambda^\alpha \\ -(-H_\delta^k\, dp_k - H_{\delta k}\, dq^k) \\ - Y_\beta(H_\alpha)\, (\delta_\delta^\beta \lambda^\alpha - \lambda^\beta \delta_\delta^\alpha) + H_\alpha r_{\delta_\gamma}^\alpha \lambda^\gamma = 0\,. \end{aligned}$$

The coefficient of dq^k (for example) must vanish. Therefore $D_\delta(p_k) = -H_{\delta k}$. Similarly $D_\delta(q^k) = -H_\delta^k$. This much already tells us that $D_\delta(f) = \{H_\delta, f\}$ which is (5.2.2). We now set equal to 0 the coefficient of λ^γ, making use of the values of $D_\delta(p_k)$ and $D_\delta(q^k)$. The resulting equation is precisely (5.2.3). Thus (5.1) is proved. Perhaps the theorem should have been formulated merely as asserting (5.1.1) and (5.1.4), because (5.1.2) and (5.1.3) follow from these, as we have seen.

Now we look into the question of the uniqueness of these H_α. As is well known, in the classical case, any two valid Hamiltonians differ by a function of t. The general case is essentially analogous.

(5.5) **Theorem.** *Let the λ be as above. Suppose another 1-form $\bar{\lambda}$ satisfies all the conditions of (5.1). Then there is a 1-form v on G such that $\bar{\lambda} = \lambda + v$ and $dv = 0$. In component form, $\bar{\lambda} = \bar{H}_\alpha \lambda^\alpha$, $\bar{H}_\alpha = H_\alpha + \varphi_\alpha$ where the functions φ_α depend only on the coordinates for G. Locally, one can find one function φ on G such that $\varphi_\alpha = Y_\alpha(\varphi)$. The converse is also true.*

The φ need not exist as a single-valued function on all of G because G is not simply connected. To prove (5.5) we begin by letting $\varphi_\alpha = \bar{H}_\alpha - H_\alpha$. From (5.2.2) we deduce that φ_α depends only on the group coordinates. From this we see that $\{\bar{H}_\alpha, \bar{H}_\beta\} = \{H_\alpha, H_\beta\}$. Writing down (5.2.3) for $\bar{H}_\alpha$, H_β on the other, and substracting, yields

(5.6) $\quad 0 = r_{\alpha\beta}^\gamma \varphi_\gamma - Y_\alpha \varphi_\beta + Y_\beta \varphi_\alpha\,.$

If we let $v = \varphi_\alpha \lambda^\alpha$ then (5.6) and (5.4) imply that $d\varrho = 0$. On any simply connected set in G we can find φ such that $v = d\varphi$, on that set. So much for (5.5).

We will say a few words about the assumption of simple connectedness for K. It is really needed to produce the generating functions. An example will illustrate the phenomenon. Let K be the real plane minus the origin. Let $\omega = dr \wedge d\Theta$. Everybody knows what this is even though Θ itself is not single-valued. Moreover, $\omega = d\mu$ where $\mu = r\, d\Theta$. The vector field

$$\frac{\partial}{\partial r}$$

certainly preserves ω, but the generating function would have to be Θ. Perhaps one could set up the theory with generating functions being replaced by closed 1-forms, but many-valued generating functions would not serve the purposes to which generating functions are usually put (see Section 6, right after (6.1)).

6. Completely Autonomous, Completely Hamiltonian Systems

Suppose the dynamics is completely autonomous. Then the infinitesimal automorphism $\Delta_{Y,T}$ evidently depends only Y. Therefore write it as Δ^Y. Let us consider the 1-parameter subgroup generated by Y. Suppose that the dynamics is completely Hamiltonian. Then this subgroup preserves the symplectic structure. We may treat it as the flow of time and the classical theory will give us a generating function H_Y for the vector field Δ^Y. Since Δ^Y does not depend on T we obtain an H^Y independently of T.

Now consider the totality of these H_Y, Y running through $\mathfrak{g}$. We will now show that $\{H_Y, H_Z\}$ is a generating function for $[Y, Z]$. Let f be any function on K. Then

$$\begin{aligned}\{\{H_Y, H_Z\}, f\} &= -\{\{H_Z, f\}, H_Y\} - \{\{f, H_Y\}, H_Z\} \\ &= -\{\Delta^Z f, H_Y\} + \{\Delta^Y f, H_Z\} \\ &= \Delta^Y \Delta^Z f - \Delta^Z \Delta^Y f \\ &= \Delta^{[Y,Z]} f .\end{aligned}$$

Of course, this latter is also $\{H_{[Y,Z]}, f\}$. Thus

$$\{H_Y, HZ\} = H_{[Y,Z]} + \text{const}.$$

One way to motivate our investigation is to say that we want to get rid of the constant here. The idea is not merely aesthetic, but one wants to use the condition

$$(6.1) \quad \{H_Y, H_Z\} = H_{[Y,Z]}$$

as a way of normalizing these functions. If this were not possible, it would be absurd to expect polynomials of the H's to represent quantities such as mass and spin [4, p. 2418]. We have already shown in [4, p. 2417] that such a normalization is possible for *elementary* systems. We solve this problem here in general, assuming things about G which are true for the Poincaré group. (For non-autonomous systems, the normalization (6.1) is unacceptable because it forces G to *act* in K, thus forcing complete autonomy. The problem of formulating an effective normalization in the non-autonomous case is surely also important, but it is not solved.)

Our assumptions about G are as follows.

(6.2) *Each exact invariant 2-form is the exteriour derivative of exactly one invariant 1-form.*

For the Poincaré group, the truth of this follows from well-known facts. Perhaps we may refer to [3] for a self-contained elementary proof. For the 2-form [3, (4.41)] the desired 1-form is $\Sigma A_{kj}\mu^{kj} + B_k\mu^k$. The proof of uniqueness rests on the observation on p. 132 of [3] that every element of the chosen basis occurs in the body of the table. For suppose μ is an invariant 1-form such that $d\mu = 0$. Now $\langle \mu, [Y, Z]\rangle = -\langle d\mu; Y, Z\rangle$ [6, pp. 103, 219] and so μ vanishes for all X in the table, thus $\mu = 0$.

(6.3) **Theorem.** *Let G be as in* (6.2), *and suppose K is simply connected. Then there is exactly one way of defining H_Y for each Y in $\mathfrak{g}$ such that*

(6.3.1) *H_Y depends only on the coordinates in K*

(6.3.2) $\{H_Y, f\} = \Delta^Y f$ *for each function on K*

(6.3.3) $\{H_Y, H_Z\} = H_{[H,Z]}$.

In terms of components, this says about the functions H_α

(6.4.1) *H_α depends only on the coordinates in K,*

(6.4.2) *H_α is a generating function for D_α*

$$\{H_\alpha, H_\beta\} = r^\gamma_{\alpha\beta} H_\gamma .$$

We begin by treating the case in which K is connected. Let H_α be a system of generating functions as provided by (5.1). We will amend them to functions $\bar{H}_\alpha$ which satisfy (6.3.1)–(6.3.3). Curiously, we have not been able to use (5.5) to do this, and give an independent argument.

We observe that each $Y_\beta(H_\alpha)$ depends only on the coordinates in G. Indeed, (5.2.2) says that

$$D_\alpha = \frac{\partial H_\alpha}{\partial p_k}\,\frac{\partial}{\partial q^k} - \frac{\partial H_\alpha}{\partial q^k}\,\frac{\partial}{\partial p_k} .$$

The complete autonomy insures that the components here depend only on the p's and q's. Therefore Y_β applied to them yields 0. Interchanging the order of differentiation shows

$$\frac{\partial}{\partial p_k} Y_\beta(H_\alpha) = 0, \qquad \frac{\partial}{\partial q_k} Y_\beta(H_\alpha) = 0 .$$

Now $Y_\beta(H_\alpha)$ can depend on the coordinates in G, but since K is connected, it does not depend on the coordinates in K.

Now select any point k_0 in K and let $G_\alpha = H_\alpha(k_0)$. This clearly depends only on the coordinates in G. Thus the same is true for $Y_\beta(G_\alpha)$. Let $F_\alpha = H_\alpha - G_\alpha$. Then $Y_\beta(F_\alpha) = Y_\beta(H_\alpha) - Y_\beta(G_\alpha)$. This depends only on the coordinates in G. So we get its value by evaluating at any k in K. Choose $k = k_0$, and see that $Y_\beta(F_\alpha) = 0$.

Observing that $\{H_\alpha, G_\beta\} = 0$ we get from (5.2.3) that

$$\{F_\alpha, F_\beta\} = r^\gamma_{\alpha\beta} F_\gamma - Y_\alpha F_\beta + Y_\beta F_\alpha + r^\gamma_{\alpha\beta} G_\gamma - Y_\alpha G_\beta + Y_\beta G_\alpha .$$

We recall that $Y_\alpha F_\beta = 0$, so

(6.4) $\quad \{F_\alpha, F_\beta\} - r^\gamma_{\alpha\beta} F_\gamma = r^\gamma_{\alpha\beta} G_\gamma - Y_\alpha G_\beta + Y_\beta G_\alpha .$

The right hand side surely depends only on the coordinates in G. As to the left hand side, recall $Y_\alpha F_\beta = 0$. If G were connected, we could conclude that F_β depends only on the coordinates in K; and if we denote each side by $c_{\alpha\beta}$ we could say that $c_{\alpha\beta}$ is constant. As it is, we can only say that $c_{\alpha\beta}$ is constant on each component of G, but this is enough for the subsequent argument. Consider G_0, the component of e, and let $c_{\alpha\beta}$ be the values assumed there. From (6.4) we see that

$$(r^\gamma_{\alpha\beta} G_\gamma - Y_\alpha G_\beta + Y_\beta G_\alpha) \lambda^\alpha \wedge \lambda^\beta = c_{\alpha\beta} \lambda^\alpha \wedge \lambda^\beta$$

whence (by virtue of (5.3) and (5.4))

(6.5) $\quad c_{\alpha\beta} \lambda^\alpha \wedge \lambda^\beta = -2d(G_\gamma \lambda^\gamma) .$

On the left side of (6.5) we have an invariant 2-form, and (6.5) tells us that it is exact. Hence (6.2) assures us that $c_{\alpha\beta} \lambda^\alpha \wedge \lambda^\beta = d(k_\gamma \lambda^\gamma)$ where these k_γ are constant. Let $k_\gamma = -2c_\gamma$. Then $r^\gamma_{\alpha\beta} c_\gamma = c_{\alpha\beta}$

(6.6) $\quad \{F_\alpha + c_\alpha, F_\beta + c_\beta\} = r^\gamma_{\alpha\beta}(F_\gamma + c_\gamma) .$

If we define $\bar{H}_\alpha$ as being $F_\alpha + c_\alpha$, then these certainly have the desired properties (6.4.1)–(6.4.3).

We now consider uniqueness. Let the systems H_α and $\bar{H}_\alpha$ (however obtained) satisfy (6.4.1)–(6.4.3). Then $\bar{H}_\alpha - H_\alpha$ must be constant, c_α. Since $\{H_\alpha, \bar{H}_\beta\} = r^\gamma_{\alpha\beta} \bar{H}_\gamma$ and $\{\bar{H}_\alpha, \bar{H}_\beta\} = \{H_\alpha, H_\beta\}$ we obtain $r^\gamma_{\alpha\beta} c_\gamma = 0$. Let $\lambda = c_\gamma \lambda^\gamma$. Now $d\lambda = -\frac{1}{2} c_\gamma r^\gamma_{\alpha\beta} \lambda^\alpha \wedge \lambda^\beta = 0$. But $d0 = 0$. Hence by the "exactly one" assumption in (6.2), we have $\lambda = 0$. Thus each c_γ is 0. Thus (6.3) is proved for connected K.

If K is not connected then the action of G either interchanges some components or it does not. In the first case it is easy to modify the action without changing it on the component of the identity of G, to produce the second case. In the second case we can apply our proof for the connected case to the *system defined by each connected component.* Thus (6.3) is proved.

A variant of (6.3) could also be stated in which the simple-connectedness of K is replaced by the weaker requirement that some system of H_α satisfying (5.1) exist. This variant would have the same conclusion as (6.3).

7. A Class of Examples

We will present some examples to show that complete Hamiltonicity does not imply complete autonomy.

Consider the cotangent bundle $T_1(M)$ of space-time. Associated with a coordinator (or indeed any coordinate system (x^1, x^2, x^3, x^4) in M there is a coordinate system $(x^1, x^2, x^3, x^4, p_1, p_2, p_3, p_4)$ in $T_1(M)$ where p_i of a generic element $a_1\,dx^1 + \cdots + a_4\,dx^4$ is just the component a_i.

By a space-time *Hamiltonian surface* we mean a hypersurface $\mathscr{S}$ in $T_1(M)$ having the property that for each coordinator $\boldsymbol{x}$, an equation for $\mathscr{S}$ can be written in the form $p_4 = -H(x^1, x^2, x^3, x^4, p_1, p_2, p_3)$ where these are the associated coordinates. This H will in general be a different function for different coordinators. An example can easily be made. Select any coordinator $\boldsymbol{x}$, and also four functions A_1, A_2, A_3, A_4, on space-time and define $\mathscr{S}$ by

$$(7.1)\quad p_4 = -A_4 + [1 + (p_1 + A_1)^2 + \cdots + (p_3 + A_3)^2]^{\frac{1}{2}}.$$

This leads to a Hamiltonian surface in the sense just defined because when (7.1) is expressed in terms of another coordinator $(y^1, \ldots, y^4)$, we get an equation like (7.1) when we solve for the 4-th "momentum" in the $\boldsymbol{y}$ system.

$T_1(M)$ has a natural 1-form Θ [6, p. 143] and this defines a symplectic structure $d\Theta$ on $T_1(M)$. By restriction to $\mathscr{S}$ we obtain a 2-form $d\Theta_{\mathscr{S}}$ on $\mathscr{S}$. This may be regarded also as $d(\Theta_{\mathscr{S}})$ where $\Theta_{\mathscr{S}}$ is the restriction of Θ. $\Theta_{\mathscr{S}}$ in coordinates has the form

$$(7.2)\quad p_1\,dx^1 + \cdots + p_3\,dx^3 - H\,dx^4$$

where $x^1, \ldots, x^4$, p_1, p_2, p_3 are being used as coordinates on $\mathscr{S}$. This 2-form $d\Theta_{\mathscr{S}}$ is singular (mainly because $\mathscr{S}$ is odd-dimensional) and thus has a singular direction defined by

$$(7.2.1)\quad dx^i = \frac{\partial H}{\partial p_i}\,dx^4, \qquad dp_i = -\frac{\partial H}{\partial x^i}\,dx^4$$

at each point of $\mathscr{S}$. The integral curves for this direction field fiber $\mathscr{S}$ into a 6-parameter family of curves. Let us call this family $[\mathscr{S}]$. These canonical Eqs. (7.2.1) show that dx^4 is not 0 along any such integral curve. Hence $p_1, p_2, p_3, x^1, x^2, x^3$ may be used as coordinates for $[\mathscr{S}]$, thereby defining a manifold structure for it. Because the fibers are singular for $d\Theta_{\mathscr{S}}$, one can define a 2-form Ω on $[\mathscr{S}]$ such that $d\Theta_{\mathscr{S}} = Q\Omega$ where Q is the map from $\mathscr{S}$ into $[\mathscr{S}]$. This is a symplectic structure, and using the six coordinates

$$(7.3)\quad \Omega = dp_1 \wedge dx^1 + \cdots + dp_3 \wedge dx^3.$$

Next, we will map $[\mathscr{S}]$ onto $K = T_1(\mathbb{R}^3)$, by various maps $\Delta^{\boldsymbol{x}}$ each indexed by a coordinator $\boldsymbol{x}$. Let $\boldsymbol{x}$ be a coordinator. Let us regard $\mathbb{R}^3$ as a subset of $\mathbb{R}^4$, namely the subspace on which the fourth coordinate is 0. Now $\boldsymbol{x}$ maps M onto $\mathbb{R}^4$ whence $\boldsymbol{x}^{-1}$ maps $\mathbb{R}^4$ onto M and by *restriction*, $\boldsymbol{x}^{-1}$ maps $\mathbb{R}^3$ into M. A point r of $\mathbb{R}^3$ maps into m in M and each vector at r maps into a vector at m. Hence the covectors at m, $T_1(M)_m$, map into the covectors, $T_1(\mathbb{R}^3)_r$ at r. These linear spaces have dimension 4 and 3 respectively. The condition we impose on Hamiltonian surfaces insures that the mapping of the *intersection*

$$(7.4) \quad T_1(M)_m \cap \mathscr{S} \to T_1(\mathbb{R}^3)_r$$

is 1 : 1 and onto. Now let ξ be an element of $[\mathscr{S}]$. So it is a subset of $\mathscr{S}$ (indeed a curve). This ξ has on it only one point for which $x^4 = 0$. This point belongs to $T_1(M)_m \cap \mathscr{S}$ for some m whose $\boldsymbol{x}(m)$ lies in $\mathbb{R}^3$. Hence we may apply (7.4) to give an element of $T_1(\mathbb{R}^3)$. This element of $\mathbb{R}^3$ we call $\Delta^{\boldsymbol{x}}(\xi)$. The dynamics is defined by [2, (2.42)]

$$(7.5) \quad \Delta_{\boldsymbol{x}}^{\boldsymbol{y}} = \Delta^{\boldsymbol{y}} \circ (\Delta^{\boldsymbol{x}})^{-1} .$$

Thus a system has been assembled. It remains to show that it is completely Hamiltonian. $T_1(\mathbb{R}^3)$ has a natural symplectic structure and

this symplectic structure is preserved by (7.5).

This follows from the fact that

(7.6) *each* $\Delta^{\boldsymbol{x}}$ *transforms* (7.3) *into the symplectic form of* $T_1(\mathbb{R}^3)$.

To prove (7.6) one must examine the map $\Delta^{\boldsymbol{x}}$, to see what happens to a curve ξ whose p_i coordinates are b_1, b_2, b_3 and whose x^i coordinates are r^1, r^2, r^3. These p and x are the coordinates on $[\mathscr{S}]$ used in (7.3). It turns out that $\Delta^{\boldsymbol{x}}(\xi)$ is a covector at (r_1, r_2, r_3) in $\mathbb{R}^3$ with Cartesian components b_1, b_2, b_3. This is simply a consequence of how we defined the x and p as coordinates on $[\mathscr{S}]$. Thus (7.6) is apparent.

We can state what the generating functions are for systems of this type. To prevent erroneous application of these results we have to be very pedantic about notation. In the first place, the dependence of these functions H on the coordinator must be explicitly indicated. About $\mathscr{S}$ we are assuming that for each coordinator $\boldsymbol{x}$ there is a function $H(\boldsymbol{x})$ of seven variables such that an element $\mathscr{S}$ of $T_1(M)$ will lie on $\mathscr{S}$ if and only if

$$(7.7) \quad p_4(\zeta) + H(\boldsymbol{x})\,(x^1(\zeta), x^2(\zeta), x^3(\zeta), x^4(\zeta), p_1(\zeta), p_2(\zeta), p_3(\zeta)) = 0 .$$

Thus we are now writing $H(\boldsymbol{x})$ where we wrote H in the original definition of Hamiltonian surface. This bold face $\boldsymbol{x}$ is nothing else but

(x^1, x^2, x^3, x^4), of course. It should be clearly understood that (7.7) does *not* say that the equation of $\mathscr{S}$ has the form

$$p_4 + H(x^1, x^2, x^3, x^4)(x^1, x^2, \ldots, p_3) = 0\,.$$

Put another way, the (x^1, x^2, x^3, x^4) which could be inserted for $\boldsymbol{x}$ in (7.7) are not numerical parameters. (In fact, $\boldsymbol{x}$ ranges over the 10-dimensional set of all coordinators!) On the other hand, the $x^i(\zeta)$ in (7.7) *are* related to the very coordinator $\boldsymbol{x}$ involved. The x^i of a covector ξ is the actual or original x^i of the base point of ζ. This is nothing but the old way of lifting coordinates from configuration space to phase space and conventionally one uses the same letter x^i. From here to the p_i is the familiar step.

An infinitesimal transformation associated with the group G is a vector field in the space on which G acts, in this case $\mathbb{R}^4$. As a matter of fact, we have the following more general result (cf. [2, (3.7)]). We will use t^1, t^2, t^3, t^4 as Cartesian coordinates in $\mathbb{R}^4$.

(7.8) **Proposition.** *Let $\mathscr{S}$ be a Hamiltonian surface. Then the dynamics it defines (as described above) is completely Hamiltonian. Let Y be an element of the Lie algebra of G and let*

$$Z = Z^1(t^1, t^2, t^3, t^4)\frac{\partial}{\partial t^1} + \cdots + Z^4(t^1, t^2, t^3, t^4)\frac{\partial}{\partial t^4}$$

be the corresponding infinitesimal transformation in $\mathbb{R}^4$. Let $\boldsymbol{x}$ be a coordinator, with $H(\boldsymbol{x})$ as the corresponding Hamiltonian. Then the following function $H_Y(\boldsymbol{x})$ is a generating function for the infinitesimal dynamorphism $\Delta_{Y,\boldsymbol{x}}$ [see (4.1)]*:*

$$H_Y(\boldsymbol{x}) = Z^1(x^1, x^2, x^3, 0)p_1 + \cdots + Z^3(x^1, x^2, x^3, 0)p_3$$
$$- Z^4(x^1, x^2, x^3, 0)\, H(\boldsymbol{x})(x^1, x^2, x^3, 0, p_1, p_2, p_3)\,.$$

To prove this, adapt the proof of [2, (2.7)], which deals with ("contravariant") vectors, to the case of covectors ("covariant" vectors). This means mainly that one uses the law for transforming covariant components rather than contravariant components. When $Z^4 = 0$ and the other components are independent of time, we have exactly the case dealt with in [2, (3.7)]. However, at first glance there seems to be an overall error in sign for the generating function. This is no error since the Poisson Bracket in [2] is the negative of the one we are using here.

Proposition (7.8) seems to contradict [2, (3.3)] which says that under certain rather natural conditions, in the Einstein-Lorentz case, there cannot be a completely Hamiltonian system which is not completely autonomous. Of course we must conclude that the examples provided as above by Hamiltonian surfaces do not satisfy the conditions alluded to. These conditions actually amount to the following. There should be some coordinator $\boldsymbol{x}$ such that if we use "his" Hamiltonian $H(\boldsymbol{x})$ to define an inverse Legendre map from $T_1(\mathbb{R}^3)$ to $T^1(\mathbb{R}^3)$, then the dynamics on $T^1(\mathbb{R}^3)$ should be that for some second order 1-particle "inter" action. (It is not *assumed* that this 1-particle motion is Poincaré invariant: it is *proved*.)

Notice that we did not say that these H_Y had the commutation relations (5.1.3). We conjecture that they do. We have verified this conjecture in the special case of the surface defined in terms of a covector potential A_1, A_2, A_3, A_4 in the manner of (7.1). This includes the motion of a charged particle in an electromagnetic field.

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