

Exact Models of Charged Black Holes

II. Axisymmetric Stationary Horizons

P. Hájíček*

Max-Planck-Institute for Physics and Astrophysics, Munich**

Received April 23, 1973

Abstract. Using the formalism developed in the preceding paper, all axisymmetric stationary horizons are described. It is found that the bifurcate-type horizons (such as Schwarzschild) are as numerous as about four functions of one variable, while the extreme-type ones (such as extreme Kerr) only as about two functions of one variable. On the other hand, there is exactly one axisymmetric stationary space-time containing a given bifurcate-type horizon, in comparison to a whole family (at least as numerous as two functions of one variable) of such space-times for a given extreme-type one.

The total mass m and angular momentum am of the corresponding black hole could in principle be computed from the invariants describing the bifurcate-type horizons, because the horizons determine their space-time uniquely, but a definite way of computation will probably be difficult to find. On the other hand, the Kerr-Newman-like parameters m and a are easily defined and computed for any extreme-type horizon, but their physical meaning remains so far obscure.

1. Introduction and Summary

In the previous paper [1], a simple classification of symmetry properties of the perfect horizons [2] has been achieved. In the present paper, we find all horizons of the first few most symmetric classes which can be imbedded in electrovacuum space-times. We use the notation introduced in [1]; this paper will be referred to as I hereafter [e.g., the Eq. (x) of [1] will be denoted I(x)].

In Section 2, the spherically symmetric horizons are investigated. They are found to form a three-parameter family that contains the Reissner-Norstrøm horizons with $m^2 > e^2 + h^2$ (m is the mass, e the electric and h the magnetic charge) and bifurcates in two two-parameter subfamilies at $m^2 = e^2 + h^2$, the first being the extreme Reissner-Nordstrøm one, and the second being formed by the horizons in the homogeneous space-times $S_m^2 \times P_m^2$. Here S_m^2 , P_m^2 is the 2-sphere and 2-pseudo-sphere, respectively, of radius m and \times denotes the Cartesian product

* A part of this work was supported by Alexander von Humboldt-Stiftung.

** Present address: Institute for Theoretical Physics, University of Berne.

of metric manifolds. Thus, all spherically symmetric horizons are static – a Birkhoff-like inversion of Israel's theorem [3].

The axially symmetric horizons with a collineation group – briefly *AC*-horizons – are easy to deal with, because they possess special space-like sections. The complete set of independent invariants for this class, found in Section 3, consists of four functions $A(\vartheta)$, $B(\vartheta)$, $C(\vartheta)$, $D(\vartheta)$ of one variable \pm finite number of parameters (cf. also [2]). Thus, *AC*-horizons are rather numerous. One could argue that this result need not have any physical relevance, because we have given only the horizons and not whole space-times and we cannot, therefore, say whether there is a physically reasonable (asymptotically flat, causal, etc.) space-time for any of our horizons. But this argument seems to break down for the following reason. We show in Section 3 that, given an *AC*-horizon \mathcal{M} , then there is only one axisymmetric and stationary space-time $\overline{\mathcal{M}}$ containing \mathcal{M} as a Killing horizon (we suppose analyticity, but this restriction is probably not very important). That is to say, if there is some matter or charge orbiting a black hole, then this *must* show up in the inner structure of the horizon (see also [4]). And it seems desirable to have physically reasonable axially symmetric stationary space-times with matter and charge flows round a black hole, since this would be the simplest model for some observable effects. On the other hand, all such space-times should be at least as numerous as four functions of two variables (stationary constraints on the flows – essentially conservation laws – together with the symmetry imply that just the φ - and t -components of any flow are non-zero and depend only on r and ϑ). Thus, we have, in fact, obtained too small a number of horizons, showing either that the flows round a black hole cannot be completely arbitrary or that different flows are compatible with the same exterior analytic fields (examples of the latter possibility are well-known: the singularity of the Schwarzschild field with a total mass m can be smoothed out by matter of density $\varrho(r)$, arbitrary up to a relation

$$\int_0^{\infty} \varrho r^2 dr = \frac{m}{4\pi} .)$$

We believe that our result does not contradict the various no-hair-theorems, because these are rigorously shown only for static electrovacuum [3] or stationary vacuum [5] in the full extent, and, in a restricted extent, they concern only the fact that all fields different from Maxwell and gravitational fields are zero at the horizon [6]. There seems also to be no contradiction to [7], where the black hole in energy and angular

momentum exchange with surrounding flows is described with just two parameters.

More difficult is the analysis of the axisymmetric horizons with a translation group – briefly *AT*-horizons (they are of the type, say, extreme Kerr). The four invariant functions A, B, C, D , which were arbitrary in the *AC*-case, now have to satisfy a complex system of differential equations. We show in Section 4 that the only solution is given by A, B, C, D corresponding to the Kerr-Newman horizons with $m^2 = a^2 + e^2 + h^2$. Thus, the parameters m, a, e, h are – formally – well-defined for any *AT*-horizon. On the other hand, some other invariant functions, which have to be identically zero for the *AC*-horizons, can, now be arbitrary; we arrive at a complete set of independent invariants consisting of two functions $\Delta_1(\vartheta), \Delta_2(\vartheta)$ of one variable \pm finite number of parameters. Another surprising property is the following. Given an *AT*-horizon \mathcal{M} , then unlike the *AC*-horizons, there is a whole family of axisymmetric stationary space-times $\tilde{\mathcal{M}}$ containing \mathcal{M} as a Killing horizon, and this family is at least as numerous as two functions of one variable \pm finite number of degrees of freedom. In other words: put all axisymmetric stationary solutions of Einstein-Maxwell equations which contain Killing horizons in a box, shake it and fish at random one space-time out of it. Then the probability to be confronted with an *AT*-horizon is *not* less than that for an *AC*-one! Of course, it is not clear what is the “quality” of all these imbedding space-times. More analysis will be necessary.

2. Spherical Symmetry

Suppose that the transversal group $\tilde{\mathcal{G}}$ is isomorphic to $SO(3)$. According to Theorem 3 of I, \mathcal{G} has a normal subgroup, \mathcal{G}' , say, with spherical space-like trajectories in \mathcal{M} and isomorphic to $SO(3)$. Then, in a sort of spherical coordinate system ϑ, φ , we can set (I(20))

$$\tilde{\mathcal{K}} = R^{-2}, \quad \Gamma = -R^{-1} 2^{-\frac{1}{2}} \operatorname{ctg} \vartheta, \quad \Psi_2 = \text{const}, \quad \Phi_1 = \text{const}, \quad R = \text{const}, \quad (1)$$

$$2\Phi_1^+ \Phi_1 - \Psi_2 - \Psi_2^+ = R^{-2}. \quad (2)$$

$(\operatorname{Im} \Psi_2) \sqrt{\tilde{g}}$ is a curl (I(23)), so the integral of $(\operatorname{Im} \Psi_2) \sqrt{\tilde{g}}$ over the whole sphere vanishes, or

$$\Psi_2^+ = \Psi_2, \quad (3)$$

and the affine coordinate α can be chosen such that (I(22))

$$\Omega = 0. \quad (4)$$

All such affine coordinates form a class, \mathcal{C} , say; any two elements α, α' from \mathcal{C} are connected by a transformation

$$\alpha' = \eta \cdot \alpha + \zeta(\mathfrak{g}, \varphi),$$

where $\eta \neq 0$ is a real number and ζ a real function. On the other hand, if $g \in \mathcal{G}'$ and $\alpha \in \mathcal{C}$, then $\alpha' = g(\alpha)$ is again from \mathcal{C} . This means that $g(\alpha) = \eta_g \cdot \alpha + \zeta_g(\mathfrak{g}, \varphi)$. But, for any $g \in \mathcal{G}'$, there is a pointwise invariant ray with coordinates, say, $\mathfrak{g}_g, \varphi_g$. Thus

$$g(\alpha, \mathfrak{g}_g, \varphi_g) = (\alpha, \mathfrak{g}_g, \varphi_g)$$

for any α , or $\eta_g = 1$. Hence the affine distance between the intersection points of any ray with two fixed orbits of \mathcal{G}' is independent of ray, if measured by an $\alpha \in \mathcal{C}$, and there is, therefore, a subclass, \mathcal{C}' of \mathcal{C} , such that any $\alpha \in \mathcal{C}'$ is constant along the orbits of \mathcal{G}' . Any two α, α' from \mathcal{C}' are related by a transformation of the form $\alpha' = \eta \cdot \alpha + \zeta$, where $\eta \neq 0$ and ζ are two reals. Choose some $\alpha \in \mathcal{C}'$, $L = \frac{\partial}{\partial \alpha}$ and M tangent to the orbits of \mathcal{G}' . Such a frame is determined by the horizon structure up to the transformation

$$\alpha' = \eta \cdot \alpha + \zeta, \quad L' = \frac{1}{\eta} L, \quad M' = e^{i\zeta} M, \quad (5)$$

where $\eta \neq 0$, ζ are real constants and $\zeta(\mathfrak{g}, \varphi)$ is a real differentiable function. Because M is tangential to a family of surfaces, it follows from I(11) that

$$\mu^+ = \mu. \quad (6)$$

Any rotation $e^{i\zeta(p)} M_p$ of M at a point p can be considered as a transformation of M_p by means of g_* , where g is an element of \mathcal{G}' keeping p fixed. It follows that any quantity of non-zero spin weight [8] must be zero:

$$\lambda = \Psi_3 = \Psi_4 = \Phi_2 = 0. \quad (7)$$

From I(14), we have

$$\mu = \mu_0 + \Psi_2 \cdot \alpha,$$

and μ_0 must be constant along the surfaces $\alpha = \text{const}$, because of (1) and the constancy of μ (μ has spin weight zero). (3) and (6) yield

$$\mu_0^+ = \mu_0. \quad (8)$$

Under the transformation (5), μ_0 transforms as follows

$$\mu'_0 = \eta \cdot \mu_0 - \Psi_2 \cdot \zeta,$$

any other quantity so far mentioned remaining unchanged. We have, therefore the following two cases

- 1) $\Psi_2 \neq 0$. Then there are always η and ξ so that $\mu'_0 = 0$.
 2) $\Psi_2 = 0$. Then there are always η such that either $\mu'_0 = 0$ or $\mu'_0 = 1$.
 There is no transformation of the form (5) bringing these two cases into each other.

Theorem 4. *The spherically symmetric horizons form the following two families:*

- 1) *There is a frame in which*

$$\Omega = 0, \quad \Gamma = -\frac{\text{ctg } \vartheta}{R\sqrt{2}}, \quad \lambda = 0, \quad \mu = 1 + \Psi_2 \cdot \alpha$$

$$R^{-2} = 2\Phi_1^+ \Phi_1 - 2\Psi_2, \quad \Psi_3 = \Psi_4 = \Phi_2 = 0.$$

This family has the additional symmetry generated by $\left(\alpha + \frac{1}{\Psi_2}\right) \frac{\partial}{\partial \alpha}$ if $\Psi_2 \neq 0$, and by $\frac{\partial}{\partial \alpha}$, if $\Psi_2 = 0$, and is identical with all Reissner-Nordström horizons with the mass m , electric charge e and magnetic charge h , where

$$\Psi_2 = \mp R^{-2}(mR - e^2 - h^2),$$

$$\text{Re } \Phi_1 = \frac{e}{R\sqrt{2}}, \quad -i \text{Im } \Phi_1 = \frac{h}{R\sqrt{2}},$$

$$R = m \pm \sqrt{m^2 - e^2 - h^2}.$$

- 2) *There is a frame in which*

$$\Omega = 0, \quad \Gamma = -\frac{\text{ctg } \vartheta}{R\sqrt{2}}, \quad \lambda = 0, \quad \mu = 0,$$

$$R^{-2} = 2\Phi_1^+ \Phi_1, \quad \Psi_2 = \Psi_3 = \Psi_4 = \Phi_2 = 0.$$

This family has the additional symmetry generated by $\alpha \frac{\partial}{\partial \alpha}$ and $\frac{\partial}{\partial \alpha}$. Its full set of invariants: two reals $\text{Re } \Phi_1$, $-i \text{Im } \Phi_1$ assuming all values from \mathbb{R}^2 except for the origin.

Corollary. (Analogy of the Theorem of Birkhoff). *Any spherically symmetric horizon has at least a one-dimensional longitudinal symmetry group.*

The horizons of class 2) are found in the homogeneous electrovacuum space-time with the metric

$$ds^2 = \frac{y^2}{e^2 + h^2} dw^2 - 2dw dy - (e^2 + h^2)(d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (9)$$

which is obtained from the Reissner-Nordström metric

$$ds^2 = \frac{\Delta}{r^2} du^2 - 2 du dr - r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2),$$

$$\Delta = (r - m)^2 - \gamma, \quad \gamma = \sqrt{m^2 - e^2 - h^2},$$

in the following limit [9]: perform the transformation

$$u = \gamma^{-1}(m + \gamma)^2 (\lg \gamma + \lg w), \quad r = (m + \gamma)^{-2} (\gamma w y + (m + \gamma)^3), \quad (10)$$

and set $\gamma = 0$. The solution is not asymptotically flat; two horizons of class 2) crossing each other are going through any point of it. The class 2) horizons are those of maximal symmetry possible (5-dimensional group).

3. Axisymmetry and Collineation (AC-horizons)

AC-horizons are defined as follows: their transversal group $\tilde{\mathcal{G}}$ is isomorphic to $SO(2)$ and the group \mathcal{H} of all longitudinal HS is just the one-dimensional two-component collineation group (see I). We have the following (cf. [2]).

Theorem 5. *The complete set of independent invariants for all AC horizons is formed by*

1) *Four smooth functions A, B, C, D on the circle $0 \leq \vartheta \leq 2\pi$ such that*

$$\begin{aligned} A(2\pi - \vartheta) &= -A(\vartheta), & B(2\pi - \vartheta) &= B(\vartheta), \\ C(2\pi - \vartheta) &= C(\vartheta), & D(2\pi - \vartheta) &= D(\vartheta), \end{aligned} \quad (11)$$

$$A(0) = A(\pi) = B(0) = B(\pi) = 0, \quad A'(0) = -A'(\pi) = 1, \quad (12)$$

and the limits

$$\lim_{\vartheta \rightarrow 0} (B' A^{-1}), \quad \lim_{\vartheta \rightarrow \pi} (B' A^{-1}), \quad \lim_{\vartheta \rightarrow 0} (A'' \cdot A^{-1}), \quad \lim_{\vartheta \rightarrow \pi} (A'' \cdot A^{-1}) \quad (13)$$

all exist and are finite.

2) *a positive real number R .*

The horizon \mathcal{M} corresponding to the value A, B, C, D and R of the invariants can be constructed as follows: on \mathcal{M} with the differentiable structure of $S^2 \times \mathbb{R}^1$, choose coordinates $\alpha, \vartheta, \varphi$,

$$\alpha = \text{const being } S^2 \quad \text{and} \quad (\vartheta = \text{const}, \varphi = \text{const})$$

being \mathbb{R}^1 ; ϑ, φ are some usually oriented spherical coordinates on S^2 with singularities at $\vartheta = 0, \vartheta = \pi$. Introduce the degenerate metric

$$ds^2 = -R^2(d\vartheta^2 + A^2 d\varphi^2), \quad (14)$$

the vector fields

$$L = \frac{\partial}{\partial \alpha}, \quad M = \frac{1}{\sqrt{2}} \frac{1}{R} \left(\frac{\partial}{\partial \vartheta} + \frac{i}{A} \frac{\partial}{\partial \varphi} + \frac{i}{A} B \alpha \frac{\partial}{\partial \alpha} \right) \quad (15)$$

and the affine connection

$$\begin{cases} \nabla_L L = 0, & \nabla_L M = 0, & \nabla_M L = \Omega L, \\ \nabla_M M = -\Gamma M, & \nabla_M M^+ = \mu L + \Gamma M^+, \end{cases} \quad (16)$$

where

$$\Omega = -\frac{i}{\sqrt{2}} \frac{1}{R} \frac{B}{A}, \quad \Gamma = \frac{1}{\sqrt{2}} \frac{1}{R} \frac{A'}{A}, \quad (17)$$

$$\mu = \left(C^2 + \frac{1}{2} R^{-2} A^{-1} A'' - \frac{i}{2} R^{-2} A^{-1} B' \right) \cdot \alpha. \quad (18)$$

The Maxwell and Weyl spinors Φ_i, Ψ_i are given by their components

$$\begin{aligned} \Psi_0 = \Psi_1 = \Phi_0 = 0, \quad \Phi_1 = C e^{iD}, \quad \Phi_2 = (M^+ \Phi_1) \cdot \alpha, \\ \Psi_2 = C^2 + \frac{1}{2} R^{-2} A^{-1} A'' - \frac{i}{2} R^{-2} A^{-1} B', \\ \Psi_3 = (M^+ \Psi_2 + \Phi_1^+ M^+ \Phi_1) \cdot \alpha \\ \Psi_4 = \frac{\alpha}{2} \{ M^+ + \Gamma^+ + \Omega^+ \} [(M^+ \Psi_2 + \Phi_1^+ M^+ \Phi_1) \cdot \alpha] \\ + \Phi_1^+ (M^+ + \Gamma^+ + \Omega^+) [(M^+ \Phi_1) \cdot \alpha] \end{aligned} \quad (19)$$

in the pseudoorthonormal tetrad whose first three vectors are L, M [10]. The HS group \mathcal{G} is generated by

$$\alpha \frac{\partial}{\partial \alpha}, \quad \frac{\partial}{\partial \varphi} \quad (20)$$

with the totally autoparallel Cauchy surface at $\alpha = 0$.

Proof. i) The invariant properties of A, B, C, D, R can be seen as follows. The coordinates ϑ, φ on \mathcal{M} are determined by the whole construction of the Theorem 5 up to transformations

$$\vartheta \rightarrow \pi - \vartheta, \quad \varphi \rightarrow -\varphi, \quad (21)$$

$$\vartheta \rightarrow \vartheta, \quad \varphi \rightarrow \varphi - \varphi_0 \quad (22)$$

where φ_0 is a constant. For (22), $A, B, C, D,$ and R remain unchanged. For (21), the new $A', B', C', D,$ and R' describing the old horizon are

$$\begin{aligned} R' = R, \quad A'(\vartheta) = A(\pi - \vartheta), \quad B'(\vartheta) = -B(\pi - \vartheta), \\ C'(\vartheta) = C(\pi - \vartheta), \quad D'(\vartheta) = D(\pi - \vartheta). \end{aligned} \quad (23)$$

As functions on the sphere $\tilde{\mathcal{S}}$, A, B, C, D, R are related to Ψ_2 and Φ_1 by (19):

$$C = \sqrt{\Phi_1^+ \Phi_1}, \quad D = -\frac{i}{2} (\lg \Phi_1 - \lg \Phi_1^+),$$

$$R^{-2} A^{-1} A'' = -2C^2 + \Psi_2 + \Psi_2^+, \quad R^{-2} A^{-1} B' = i(\Psi_2 - \Psi_2^+),$$

from which and from the boundary Condition (12) they are uniquely determined (if existing at all for a given set of Ψ_2, Φ_1). On the other hand, Ψ_2 and Φ_1 are invariant (see I)¹.

Thus, the invariance against all coordinate and triad transformations with the exception of (21), (23) is established.

ii) The independence of A, B, C, D, R means that the horizon data given in the Theorem 5 satisfy the characteristic initial value constraints [Eqs. I(14)–I(16)] identically. A straightforward calculation, which will not be given here, proves this easily.

iii) The symmetry claimed can be verified, if we introduce another frame field L', M' , which is invariant under the symmetries. Then, the components of all magnitudes in the new frame must be independent of φ and α . Such a frame is given by

$$L' = \alpha L, \quad M' = M - \frac{i}{\sqrt{2}} \frac{1}{R} \frac{B}{A} \alpha L; \quad (24)$$

the new metric component are equal to the old ones (14), and the new rotation coefficients are given by

$$V_{L'} L' = L', \quad V_{L'} M' = \Omega L', \quad V_{M'} L' = \Omega' L', \quad (25)$$

$$V_{M'} M' = (M\Omega + \Gamma\Omega + \Omega^2) L' - \Gamma M',$$

$$V_{M'} M'^+ = (\Psi_2 + M\Omega^+ - \Gamma\Omega^+ + \Omega\Omega^+) L' + \Gamma M'^+, \quad (26)$$

where Γ, Ω is given by (17). The transformation formulas for the components of the Weyl and Maxwell spinor under the transformation ([10])

$$l'^i = \eta l^i, \quad m'^i = m^i + \xi l^i, \quad n'^i = \frac{1}{\eta} (n^i + \xi^+ m^i + \xi m^{i+} + \xi \xi^+ l^i), \quad (27)$$

¹ We could choose Ψ_2 and Φ_1 instead of A, B, C, D (a proposal by S. Hawking, private communication). The set of invariants would, then, be more homogeneous; on the other hand, Ψ_2 and Φ_1 must satisfy more complicated constraints than A, B, C, D and it is not so simple to construct the corresponding horizon (differential equations had to be solved). Therefore, we preferred the A, B, C, D .

where $\eta \neq 0$ is some real and ξ some complex function, are

$$\begin{aligned}\Phi'_0 &= 0, \quad \Phi'_1 = \Phi_1, \quad \Phi'_2 = \frac{1}{\eta}(\Phi_2 + 2\xi^+ \Phi_1), \\ \Psi'_0 &= 0, \quad \Psi'_1 = 0, \quad \Psi'_2 = \Psi_2, \quad \Psi'_3 = \frac{1}{\eta}(\Psi_3 + 3\xi^+ \Psi_2), \\ \Psi'_4 &= \frac{1}{\eta^2}(\Psi_4 + 4\xi^+ \Psi_3 + 6\xi^{+2} \Psi_2), \\ &\quad (\text{if } \Phi_0 = \Psi_0 = \Psi_1 = 0).\end{aligned}\tag{28}$$

Using (28) with $\eta = \alpha$ and $\xi = -\frac{i}{\sqrt{2}} \frac{1}{R} \frac{B}{A} \alpha$, we obtain

$$\begin{aligned}\Phi'_2 &= M^+ \Phi_1 + 2\Omega^+ \Phi_1, \quad \Psi'_3 = M^+ \Psi_2 + 3\Omega^+ \Psi_2 + \Phi_1^+ M^+ \Phi_1, \\ \Psi'_4 &= \frac{1}{2}(M^+ + \Gamma^+)(M^+ \Psi_2 + \Phi_1^+ M^+ \Phi_1) + 4\Omega^+(M^+ \Psi_2 + \Phi_1^+ M^+ \Phi_1) \\ &\quad + \frac{1}{2} \Phi_1^+ (M^+ + \Gamma^+) M^+ \Phi_1 + 6\Omega \Omega^+ \Psi_2.\end{aligned}\tag{29}$$

Each of (14), (28), and (29) show the required symmetry.

iv) Finally, we must show that all horizons with axial and collineation symmetry are described by the Theorem. Suppose \mathcal{M} is a horizon with these symmetries. We know from I, Lemma 6, that there is a unique totally autoparallel Cauchy surface \mathcal{S}_0 in \mathcal{M} . On the other hand, the Theorem 3 of I guarantees the existence and uniqueness of the subgroup \mathcal{G}' of the group \mathcal{G} of all HS of \mathcal{M} , which is isomorphic to $SO(2)$ and satisfies $\mathcal{G} = \mathcal{G}' \times \mathcal{H}$. $\mathcal{G}' \mathcal{S}_0 = \mathcal{S}_0$ holds, because \mathcal{G}' and \mathcal{H} commute, so the trajectories of \mathcal{G}' starting in \mathcal{S}_0 remain in \mathcal{S}_0 , and are closed. \mathcal{S}_0 is, therefore, a Riemannian surface with axial symmetry (and spherical topology), and we can choose coordinates ϑ, φ on it as follows. There must be two fixed points p and q of \mathcal{G}' on \mathcal{S}_0 ; because \mathcal{S}_0 is complete, there is at least one geodesic γ of length πR joining p with q . Choose ϑ along γ so that $\vartheta = \frac{1}{R} \cdot s$, s being the distance from p , say. The parameter of \mathcal{G}' is φ , $0 \leq \varphi \leq 2\pi$; the curves $g_\varphi \cdot \gamma$ form the $\varphi = \text{const}$ -curves, $\mathcal{G}' \cdot x$, $x \in \mathcal{S}_0$ the $\vartheta = \text{const}$ -curves of the coordinate system. The metric has clearly the form

$$ds^2 = -R^2(d\vartheta^2 + A^2 d\varphi^2),\tag{30}$$

where A satisfies the boundary conditions

$$A(0) = A(\pi) = 0, \quad A'(0) = -A'(\pi) = 1,$$

because \mathcal{S}_0 must be a smooth surface (a cusp would mean that the set of tangent vectors orthogonal to the ray going through the cusp does

not form a null hyperplane, which is impossible in regular space-times; I am indebted to S. Hawking for this remark). Now, we can extend the functions ϑ and φ to the whole of \mathcal{M} keeping them constant along rays; and we introduce an affine coordinate α such that $\alpha=0$ at \mathcal{S}_0 and the lines $\alpha=\text{const}$, $\vartheta=\text{const}$ are trajectories of \mathcal{G}' . Then, ϑ and φ are coordinates on $\tilde{\mathcal{S}}$ as well, so the metric \tilde{g} can be defined by (30). The vector \tilde{M} on $\tilde{\mathcal{S}}$ (see I) can be normalized as follows

$$\tilde{M} = \frac{1}{\sqrt{2}} \frac{1}{R} \left(\frac{\partial}{\partial \vartheta} + \frac{i}{A} \frac{\partial}{\partial \varphi} \right), \quad (31)$$

so that

$$\Gamma = \tilde{\Gamma} = -\frac{1}{\sqrt{2}} \frac{1}{R} \frac{A'}{A}, \quad \tilde{\mathcal{K}} = -\frac{1}{R^2} \frac{A''}{A}. \quad (32)$$

Clearly, $A^{-1}A''$ must be regular at the poles p, q .

Because $\frac{\partial}{\partial \varphi}$ is a generator of \mathcal{G}' , the scalars Ψ_2 and Φ_1 do not depend on φ . Then, the Eq. I(23) is

$$\frac{\partial \Omega_{\vartheta}}{\partial \varphi} - \frac{\partial \Omega_{\varphi}}{\partial \vartheta} = F(\vartheta), \quad (33)$$

where $F(\vartheta) = -i(\Psi_2 - \Psi_2^+)R^2A$. The vector field Ω_A given by

$$\Omega_{\vartheta} = 0, \quad \Omega_{\varphi} = B(\vartheta) \quad (34)$$

with $B' = F$ is a solution of (33). The Eq. I(22) then implies the existence of such a rescaling of α that Ω_A is given by (34) or

$$\Psi_2 - \Psi_2^+ = -iR^{-2}A^{-1}B'. \quad (35)$$

Ω_A is a continuous vector field, so it must be zero at the poles $\vartheta=0, \pi$:

$$B(0) = B(\pi) = 0; \quad (36)$$

$A^{-1}B'$, too, must be regular at the poles. (32), (35), and I(20) yield

$$\Psi_2 = C^2 + \frac{1}{2}R^{-2}A^{-1}A'' - \frac{i}{2}R^{-2}A^{-1}B', \quad (37)$$

where we have set

$$\Phi_1 = Ce^{iD} \quad (38)$$

$C(\vartheta)$ and $D(\vartheta)$ being real smooth functions on the circle satisfying

$$C(\vartheta) = C(2\pi - \vartheta), \quad D(\vartheta) = D(2\pi - \vartheta). \quad (39)$$

By rescaling α so that (34) holds we have arrived at an affine coordinate α' which is determined up to a transformations

$$\alpha'' = \eta \cdot \alpha'$$

where η is a constant. Geometrical meaning of α' is as follows: choose a vector L at $p \in \mathcal{S}_0$, use the parallel transport along the curves $\varphi = \text{const}$, and \mathcal{G}'_* -maps along $\vartheta = \text{const}$. A smooth, well-defined vector field L along \mathcal{S}_0 results because of (33). Parallel transport of L along the rays completes the definition of L on \mathcal{M} . α' is then the parameter of integral curves of L such that $\alpha' = 0$ at \mathcal{S}_0 .

From (31), I(21) and I(6), we obtain

$$\Omega = -\frac{i}{\sqrt{2}} R^{-1} A^{-1} B \quad (40)$$

and I(14) together with (37) implies

$$\mu = \mu_0 + \left(C^2 + \frac{1}{2} R^{-2} A^{-1} A'' - \frac{i}{2} R^{-2} A^{-1} B' \right) \cdot \alpha', \quad \lambda = \lambda_0,$$

where μ_0 and λ_0 are independent of α . But \mathcal{S}_0 is totally autoparallel; hence $\mu_0 = \lambda_0 = 0$ and

$$\mu = \left(C^2 + \frac{1}{2} R^{-2} A^{-1} A'' - \frac{i}{2} R^{-2} A^{-1} B' \right) \cdot \alpha', \quad \lambda = 0. \quad (41)$$

On the other hand, (40) and (31) yield

$$L = \frac{\partial}{\partial \alpha'}, \quad M = \frac{1}{\sqrt{2}} \frac{1}{R} \left(\frac{\partial}{\partial \vartheta} + \frac{i}{A} \frac{\partial}{\partial \varphi} + \frac{i}{A} B \alpha' \frac{\partial}{\partial \alpha'} \right). \quad (42)$$

We have arrived at the metric (30) and the affine connection (32), (40), and (41) in the frame (42) exactly as required by Theorem 5. In the frame (24) they have components independent of α' and φ . Now, we require that Ψ_3 , Ψ_4 and Φ_2 have the same property, or, using the transformation formulas (28) for $\eta = \frac{1}{\alpha'}$, $\xi = -\Omega$:

$$\begin{aligned} \Phi_2 &= \alpha' (\Phi'_2(\vartheta) - 2\Omega^+ \Phi_1), \\ \Psi_3 &= \alpha' (\Psi'_3(\vartheta) - 3\Omega^+ \Psi_2), \\ \Psi_4 &= \alpha'^2 (\Psi'_4(\vartheta) - 4\Omega^+ \Psi'_3(\vartheta) + 6\Omega^+ \Omega^+ \Psi_2), \end{aligned} \quad (43)$$

where $\Phi'_2(\vartheta)$, $\Psi'_3(\vartheta)$ and $\Psi'_4(\vartheta)$ are the α' , φ -independent components of Maxwell and Weyl spinors in the frame (24). Setting (43) into I(14) we

obtain

$$\begin{aligned}
\Phi'_2 &= M^+ \Phi_1 + 2\Omega^+ \Phi_1, \\
\Psi'_3 &= M^+ \Psi_2 + 3\Omega^+ \Psi_2 + \Phi_1^+ M^+ \Phi_1, \\
\Psi'_4 &= 4\Omega^+ \Psi_3 - 6\Omega^+ \Omega^+ \Psi_2 \\
&\quad + \frac{1}{2\alpha'} \{ (M^+ + \Gamma^+ + \Omega^+) [(M^+ \Psi_2 + \Phi_1^+ M^+ \Phi_1)\alpha'] \\
&\quad + \Phi_1^+ (M^+ + \Gamma^+ + \Omega^+) [(M^+ \Phi_1)\alpha'] \},
\end{aligned}$$

and (43) yields (19) at once, q.e.d.

Theorem 6. *For any analytic AC horizon \mathcal{M} , there is only one space-time $\bar{\mathcal{M}}$ satisfying the following conditions*

- 1) $\bar{\mathcal{M}}$ is analytic, symmetric under the action of a one-dimensional group of motions $\bar{\mathcal{G}}$, and contains a horizon \mathcal{M}' isomorphic to \mathcal{M}
- 2) The trajectories of $\bar{\mathcal{G}}$ starting at \mathcal{M}' do not leave \mathcal{M}' and are neither closed nor parallel to the rays of \mathcal{M}' .

Theorem 6 is stronger than the corresponding one given in [2] except for the analyticity requirement. This is likely not to be necessary, but without it we were, in fact, unable to prove anything². This new version has been proposed by S. Hawking.

Proof. With the same tools as S. Hawking used in [13] and [14], we show from 1), 2) that $\bar{\mathcal{M}}$ must, in fact, be axisymmetric and stationary. The group $\bar{\mathcal{H}}$ of motions of $\bar{\mathcal{M}}$ acting along the rays of \mathcal{M}' keeps all points of the totally autoparallel Cauchy surface \mathcal{S}'_0 of \mathcal{M}' fixed. According to the classification of such fixed points, given by Ehlers in [15], the two null directions orthogonal to \mathcal{S}'_0 at \mathcal{S}'_0 are fixed directions of $\bar{\mathcal{H}}$, and, according to Boyer's theorem [15], the two null hypersurfaces generated by the rays corresponding to these initial points and directions form a bifurcate Killing horizon of $\bar{\mathcal{M}}$. One of them is \mathcal{M}' , the second one can be denoted by \mathcal{M}'' . On \mathcal{M}' , $\bar{\mathcal{H}}$ acts along rays and the axisymmetric group $\bar{\mathcal{G}}$ of \mathcal{M}' does not move \mathcal{M}'' , because it does not move \mathcal{S}'_0 , and commutes with $\bar{\mathcal{H}}$. Thus, \mathcal{M}'' must be of the type AC. We compute the corresponding invariants. Clearly, A and R must be the same as for \mathcal{M}' , because the metric in \mathcal{M}'' is given by that of \mathcal{S}'_0 . At \mathcal{S}'_0 , the vectors L', M'' on \mathcal{M}'' are identical with n^i and m^{i+} of the pseudo-orthonormal tetrad l^i, m^i, m^{i+}, n^i , where $l^i = \theta_{*}(L)$ and $m^i = \theta_{*}(M)$ ([10]).

Therefore

$$V_{\mathcal{M}''} L' = n^i{}_{;j} m^{j+} = \Omega_{\mathcal{M}''} n^i,$$

² The analyticity in the conditions of the Theorem 6 could probably be completely discarded, because the stationary electrovacuum space-times have been shown to be analytic [12] (I am indebted to B. G. Schmidt for this remark).

or

$$\Omega_{\mathcal{M}''} = l^i n^i ;_j m^{j+} = -l_{i;j} m^{j+} n^i = -\Omega_{\mathcal{M}'}^+$$

thus, B is the same as for \mathcal{M}' . Finally, Φ_1 changes only its sign under the transformation $l_i \rightarrow n_i$, $m_i \rightarrow m_i^+$, so $C_{\mathcal{M}''} = C_{\mathcal{M}'}$, $D_{\mathcal{M}''} = D_{\mathcal{M}'} + \pi$. But then, \mathcal{M}'' , together with all initial data, is uniquely determined according to Theorem 5 and we have a well-posed initial value problem for the Einstein-Maxwell equations on the pair of null hypersurfaces \mathcal{M}' and \mathcal{M}'' . In the analytic case [11], there is one and only one development, which must, therefore, be $\overline{\mathcal{M}}$, q.e.d.

Theorem 6 implies, e.g., that the space-time given by (9) is the only axisymmetric and stationary one containing a horizon of type 2) of the Theorem 4. Another interesting consequence of the Theorem is that the total energy, total angular momentum and angular velocity of the black hole corresponding to an AC horizon, which are defined as some global functions on stationary axisymmetric asymptotically flat space-times containing the hole [7], must be some functionals of the horizon variables R, A, B, C, D . The explicit form of these functionals would be of great interest, but, unfortunately, the problem is not simple to solve.

We finish Section 3 by giving an example. The invariants R, A, B, C, D for the Kerr-Newman horizons with $m^2 > a^2 + e^2 + h^2$, $h=0$, ([16]) are determined by the following parametric relations:

$$\begin{aligned} R &= \frac{1}{\pi} \int_0^\pi \sqrt{r^2 + a^2 \cos^2 x} \, dx, \\ B(\psi) &= \frac{1}{R} \frac{r^2 + a^2}{\sqrt{r^2 + a^2 \cos^2 \psi}} \left(\frac{r-m}{r^2 + a^2} + \frac{r}{r^2 + a^2 \cos^2 \psi} \right), \\ C(\psi) &= \frac{e}{r^2 + a^2 \cos^2 \psi}, \quad D(\psi) = -i \lg \left(-\frac{r - ia \cos \psi}{r + ia \cos \psi} \right), \\ r &= m \pm \sqrt{m^2 - a^2 - e^2}, \quad A(\psi) = \frac{1}{R} \sqrt{\frac{r^2 + a^2}{r^2 + a^2 \cos^2 \psi}} \sin \psi. \end{aligned}$$

4. Axial Symmetry and Translation (AT -Horizons)

If the translation is generated by $L = \frac{\partial}{\partial \alpha}$, the Eqs. I(31), I(32) must be satisfied, and we have, in the frame L, M' ,

$$\begin{aligned} L &= L, \quad M' = M + \alpha \Omega L : \\ \nabla_L L &= 0, \quad \nabla_L M' = \Omega L, \quad \nabla_{M'} L = \Omega L, \\ \nabla_{M'} M' &= \lambda^+ L - \Gamma M', \quad \nabla_{M'} M'^+ = \mu_0 L + \Gamma M'^+. \end{aligned}$$

Also, Ψ'_3 , Ψ'_4 and Φ'_2 , the components of Weyl and Maxwell spinors in the frame L , M' , must be independent of α . Using (28) with $\eta = 1$, $\xi = -\Omega\alpha$, we obtain

$$\begin{cases} \Psi_3 = \Psi'_3 - 3\Omega^+ \Psi_2 \cdot \alpha, \\ \Psi_4 = \Psi'_4 - 4\Omega^+ \Psi_3 \cdot \alpha + 6\Omega^{+2} \Psi_2 \cdot \alpha^2, \\ \Phi_2 = \Phi'_2 - 2\Omega^+ \Phi_1 \cdot \alpha. \end{cases} \quad (44)$$

Then, I(14) implies

$$M^+ \Psi_2 + \Phi_1^+ M^+ \Phi_1 = -3\Omega^+ \Psi_2, \quad (45)$$

$$\begin{aligned} & (M^+ + \Gamma^+ + \Omega^+) \Psi_3 + \Phi_1^+ (M^+ + \Gamma^+ + \Omega^+) \Phi_2 \\ &= 3\lambda \Psi_2 + 2\lambda \Phi_1^+ \Phi_1 - 4\Omega^+ \Psi'_3 + 12\Omega^{+2} \Psi_2 \cdot \alpha, \end{aligned} \quad (46)$$

$$M^+ \Phi_1 = -2\Omega^+ \Phi_1. \quad (47)$$

Applying the operator M^+ on both sides of Eq. I(32), using I(12), I(31), I(32) again, the first relation of I(15) and (47), we obtain (45). Similarly, from I(31), (45), (47), and (44), we derive Eq. L (46) [which results by applying L on both sides of (46)]. Therefore, the only independent part of (46) is what remains after setting $\alpha = 0$ in it

$$\begin{aligned} & (M^+ + \Gamma^+ + 5\Omega^+) \Psi'_3 + \Phi_1^+ (M^+ + \Gamma^+ + \Omega^+) \Phi'_2 \\ & - (3\Psi_2 + 2\Phi_1^+ \Phi_1) \lambda = 0. \end{aligned} \quad (48)$$

The same holds for I(16), L (I(16)) being a consequence of (45), so there remains

$$(M^+ + \Omega^+) \mu_0 - (M - 2\Gamma + \Omega) \lambda - \Psi'_3 + \Phi_1^+ \Phi'_2 = 0. \quad (49)$$

We observe that the quantities and the independent equations separate into three groups:

- I. The quantities Γ , Ω , Ψ_2 , Φ_1 ,
The equations: I(31), I(32), I(15), (47).
- II. The quantities μ_0 , λ , Ψ'_3 , Φ'_2 ,
The equations: (48), (49).
- III. The quantities Ψ'_4 ,
The equations: none.

If we find a set of these quantities satisfying all these equations, then we have an AT -horizon. The equations of group I. can be solved separately, because they contain only the quantities of group I. Keeping such a solution Γ , Ω , Ψ_2 , Φ_1 fixed, the equations of group II. turn out

to be four linear differential equations of the first order for the eight variables of group II. Thus, there will be a whole family of horizons with fixed Γ , Ω , Ψ_2 , Φ_1 . Similarly, having fixed a solution of the group I. and II. equations, there will be some freedom in choosing Ψ_4 .

The solution of the group I. equations is given by

Lemma 13³. *If the horizon \mathcal{M} is axially symmetric and $\tilde{\mathcal{G}}$ has spherical topology, all solutions of the group I. equations form a three-parameter family given by*

$$\Gamma = -\frac{1}{\sqrt{2}} \frac{1}{R} \frac{A'}{A}, \quad \Omega = \frac{1}{\sqrt{2}} \frac{1}{R} \left(\frac{E'}{E} - ib \frac{A}{E^2} \right), \quad (50)$$

$$\Psi_2 = -\frac{1}{R^2} \frac{E''}{E} + \frac{ib}{R^2} \frac{AE' - A'E}{E^3}, \quad (51)$$

$$\Phi_1 = \frac{e}{E^2} \exp i \left(d - 2b \int_0^{\vartheta} AE^{-2} dx \right),$$

where

$$e = \frac{1}{\sqrt{2}} \frac{1}{R} \frac{2}{\pi} \mathbf{E}(v) \sqrt{1-2v^2}, \quad b = -\left(\frac{2}{\pi} \mathbf{E}(v) \right)^2 v \sqrt{1-v^2}, \quad (52)$$

$$0 \leq v \leq \frac{1}{\sqrt{2}},$$

v , R , d being the parameters, and $A(\vartheta)$, $E(\vartheta)$ are defined by

$$A = \frac{\pi}{2} \frac{1}{\mathbf{E}(v)} \frac{\sin \zeta}{\sqrt{1-v^2 \sin^2 \zeta}}, \quad E = \sqrt{1-v^2 \sin^2 \zeta}, \quad (53)$$

$$\vartheta = \frac{\pi}{2} \frac{1}{\mathbf{E}(v)} \int_0^{\zeta} \sqrt{1-v^2 \sin^2 \alpha} d\alpha, \quad 0 \leq \zeta \leq \pi; \quad (54)$$

$\mathbf{E}(v)$ is the complete elliptic integral

$$\mathbf{E}(v) = \int_0^{\frac{\pi}{2}} \sqrt{1-v^2 \sin^2 \alpha} d\alpha. \quad (55)$$

Reparametrizing

$$R = \frac{2}{\pi} \left(\mathbf{E} \left(\frac{a}{\sqrt{m^2+a^2}} \right) \right) \sqrt{m^2+a^2}, \quad v = \frac{a}{\sqrt{m^2+a^2}}, \quad (56)$$

we recover the well-known expressions for the Kerr-Newman extreme horizons [16] ($m^2 = a^2 + e^2 + h^2$) with the electric and magnetic charges

$$e = \sqrt{2}(m^2+a^2)c \cos d, \quad h = \sqrt{2}(m^2+a^2)c \sin d.$$

³ Similar results for the non-charged case have been obtained by J. Bardeen [17].

Proof. The spherical topology and axial symmetry enable us to choose the coordinates ϑ , φ exactly as in Section 3; ϑ , φ and α are well-defined because the translation commutes with the axial symmetry. The metric is then given by (30), which implies (31) and (32). On the other hand, Ω must have the form

$$\Omega_{\vartheta} = -\frac{E'}{E}, \quad \Omega_{\varphi} = B(\vartheta), \quad (57)$$

where $E(\vartheta)$ must satisfy the boundary conditions

$$E'(0) = E'(\pi) = 0, \quad (58)$$

and can be normalized so that

$$E(0) = 1; \quad (59)$$

$$L = \frac{\partial}{\partial \alpha}, \quad M = \frac{1}{\sqrt{2}} \frac{1}{R} \left(\frac{\partial}{\partial \vartheta} + iA^{-1} \frac{\partial}{\partial \varphi} \right) - \alpha \cdot \Omega \frac{\partial}{\partial \alpha}. \quad (60)$$

Finally, we set as in Section 3,

$$\Phi_1 = C e^{iD}. \quad (61)$$

(32), I(19), (61), and I(15) yield

$$\frac{1}{2} (\Psi_2 + \Psi_2^+) = C^2 + \frac{1}{2} \frac{1}{R^2} \frac{A''}{A}, \quad (62)$$

(47), together with (61) and (60), implies

$$\Omega = -\frac{1}{2\sqrt{2}} \frac{1}{R} \frac{C'}{C} + \frac{i}{2\sqrt{2}} \frac{D'}{R}.$$

On the other hand, from (57), (60), and I(21), we have

$$\Omega = \frac{1}{\sqrt{2}} \frac{1}{R} \frac{E'}{E} - \frac{i}{\sqrt{2}} \frac{1}{R} \frac{B}{A}, \quad (63)$$

or

$$C = cE^{-2}, \quad D' = -2A^{-1}B \quad (64)$$

where c is a real constant. Setting (60) and (63) in I(31) and I(32) we have

$$B = b \cdot \frac{A^2}{E^2}, \quad \Psi_2 = -\frac{1}{R^2} \left(\frac{A'E'}{AE} + b^2 \frac{A^2}{E^4} + ib \frac{A'E - AE'}{E^2} \right),$$

$$E'' - A^{-1}A'E' - b^2A^2E^{-3} = 0, \quad (65)$$

where b is a real constant; (62) and (64) then imply

$$A'' + 2E^{-1}A'E' + 2b^2A^3E^{-4} + 2c^2R^2AE^{-4} = 0. \quad (66)$$

Hence, the functions $E(\vartheta)$ and $A(\vartheta)$ must satisfy the Eqs. (65) and (66) supplemented by the boundary conditions

$$E(0)=1, A(0)=0, E'(0)=0, A'(0)=1, \quad A^{-1}A'', A^{-1}E \text{ regular at } 0, \quad (67)$$

$$A(\pi)=0, E'(\pi)=0, A'(\pi)=-1, \quad A^{-1}A'', A^{-1}E \text{ regular at } \pi, \quad (68)$$

Ψ_2, Φ_1, Γ and Ω being then given by (50) and (51).

General properties of the system (65), (66) can most clearly be seen if we write it in the form

$$E' = A^{-1}V_E, \quad A' = A^{-1}V_A, \quad F' = A^{-1}V_F, \quad G' = A^{-1}V_G,$$

where

$$V_E = AF, \quad V_A = AG,$$

$$V_F = FG + b^2 A^3 E^{-3}, \quad V_G = 2AE^{-1}FG + 2b^2 A^4 E^{-4} + 2c^2 R^2 A^2 E^{-4}.$$

The integral curves of the vector field \vec{V} in the space of all (E, A, F, G) are identical with the integral curves of the original system up to a parametrization. \vec{V} is analytic about the point $P=(1, 0, 0, 1)$ corresponding to (67), so there is at least one analytic integral curve of \vec{V} through P . But P is a zero point of \vec{V} . Setting

$$E = 1 + \varepsilon n_E, \quad A = \varepsilon n_A, \quad F = \varepsilon n_F, \quad G = 1 + \varepsilon n_G,$$

we have

$$V_E = O(\varepsilon^2), \quad V_A = n_A \cdot \varepsilon + O(\varepsilon^2), \quad V_F = n_F \cdot \varepsilon + O(\varepsilon^2), \quad V_G = O(\varepsilon^2),$$

hence, P is approached by a whole one-parametric family of integral curves of \vec{V} , the parameter being (because $A > 0$ for $\vartheta > 0$ at $\vartheta = 0$)

$$f = \frac{n_F}{n_A} = \lim_{\vartheta \rightarrow 0} \frac{E'}{A}. \quad (69)$$

We have, therefore, exactly one solution of (65), (66) satisfying (67) for a fixed values of b, cR and f . We find all these solutions and then look to see which of them satisfy (68).

Because (65), (66) do not contain ϑ explicitly, we can choose E as a new independent variable. (By this step, we could omit all solutions with $E = \text{const}$, but we shall see that this is not the case here.) Then,

$$E' = p(E), \quad E'' = p\dot{p}, \quad A = q(E), \quad A' = \dot{q}p, \quad A'' = \ddot{q}p^2 + \dot{q}\dot{p}p,$$

the dot denoting E -derivative, and (65), (66) become

$$p\dot{p} - p^2 q^{-1} \dot{q} - b^2 E^{-3} q^2 = 0, \\ p^2 \ddot{q} + p\dot{p}\dot{q} + 2E^{-1} p^2 \dot{q} + 2b^2 E^{-4} q^3 + 2c^2 R^2 E^{-4} q = 0.$$

The solution of the first equation under the condition $\lim_{E \rightarrow 1} \frac{P}{q} = f$ is

$$p^2 = E^{-2} q^2 [(b^2 + f^2) E^2 - b^2]. \quad (70)$$

Putting this into the second equation yields

$$E^2 [(b^2 + f^2) E^2 - b^2] \ddot{x} + E [2(b^2 + f^2) E^2 - b^2] \dot{x} + 4b^2 x + 4b^2 P^2 = 0 \quad (71)$$

where $x = q^2$, $P = \frac{cR}{b}$. We assume $b \neq 0$. The case $b = 0$ is simple to deal with, but all the solutions obtained there are limiting cases of our solutions. A general solution of (71) is

$$x = -P^2 + Q_1 \cos 2\xi + Q_2 \sin 2\xi, \\ E = \frac{b}{\sqrt{b^2 + f^2}} \frac{1}{\sin \xi}, \quad (72)$$

where ξ is a parameter Q_1 and Q_2 some real constants. From $E' = \sqrt{y}$ (setting $E' = -\sqrt{y}$ reduces to reversing the ϑ -axis), (70), (72), and (67), we obtain

$$d\vartheta = -\frac{b}{b^2 + f^2} \frac{d\xi}{\sin^2 \xi \sqrt{x}}, \\ \vartheta(\xi_0) = 0: \sin \xi_0 = \frac{b}{\sqrt{b^2 + f^2}}, \quad \cos \xi_0 = \frac{f}{\sqrt{b^2 + f^2}}. \quad (73)$$

Because of $\frac{dx}{d\xi} = 2\sqrt{x} A' \frac{d\vartheta}{d\xi}$, we have

$$\frac{dx}{d\xi} = -\frac{2b}{b^2 + f^2} \frac{A'}{\sin^2 \xi}, \quad (74)$$

therefore, $x(\xi_0) = 0$, $\frac{dx}{d\xi}(\xi_0) = -\frac{2}{b}$, and the general solution for the initial conditions (67) is

$$x = 2Q \sin(\xi - \xi_0) \sin(\xi_0 + \eta_0 - \xi), \quad (75)$$

where

$$Q = \sqrt{P^4 + \frac{1}{b^2}}, \quad \cos \eta_0 = \frac{P^2}{Q}, \quad \sin \eta_0 = -\frac{1}{bQ}.$$

Distinguish the cases

$$\text{I. } b > 0: \frac{d\xi}{d\vartheta} < 0, \quad -\frac{\pi}{2} \leq \eta_0 < 0; \quad x > 0: \xi_0 + \eta_0 < \xi < \xi_0, \\ x(\xi_0 + \eta_0) = x(\xi_0) = 0.$$

$$\text{II. } b < 0: \frac{d\xi}{d\vartheta} > 0, \quad 0 < \eta_0 \leq \frac{\pi}{2}; \quad x > 0: \xi_0 < \xi < \xi_0 + \eta_0, \\ x(\xi_0) = x(\xi_0 + \eta_0) = 0.$$

In both cases, we must have $\xi(\pi) = \xi_0 + \eta_0$ because of the first condition of (68), or

$$-\frac{b}{b^2 + f^2} \int_{\xi_0}^{\xi_0 + \eta_0} \frac{d\xi}{\sin^2 \xi \sqrt{x}} = \pi. \quad (76)$$

The second condition of (68) is satisfied automatically. The third one is in both cases equivalent to

$$\sin^2(\xi_0 + \eta_0) = \sin^2(\xi_0) = \frac{b^2}{b^2 + f^2}. \quad (77)$$

Now, dividing I and II into subcases according as $f \geq 0$, $f < 0$, $\xi_0 + \eta_0 \geq 0$, $\xi_0 + \eta_0 < 0$, a detailed analysis shows that all solutions can be found under the subcase $b < 0$, $f < 0$. Then

$$\pi < \xi_0 < \frac{3\pi}{2}, \quad 0 < \eta_0 \leq \frac{\pi}{2}, \quad \pi < \xi_0 + \eta_0 < 2\pi$$

and (77) yields $\eta_0 = 3\pi - 2\xi_0$, or $\pi + \frac{\pi}{4} \leq \xi_0 < \pi + \frac{\pi}{2}$, so $b \leq f$ and

$$P^2 = Q \frac{b^2 - f^2}{b^2 + f^2}.$$

Solving this for P^2 , we obtain finally

$$P^2 = \frac{1}{b} \frac{f^2 - b^2}{2bf}, \quad Q = -\frac{1}{b} \frac{f^2 + b^2}{2bf}, \quad b \leq f < 0, \\ x = \frac{1}{b} \frac{b^2 - f^2}{2bf} + \frac{1}{b} \frac{b^2 + f^2}{2bf} \cos 2\xi. \quad (78)$$

Being given this special shape of x , we can greatly simplify all calculations by introducing a new variable ζ defined by

$$\frac{b}{\sqrt{b^2 + f^2}} \frac{1}{\sin \xi} = \sqrt{1 - \frac{f^2}{b^2 + f^2}} \sin \zeta = E(\zeta), \quad 0 \leq \zeta \leq \pi. \quad (79)$$

Then,

$$x = \frac{-f}{b^2 + f^2} \frac{\sin^2 \zeta}{E^2}, \quad \frac{d\vartheta}{d\zeta} = \frac{\sqrt{-f}}{\sqrt{b^2 + f^2}} \cdot E, \quad (80)$$

and (76) reduces to

$$2 \frac{\sqrt{-f}}{\sqrt{b^2 + f^2}} \mathbf{E}(v) = \pi, \quad (81)$$

where

$$v = \frac{-f}{\sqrt{b^2 + f^2}}, \quad 0 < v \leq \frac{1}{\sqrt{2}}, \quad (82)$$

$\mathbf{E}(v)$ being given by (55). Equations (81) and (82) can be solved for b and f , giving

$$b = -\left(\frac{2}{\pi} \mathbf{E}(v)\right)^2 v \sqrt{1 - v^2}, \quad f = -\left(\frac{2}{\pi} \mathbf{E}(v)\right)^2 \cdot v^2,$$

whereas the formulas (78) and (80) yield the rest of (52), (53) and (54), if $v > 0$. But the functions $A_v(\zeta)$, $E_v(\zeta)$ and $\vartheta_v(\zeta)$ have well-defined values for $v = 0$, too, which corresponds to $b = 0$. Now, setting $b = 0$ in the original systems (65), (66), it is very easy to show that the only solution of it satisfying (67), (68) is given by $A_0(\zeta)$, $E_0(\zeta)$, $\vartheta_0(\zeta)$. Moreover, $E = \text{const}$ is a subcase of $b = 0$, so Lemma 13 is shown.

It follows that any AT -horizon is characterized by well-defined ‘‘Kerr-Newman’’ parameters a , e , h . From now on, we shall suppose a , e , h fixed and turn our attention to the group II. quantities and equations. The main difficulty at this step is not solving the Eqs. (48), (49) which are rather trivial, but finding independent invariants describing the degrees of freedom in these quantities. On AT -horizons, we can choose $\text{Im}M$ in the direction of $\frac{\partial}{\partial\varphi}$, but there is no simple invariant manner of prescribing the direction of $\text{Re}M$, while L is determined up to a constant factor:

$$L' = \eta L, \quad M' = M + \xi L. \quad (83)$$

Under this transformation, the group I. quantities remain invariant, while the second group changes like this:

$$\begin{aligned} \lambda' &= \frac{1}{\eta} (\lambda + M^+ \xi^+ + \xi^+ L \xi^+ + \Gamma^+ \xi^+ + \Omega^+ \xi^+), \\ \Psi'_3 &= \frac{1}{\eta} (\Psi_3 + 3\xi^+ \Psi_2), \\ \mu' &= \frac{1}{\eta} (\mu + M \xi^+ + \xi L \xi^+ - \Gamma \xi^+ + \Omega \xi^+), \\ \Phi'_2 &= \frac{1}{\eta} (\Phi_2 + 2\xi^+ \Phi_1). \end{aligned} \quad (84)$$

Our way out of the difficulty is based on the observation that the differences $(\Delta\lambda, \Delta\mu, \Delta\Psi_3, \Delta\Phi_2)$ between two sets of these quantities transform much more simply. We choose some standard horizons, e.g. the extreme Kerr-Newman ones, and consider only the excesses of the second group quantities over their standard values. This approach has one more advantage that the resulting invariants measure the departure from a well-known situation, being zero for Kerr-Newman horizons. A correct performing of this simple program can be based on the following

Lemma 14. *Let (\mathcal{M}, θ) be a perfect horizon in a space-time $\bar{\mathcal{M}}$ (see I). Then, for any autoparallel γ on \mathcal{M} parametrized by an affine parameter*

- 1) $\bar{\gamma} = \theta \circ \gamma$ is a geodesic in $\bar{\mathcal{M}}$ parametrized by an affine parameter,
- 2) $\tilde{\gamma} = \pi \circ \gamma$ is a geodesic in $\tilde{\mathcal{S}}$ parametrized by an affine parameter.

Moreover, if γ is not a ray, and coordinates β, s on $\pi^{-1}(\tilde{\gamma})$ are chosen such that

$$g\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) = -1, \quad g\left(\frac{\partial}{\partial \beta}, \frac{\partial}{\partial \beta}\right) = 0$$

$$\nabla_{\frac{\partial}{\partial \beta}} \frac{\partial}{\partial \beta} = 0, \quad \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial \beta} = 0$$

and the fields L, M on \mathcal{M} such that, at $\pi^{-1}(\tilde{\gamma})$,

$$L = \frac{\partial}{\partial \beta}, \quad \text{Re } M = \frac{\partial}{\partial s},$$

M tangential to the surface $\beta = 0$, then γ satisfies the equation

$$\frac{d^2 \beta(\gamma(s))}{ds^2} + \text{Re } \Psi_2 \cdot \beta(\gamma(s)) + \text{Re}(\mu_0 + \lambda) = 0. \quad (85)$$

Within the formalism of I, the proof is very simple and we omit it. Now, choose a point p on the ray with $\vartheta = 0$, and a direction

$$\frac{d\alpha}{ds} = 0, \quad \frac{d\vartheta}{ds} = \frac{1}{R}, \quad \varphi = \varphi_0 \quad (86)$$

at p (values of φ at $\vartheta = 0$ describe directions rather than points). From Lemma 14, it follows that $\varphi = \varphi_0$, $\frac{d\vartheta}{ds} = \frac{1}{R}$ all along the autoparallel γ_{0,p,φ_0} determined by the initial data (86). Thus γ_{0,p,φ_0} must intersect the ray $\vartheta = \pi$. On the other hand, any two autoparallels $\gamma_{0,p,\varphi_0}, \gamma_{0,p,\varphi_0+\Delta\varphi}$ intersect $\vartheta = \pi$ at the same point, say q , because their initial data are transformed into each other by a rotation, therefore, so must the auto-

parallels themselves. Moreover, if l is the α -component $\left. \frac{d\alpha}{ds} \right|_q$ of the unit tangent vector to γ_{0,p,φ_0} at $\vartheta = \pi$, then l depends neither on p nor on φ_0 , because γ_{0,p,φ_0} is transformed into γ_{0,p',φ'_0} by a combination of translation and rotation [l does not depend even on whether the autoparallel starts at $\vartheta = \pi$ (γ_{π,p,φ_0}) and l is measured at $\vartheta = 0$, as is easily computed from (85), (50), (51), and I(22)]. On the other hand, under any rescaling of α , l is multiplied by a positive constant. Thus, α can always be rescaled in such a way that either $l = 1$, or $l = 0$, or $l = -1$; the cases $l = 1$ and $l = -1$ go over into each other by time inversion.

An example of an (a, e, h) -horizon with $l = 1$ ($l = -1$) is the future (past) extreme Kerr-Newman horizon ($m^2 = a^2 + e^2 + h^2$). The most simple (= most symmetric) example of the $l = 0$ case is constructed as follows. The maximal extension of any $\gamma_{0,p,\varphi}$ is a closed autoparallel. The rotation surface \mathcal{S} defined by it can be chosen to be totally autoparallel. Then, we have the maximal longitudinal symmetry: a group generated by the translation and the collineation corresponding to \mathcal{S} . Horizons of this type are fully described by three parameters a, e, h , and they can be imbedded, under the conditions of Theorem 6, only in space-times $\tilde{\mathcal{S}} \times P_m^2$, where $\tilde{\mathcal{S}}$ is a space-like section of an extreme (a, e, h) -Kerr-Newman horizon, $m^2 = a^2 + e^2 + h^2$, P_m^2 is the pseudosphere with signature zero, radius m and topology of \mathbb{R}^2 and \times denotes the Cartesian product of two metric manifolds, analogously to the spherically symmetric situation (this easily follows from the Theorem 6). We choose these three examples as our standards.

Consider a horizon with $l = 1$. The scaling of the affine coordinate α and the field L is fixed. Choose the origin of α at the rotation surface \mathcal{S} based on an autoparallel $\gamma_{0,p,0}$, and the field M tangential to \mathcal{S} , and parallel-propagated along the rays. Any two such so-called canonical coordinate systems are transformed into each other by means of a translation and possibly $\vartheta \rightarrow \pi - \vartheta$. In any such triad field L, M we must have

$$\text{Im } \mu_0 = 0, \quad \text{Re}(\mu_0 + \lambda) = 0, \quad (87)$$

because 1) M is tangential to a surface $\alpha = 0$ and 2) $\alpha = 0$, $\varphi = \text{const}$ is an autoparallel, so Lemma 14 implies the second identity. On the other hand, the identities (87) imply that $\alpha = 0$, $\varphi = \text{const}$ is an autoparallel and M is tangential to the surface $\alpha = 0$, which, with the symmetry of the horizon, implies the very same canonical system. Thus, (87) are the only constraints on λ and μ_0 .

Given two (a, e, h) -horizons \mathcal{M}, \mathcal{N} with $l = 1$ and some canonical systems on them, there is a unique diffeomorphism $\psi: \mathcal{M} \rightarrow \mathcal{N}$ mapping

the canonical system of \mathcal{M} into the canonical system of \mathcal{N} . Moreover, any two such maps ψ_1, ψ_2 differ at most by a horizon symmetry combined possibly with $\vartheta \rightarrow \pi - \vartheta$. That is to say, we can compare the group II. quantities on \mathcal{M} and \mathcal{N} by means of ψ . Choosing for \mathcal{N} the future Kerr-Newman (a, e, h) -horizon, we can describe \mathcal{M} by the differences

$$\Delta\mu_0 = \psi(\mu_0) - \mu_{0KN}, \quad \Delta\lambda = \psi(\lambda) - \lambda_{KN},$$

μ_0 and λ being the values for \mathcal{M} and KN denoting Kerr-Newman. From (87) it follows

$$\text{Im} \Delta\mu_0 = 0, \quad \text{Im}(\Delta\lambda + \Delta\mu_0) = 0,$$

so we have

$$\Delta\lambda = \Delta_1 + i\Delta_2, \quad \Delta\mu_0 = -\Delta_1,$$

where Δ_1, Δ_2 are two real functions of ϑ determined up to $\vartheta \rightarrow \pi - \vartheta$. The $\Delta_i(\vartheta)$ are not completely arbitrary at the endpoints $\vartheta=0, \vartheta=\pi$, because, in a special coordinate system, they coincide with real and imaginary part of a spin-weight-(-2)-quantity. Therefore

$$\Delta_1(0) = \Delta_2(0) = \Delta_1(\pi) = \Delta_2(\pi) = 0.$$

There are some more relations including derivatives; the best method to incorporate them is to write $\Delta_1 + i\Delta_2$ as a series in the spin weighted harmonics [7].

The Eqs. (48), (49) are linear homogeneous and they hold, therefore, for the differences $\Delta\lambda, \Delta\mu_0, \Delta\Psi'_3$ and $\Delta\Phi'_2$ as well. We obtain

$$\Delta\Phi'_2 = g \cdot \frac{A\Phi_1^{3/2}}{(\Phi_1^+)^{1/2}} + \frac{R}{\sqrt{2}} \frac{A\Phi_1^{3/2}}{(\Phi_1^+)^{1/2}} \int_0^\vartheta \frac{\mathcal{F}(x) dx}{A\Phi_1^{3/2}(\Phi_1^+)^{1/2}},$$

$$\Delta\Psi'_3 = \Phi_1^+ \Delta\Phi'_2 - (M^+ + \Omega^+) \Delta_1 - (M - 2\Gamma + \Omega)(\Delta_1 + i\Delta_2),$$

$$\begin{aligned} \mathcal{F}(x) = & (3\Psi_2 + 2\Phi_1^+ \Phi_1)(\Delta_1 + i\Delta_2) \\ & + (M^+ + \Gamma^+ + \Omega^+) [(M^+ + \Omega^+) \Delta_1 + (M - 2\Gamma + \Omega)(\Delta_1 + i\Delta_2)], \end{aligned}$$

where g is a real constant. Summarizing: one parameter g and two functions $\Delta_1(\vartheta), \Delta_2(\vartheta)$ represent (up to the transform $\vartheta \rightarrow \pi - \vartheta$) all the degrees of freedom in the second group quantities in case $l=1$. For other cases, the results are similar. The group III. quantities contain obviously another two functions of ϑ .

Acknowledgement. The author thanks Dr. S. W. Hawking, Dr. J. Bardeen, Prof. W. Kundt, Prof. J. Ehlers and Dr. B. G. Schmidt for many suggestions and discussions as well as the relativity group of the Max-Planck-Institut in Munich where a part of this work was done, for their nice hospitality.

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P. Hájíček
Institute for Theoretical Physics
University of Berne
Berne, Switzerland