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Exact Models of Charged Black Holes

I. Geometry of Totally Geodesic Null Hypersurface*

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Abstract. Inner geometry and embedding formulas for a totally geodesic null hypersurface $\mathscr{M}$ in an electrovacuum space-time are given. The structure of all possible symmetry groups of the geometry is described in case that the space-like sections $\mathscr{S}$ of $\mathscr{M}$ are compact orientable surfaces and $\mathscr{M}$ is, topologically, $\mathscr{S} \times R^1$. The result is $\mathscr{G} = \tilde{\mathscr{G}} \times \mathscr{H}$, where $\tilde{\mathscr{G}}$ are the well-known isometry groups of $\mathscr{S}$, and $\mathscr{H}$ is an at most two-dimensional group acting along rays, which is fully specified in the paper. It is not shown that all these symmetry types exist, but this will be done in the next papers where all horizons of a given symmetry type will be explicitly written down.

1. Introduction

The interest in the theory of collapsed stars – black holes – has recently increased, not only because of a theoretical appeal of these objects. It seems that some phenomena observed in the sky, e.g. the large X-ray source in Cygnus [1], can most naturally be explained by the accretion of matter onto a black hole [2].

In the present and subsequent papers, we describe a way of obtaining exact models of charged (electrically and magnetically) black holes surrounded by matter and charge currents without solving explicitly the Einstein-Maxwell equations. The method is akin to that of [3], [4], and [5], in which the initial value constraints of Einstein's equations are solved along a space-like hypersurface of time symmetry, or, geometrically, along a totally geodesic space-like hypersurface. What we are doing is to solve the initial value constraint of the *characteristic* Cauchy problem for Einstein-Maxwell equations along a totally geodesic *null* hypersurface (TGNH). The relation to black holes is as follows. Hawking [6] has shown that the area of black holes cannot decrease. In a limiting case, it will remain constant, the convergence ϱ and the shear σ will be zero and the horizon will be TGNH; such a black hole cannot swallow anything and can be, therefore, called "a fasting black hole" (a name proposed by Kundt). In the "thermodynamic" language of [7], one could also call the holes "adiabatic", but we should like to

* Devoted to Professor André Mercier who is sixty in April, 1973.

preserve this word for a sort of slow change processes with our models. All stationary black holes are fasting, but the converse is not true, as we shall see later. Thus, restricting ourselves to the fasting holes, we still deal with a more general situation than any known model ever described.

All TGNH have a nice inner geometrical structure induced by that of the embedding space-time and consisting of 1) degenerate metric (as any null hypersurface) 2) three-dimensional complete affine connection (only on TGNH). The metric and the connection are not related in the usual way by means of the Christoffel symbols; rather, they form, to a degree, two independent structures, so that there are horizons with the same metric and different connections and vice versa. We look for the symmetries of this structure, i.e. the Lie groups of diffeomorphisms which preserve the metric and the connection being, thus, simultaneously isometry and affine transformation groups ([8], p. 225–236). Properties of these so-called horizon symmetries (HS) are investigated in Section 3, where the structure of all HS groups is found.

The first steps in the theory of HS have been done by Hawking, who has shown that a stationary horizon is either static or axisymmetric. We rediscover many results and methods of [9] and [10], mainly in the proofs of the Lemmas 7 and 11. But the whole approach as well as our final Theorems are believed to be original.

The notation and all undefined notions are taken over from [8] and [9], which can be regarded as standarts for mathematical relativists.

2. Geometry of *TGNH*

a) Embedding Formulae

A null hypersurface in a space-time $\bar{\mathcal{M}}$ is a three-dimensional manifold $\mathcal{M}$ together with an embedding $\Theta : \mathcal{M} \to \bar{\mathcal{M}}$, so that the induced metric form $g = \Theta^*(\bar{g})$ is degenerate. Θ_* maps vector fields on $\mathcal{M}$ into vector fields on $\Theta(\mathcal{M})$. If there is a vector field Z on $\mathcal{M}$ for any two vector fields X, Y on $\mathcal{M}$ such that

$$\Theta_*(Z) = \bar{\nabla}_{\Theta_*(Y)} \Theta_*(X), \tag{1}$$

where $\bar{\nabla}_X$ denotes the covariant derivative of $\bar{\mathcal{M}}$, then $(\mathcal{M}, \Theta)$ is called a totally geodesic null hypersurface. On any TGNH $(\mathcal{M}, \Theta)$, there is a unique affine connection ∇ such that

$$\Theta_*(\nabla_Y X) = \bar{\nabla}_{\Theta_*(Y)} \Theta_*(X). \tag{2}$$

We call ∇ the induced affine connection.

Any vector field L on $\mathcal{M}$ tangent to the rays of $\mathcal{M}$ (we prefer to call the null geodesics in $\mathcal{M}$ rays instead of the more common "generators", preserving the latter word for symmetry groups generators) is orthogonal to any vector field on $\mathcal{M}$, $g(L, X) = 0$. We denote such a field always by L.

Lemma 1. *The necessary and sufficient condition for the null hypersurface $(\mathcal{M}, \Theta)$ to be a TGNH is that*

$$\bar{g}(\Theta_*(X), \bar{\nabla}_{\Theta_*(Y)} \Theta_*(L)) = 0 \tag{3}$$

be satisfied for any two vector fields X, Y and any L on $\mathcal{M}$.

Proof. Iff a vector field $\bar{Z}$ on $\Theta(\mathcal{M})$ in $\bar{\mathcal{M}}$ satisfies the condition $\bar{g}(\bar{Z}, \Theta_*(L)) = 0$, then there is a vector field Z on $\mathcal{M}$ such that $\Theta_*(Z) = \bar{Z}$. The equation

$$\bar{g}(\bar{\nabla}_{\Theta_*(Y)} \Theta_*(X), \Theta_*(L)) = 0 \tag{4}$$

is, therefore, equivalent to (1). On the other hand, (4) is equivalent to (3), because $\bar{\nabla}$ is the Riemannian connection of $\bar{g}$ ([8]), q.e.d.

Choose a real vector field L on $\mathcal{M}$ satisfying

$$\nabla_L L = 0, \tag{5}$$

and a complex vector field M on $\mathcal{M}$ satisfying

$$g(M^+, M) = -1, \qquad \nabla_L M = 0, \tag{6}$$

where "+" denotes the complex conjugate. The connection ∇ can then be described through the four complex coefficients Γ, Ω, λ, μ as follows

$$\left.\begin{aligned} &\nabla_L L = \nabla_L M = \nabla_L M^+ = 0, \\ &\nabla_M L = \Omega L, && \nabla_{M^+} L = \Omega^+ L, \\ &\nabla_M M = \lambda^+ L - \Gamma M, && \nabla_{M^+} M = \mu^+ L + \Gamma^+ M, \\ &\nabla_M M^+ = \mu L + \Gamma M^+, && \nabla_{M^+} M^+ = \lambda L - \Gamma^+ M^+. \end{aligned}\right\} \tag{7}$$

The self-consistence and full generality of (7) is guaranteed by (5), (6) and the relation

$$g(X, \nabla_Z Y) = -g(Y, \nabla_Z X), \tag{8}$$

following from $g = \Theta^*(\bar{g})$ and (2). The Lemma 1 and (7) implies that the existence of affine connection ∇ on $(\mathcal{M}, \Theta)$ satisfying (8) is only possible, if $(\mathcal{M}, \Theta)$ is totally geodesic ([11]).

The curvature tensor R for ∇ is easily computed from the well-known relation

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z \tag{9}$$

and (7):

$$\left.\begin{aligned} R(M, M^+)L &= (M\Omega^+ - M^+\Omega - \Gamma\Omega^+ + \Gamma^+\Omega)L, \\ R(M, M^+)M &= (M\mu^+ - M^+\lambda^+ + 2\Gamma^+\lambda^+ + \mu^+\Omega - \lambda^+\Omega^+)L \\ &\quad + (M\Gamma^+ + M^+\Gamma - 2\Gamma\Gamma^+)M, \\ R(L, M)L &= (L\Omega)L, \\ R(L, M)M &= (L\lambda^+)L - (L\Gamma)M, \\ R(L, M)M^+ &= (L\mu)L + (L\Gamma)M^+. \end{aligned}\right\} \tag{10}$$

We have used

$$[M, M^+] = \nabla_M M^+ - \nabla_{M^+} M = (\mu - \mu^+) L + \Gamma M^+ - \Gamma^+ M\,, \quad (11)$$

$$[L, M] = \nabla_L M - \nabla_M L = -\Omega L\,, \quad (12)$$

the symbol Xf denoting the derivation of the function f in the direction X.

The curvature tensor of a non-metric connection in a three-dimensional space has, generally, 24 independent components and the following symmetries

$$R(X, Y)Z + R(Y, X)Z = 0\,,$$

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0\,,$$

([8]). The first relation is automatically incorporated in (9). The second one yields

$$L\Gamma = 0\,,$$

$$-L(\mu - \mu^+) = M\Omega^+ - M^+\Omega - \Gamma\Omega^+ + \Gamma^+\Omega\,.$$

Thus, our curvature R has just 9 independent (real) components in comparison with the twenty four possible for general non-Riemannian connections and 6 possible for Riemannian connections in three dimensions. The 15 extra relations are

$$g(X, R(Y, U)V) + g(V, R(Y, U)X) = 0\,,$$

for any four vector fields X, Y, U, V and are related to the existence of the degenerate metric.

From the definition of $\nabla(2)$ and from (9), the Gauss-Coddazzi equation follows

$$\bar{R}(\Theta_*(X), \Theta_*(Y))\,\Theta_*(Z) = \Theta_*(R(X, Y)Z)\,.$$

It allows to compute 14 of the 20 independent components of $\bar{R}$ from the components of R, at every point of $\Theta(\mathscr{M})$. But the components of $\bar{R}$ must satisfy the Einstein-Maxwell equations and Bianchi identities. We shall not compute the implications of these relations from the Gauss-Coddazzi equation, which would be the most direct method, but use rather the Newman-Penrose equations [12], which will be more rapid.

Introduce a pseudo-orthonormal tetrad field X_1, X_2, X_3, X_4 along $\Theta(\mathscr{M})$ such that

$$\bar{X}_1 = \Theta_*(L),\ \bar{X}_3 = \Theta_*(M),\ \bar{X}_4 = \Theta_*(M^+) \quad (13)$$

and extend it differentiably in a neighbourhood of $\Theta(\mathscr{M})$. The corresponding spin coefficients must then satisfy, because of (7), (13) and (4.1 a) of [12]

$$\varkappa = \varepsilon = \pi = \varrho = \sigma = 0,\ \bar{\alpha} + \beta = \Omega,\ \bar{\alpha} - \beta = \Gamma\,,$$

the coefficients γ, ν remaining undetermined. A straightforward calculation yields then the following relations:

$$\left.\begin{aligned} &L\Gamma = L\Omega = L\lambda = 0,\ L\mu = \Psi_2\,,\\ &\Phi_0 = 0,\ L\Phi_1 = 0,\ L\Phi_2 = M^+\Phi_1\,,\\ &\Psi_0 = \Psi_1 = 0,\ L\Psi_2 = 0,\ L\Psi_3 = M^+\Psi_2 + \Phi_1^+ M^+\Phi_1\,,\\ &L\Psi_4 = (M^+ + \Gamma^+ + \Omega^+)\Psi_3 - 3\lambda\Psi_2 + \Phi_1^+(M^+ + \Gamma^+ + \Omega^+)\Phi_2\\ &\qquad - 2\lambda\Phi_1\Phi_1^+\,, \end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned} M\Gamma^+ + M^+\Gamma - 2\Gamma\Gamma^+ &= -\Psi_2 - \Psi_2^+ + 2\Phi_1\Phi_1^+\,,\\ M\Omega^+ - M^+\Omega - \Gamma\Omega^+ + \Gamma^+\Omega &= -(\Psi_2 - \Psi_2^+) = -L(\mu - \mu^+)\,, \end{aligned}\right\} \tag{15}$$

$$M\mu^+ + \Omega\mu^+ - M^+\lambda^+ + 2\Gamma^+\lambda^+ - \Omega^+\lambda^+ = \Psi_3^+ - \Phi_1\Phi_2^+\,, \tag{16}$$

where Ψ_i and Φ_i are the components of the Weyl and Maxwell spinor, respectively, as defined in [12].

From these equations, it can be immediately read that the horizon is determined, if one prescribes

1) its metric and affine structure,
2) the phase of Φ_1 at a Cauchy surface $\mathscr{S}$,
3) Φ_2 and Ψ_4 at the Cauchy surface $\mathscr{S}$.

By a Cauchy surface, any space-like properly embedded two-dimensional submanifold of $\mathscr{M}$ is meant that intersects all rays of $\mathscr{M}$.

b) Quotient Surface

The set $\tilde{\mathscr{S}}$ of all rays of $\mathscr{M}$ can be given a differentiable structure assuming the projection map $\pi : \mathscr{M} \to \tilde{\mathscr{S}}$ differentiable; the point $p \in \mathscr{M}$ is mapped by π in the ray $\tilde{p}$ going through p. The subspace of $T_p(\mathscr{M})$ tangent to this ray, i.e. the space $T_p(\tilde{p})$ of all vectors of the form aL, is mapped by π_* into the zero vector of $T_{\tilde{p}}(\tilde{\mathscr{S}})$. There is one-to-one correspondence, therefore, between points of $T_{\tilde{p}}(\tilde{\mathscr{S}})$ and the classes $X_p + aL$ of $T_p(\mathscr{M})/T_p(\tilde{p})$. All forms $\omega \in T_p^*(\mathscr{M})$ such that $\omega(L) = 0$ are linear forms on classes of $T_p(\mathscr{M})/T_p(\tilde{p})$ and determine, via the correspondence, forms in $T_{\tilde{p}}^*(\tilde{\mathscr{S}})$. It holds

$$\pi^*(\tilde{\omega})(X) = \tilde{\omega}(\pi_*(X))$$

([8]). The metric g in $\mathscr{M}$ induces a non-degenerate metric on the classes $T_p(\mathscr{M})/T_p(\tilde{p})$, because $T_p(\tilde{p})$ is the maximal subspace of $T_p(\mathscr{M})$ orthogonal to $T_p(\mathscr{M})$. Thus, we have a unique negative definite metric $\tilde{g}$ on $\tilde{\mathscr{S}}$ satisfying $\pi^*(\tilde{g}) = g$ at p, i.e. $g(X_p, Y_p) = \tilde{g}(\pi_*(X_p), \pi_*(Y_p))$. The metric $\tilde{g}$ is well-defined on $\tilde{\mathscr{S}}$, i.e. independent of the choice of p along $\tilde{p}$, because of

$$\mathscr{L}_L g = 0\,, \tag{17}$$

$\mathscr{L}_X$ denoting the Lie derivative along X.

Lemma 2. *Let X be a vector field on $\mathcal{M}$. There is a vector field $\tilde{X}$ on $\tilde{\mathcal{S}}$ π-related to X, iff $[L, X] = aL$, where a is an arbitrary function.*

Proof. 1) L is π-related to the zero vector field on $\tilde{\mathcal{S}}$. Suppose there is $\tilde{X} = \pi_*(X)$. Then $\pi_*([L, X]_p) = [0, \tilde{X}_{\pi(p)}] = 0$. Therefore, $[L, X]_p \in T_p(\tilde{p})$, for any $p \in \mathcal{M}$.

2) If $[L, X] = aL$, then there is X' and a function b such that $X' = X + bL$ and $[L, X'] = 0 : b$ satisfies the ordinary differential equation $Lb = 0$. Because $\mathcal{L}_L(X') = 0$, the integral curves of X' are dragged along by the rays; hence, the two integral curves of X' starting at any two points p_1, p_2 on the same ray have the same projection. The tangent vector $\tilde{X}_{\pi(p_i)}$ to this projection at $\pi(p_i)$ is $\pi_*(X'_{p_i})$. Thus, for X', there is a unique π-related field $\tilde{X}$ on $\tilde{\mathcal{S}}$. Now, $\pi_*(X) = \pi_*(X')$, and $\tilde{X}$ is π-related to X as well, q.e.d.

It follows, that any real vector field X on $\mathcal{M}$ for which there is a π-related field in $\tilde{\mathcal{S}}$ can be written in the form $X = xL + z^+ M + zM^+$, where x is an arbitrary real function on $\mathcal{M}$ and z is a complex one satisfying $Lz = 0$.

For any vector field $\tilde{X}$ on $\tilde{\mathcal{S}}$, there is an operator $\tilde{\nabla}_{\tilde{X}}$ on vector fields in $\tilde{\mathcal{S}}$ defined as follows. Let $\tilde{X} = u^+\tilde{M} + u\tilde{M}^+$, $\tilde{Y} = v^+\tilde{M} + v\tilde{M}^+$, where $\tilde{M} = \pi_*(M)$. Choose two fields X, Y on $\mathcal{M}$ such that

$$X = xL + (\pi^*(u^+))M + (\pi^*(u))M^+ ,$$

$$Y = yL + (\pi^*(v^+))M + (\pi^*(v))M^+ ,$$

x, y are arbitrary functions on $\mathcal{M}$. Then, $\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \pi_*(\nabla_X Y)$. It is easy to show that $\tilde{\nabla}_{\tilde{X}}\tilde{Y}$ so defined does not depend on x and y,

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = (u^+(\tilde{M} - \tilde{\Gamma})v^+ + u(\tilde{M}^+ + \tilde{\Gamma}^+)v^+)\tilde{M} + (u^+(\tilde{M} + \tilde{\Gamma})v + u(\tilde{M}^+ - \tilde{\Gamma}^+)v)\tilde{M}^+ ,$$

where $\tilde{\Gamma} = \pi(\Gamma)$, and that $\tilde{\nabla}$ is the symmetric Riemannian connection on $\tilde{\mathcal{S}}$ corresponding to $\tilde{g}$. In particular, we have

$$\tilde{\nabla}_{\tilde{M}}\tilde{M} = -\tilde{\Gamma}\tilde{M}, \qquad \tilde{\nabla}_{\tilde{M}}\tilde{M}^+ = \tilde{\Gamma}\tilde{M}^+ ,$$

$$\tilde{R}_{ABCD} = \tilde{K}(\tilde{g}_{AD}\tilde{g}_{BC} - \tilde{g}_{AC}\tilde{g}_{BD}), \tag{18}$$

$$\tilde{K} = \tilde{M}\tilde{\Gamma}^+ + \tilde{M}^+\tilde{\Gamma} - 2\tilde{\Gamma}\tilde{\Gamma}^+ , \tag{19}$$

where $\tilde{K}$ is the scalar curvature of $\tilde{\mathcal{S}}$. With (19), the first equation of (15) is

$$\tilde{K} = -\Psi_2 - \Psi_2^+ + 2\Phi_1\Phi_1^+ . \tag{20}$$

We shall often need the component transcription of the last equation of (15), too. Defining the real vector Ω_A by

$$\Omega_A = \Omega\tilde{M}_A^+ + \Omega^+\tilde{M}_A , \tag{21}$$

we find that Ω_A is invariant under space rotation $M' = e^{i\varphi} \cdot M$, under null rotation $M' = M + \xi L$ and under rescaling $L' = \eta L$, it behaves like

$$\Omega'_A = \Omega_A - \frac{\partial(\lg \eta)}{\partial x^A}. \tag{22}$$

The second equation of (15) is, then, equivalent to

$$\frac{\partial \Omega_1}{\partial x^2} - \frac{\partial \Omega_2}{\partial x^1} = -i(\Psi_2 - \Psi_2^+)\sqrt{\tilde{g}} \tag{23}$$

where x^A are some coordinates on the manifold $\tilde{\mathscr{S}}$ and $\tilde{g}$ is the determinant of the matrix $\tilde{g}_{AB}$ of the metric components in these coordinates.

A straightforward verification from the definition yields that Φ_1 and Ψ_2 are independent of the choice of L, M, and are, therefore, invariants on $\mathscr{M}$.

In what follows, we shall consider only those TGNH $\mathscr{M}$ with the following properties

1) the quotient surface $\tilde{\mathscr{S}}$ of $\mathscr{M}$ is compact orientable surface,

2) $\mathscr{M}$ is, topologically, $\tilde{\mathscr{S}} \times R^1$.

(The last condition excludes, e.g. the TGNH separating the Taub from NUT part in Taub-NUT space.)

Any TGNH satisfying 1) and 2) is called a horizon.

3. Horizon Symmetries

Let $\varphi : \mathscr{M} \to \mathscr{M}$ be a diffeomorphism onto satisfying

$$g_p(X, Y) = g_{\varphi(p)}(\varphi_*(X), \varphi_*(Y)), \tag{24}$$

$$\varphi_*(\nabla_X Y) = \nabla_{\varphi_*(X)} \varphi_*(Y), \tag{25}$$

for arbitrary $p \in \mathscr{M}$ and arbitrary vector fields X, Y on $\mathscr{M}$. Then, φ is called a horizon symmetry, because it preserves all structures of the horizon.

Let us call φ logitudinal HS, if it does not move the rays of the horizon. Then, (24) is satisfied because of (17) and the only non-trivial condition is (25). We find some implications of it.

Choose a ray $\tilde{p}$, and four different points a, b, c, d, on it. The number $(\alpha(b) - \alpha(a)) : (\alpha(d) - \alpha(c))$ is independent of the choice of the affine parameter α along $\tilde{p}$, and is, therefore, an affine invariant of the figure consisting of a, b, c, d. φ must preserve that invariant, or

$$\frac{\alpha(\varphi(b)) - \alpha(\varphi(a))}{\alpha(b) - \alpha(a)} = \frac{\alpha(\varphi(d)) - \alpha(\varphi(c))}{\alpha(d) - \alpha(c)} = k \tag{26}$$

where k is a non-zero real, dependent only on φ and $\tilde{p}$, independent of the choice of a and b, because of the first half of (26).

Let $\tilde{p}_1$ and $\tilde{p}_2$ be two different rays, a, b points on p_1, c, d points on p_2 and $C:[0,1]\to\mathscr{M}$ a smooth curve with $C(0)=a$, $C(1)=c$. Parallel transport of the vector $\exp_a^{-1}(b)$ along C defines a unique vector X at c and X a unique point $b'=\exp_c(X)$ on $\tilde{p}_2$. Again, the real $(\alpha(b')-\alpha(c))$: : $(\alpha(d)-\alpha(c))$, where α is an affine parameter along $\tilde{p}_2$, is an affine invariant of the constructed figure, and is, therefore, preserved by φ. Using the relation $L\Omega=0$, we obtain easily that $k_1=k_2$ with k_l defined by (26) on $\tilde{p}_l$. We have shown

Lemma 3. *Any longitudinal HS φ defines a unique non-zero real k, called affine magnification of φ, so that*

$$k=\frac{\alpha(\varphi(b))-\alpha(\varphi(a))}{\alpha(b)-\alpha(a)}$$

for any two points a, b on any ray $\tilde{p}$, and for any affine parameter α along $\tilde{p}$.

We call φ a collineation, if $k\neq 1$, and a translation, if $k=1$. Let φ be a collineation and $|k|<1$ (if $|k|>1$, choose φ^{-1} instead of φ). Then, the series $\varphi^n(a)$ is convergent for any point $a\in\mathscr{M}$, and $p_0=\lim_{n\to\infty}\varphi^n(a)$ is a well-defined point on any ray $\tilde{p}$, for $a\in\tilde{p}$, because $\lim_{n\to\infty}\varphi^n(a)=\lim_{n\to\infty}\varphi^n(b)$ for any two points a, b on $\tilde{p}$. All such p_0 form a Cauchy surface $\mathscr{S}_0$ in $\mathscr{M}$, satisfying $\varphi(\mathscr{S}_0)=\mathscr{S}_0$, and it is the maximal subset of $\mathscr{M}$ kept pointwise fixed by φ. It holds [8]

Lemma 4. *Let $\varphi:\mathscr{M}\to\mathscr{M}$ be a diffeomorphism onto, X_1, X_2, X_3 a frame field on $\mathscr{M}$ and Γ^l_{jk}, Γ'^l_{jk} defined by*

$$\nabla_{X_i}X_j=\Gamma^k_{ij}X_k\,,\qquad \nabla_{\varphi_*(X_i)}\varphi_*(X_j)=\Gamma'^k_{ij}\varphi_*(X_k)\,.$$

Then, φ is a longitudinal HS, iff

1) $\pi\circ\varphi\circ\pi^{-1}$ *= identity on* $\tilde{\mathscr{S}}$,

2) $\Gamma^l_{jk}=\Gamma'^l_{jk}$ *for all* $i, j, k=1,2,3$.

Lemma 5. *Let Z be a vector field, X_1, X_2, X_3 a frame field along $\mathscr{M}$ satisfying $\mathscr{L}_Z(X_l)=0$, and Γ^l_{jk} the corresponding rotation coeff.*

$$\nabla_{X_i}X_j=\Gamma^k_{ij}X_k\,.$$

Then, Z generates a one-dimensional group of longitudinal HS, iff

1) *Z is tangential to rays,*

2) $Z\Gamma^i_{jk}=0$.

Proof. 1) Suppose that Z generates a one-dimensional group of affine transformations. According to [8], p. 231,

$$\mathscr{L}_Z(\nabla_X Y)-\nabla_X\mathscr{L}_Z(Y)=\nabla_{[Z,X]}Y \tag{27}$$

for any X, Y, in particular for $X=X_i$, $Y=X_j$. But $\mathscr{L}_Z(X_j)=[Z,X_j]=0$, hence $0=\mathscr{L}_Z(\nabla_{X_i}X_j)=\mathscr{L}_Z(\Gamma^k_{ij}X_k)=(Z\Gamma^k_{ij})X_k$, or $Z\Gamma^k_{ij}=0$.

2) Suppose that $Z\Gamma^k_{ij}=0$. We show that (27) is satisfied for arbitrary $X=x^iX_i$, $Y=y^iY_i$: From the identity

$$\begin{aligned}(Zx^i)(X_iy^j)X_j+x^i(Z(X_iy^j))X_j+(Zx^i)y^j\Gamma^k_{ij}X_k+x^i(Zy^j)\Gamma^k_{ij}X_k\\ -x^i(X_i(Zy^j))X_j \qquad\qquad -x^i(Zy^j)\Gamma^k_{ij}X_k\\ =(Zx^i)(X_iy^j)X_j+(Zx^i)y^j\Gamma^k_{ij}X_k\end{aligned}$$

we have

$$\mathscr{L}_Z(x^i(X_iy^j)X_j+x^iy^j\Gamma^k_{ij}X_k)-x^i\nabla_{X_i}((Zy^j)X_j)=(Zx^i)\nabla_{X_i}(y^jX_j),$$

$$\mathscr{L}_Z(x^i\nabla_{X_i}(y^jX_j))-x^i\nabla_{X_i}(\mathscr{L}_Z(y^jX_j))=\nabla_{(Zx^i)X_i}(y^jX_j),$$

which is nothing but (27), q.e.d.

Let φ be a collineation with a fixed-point Cauchy surface $\mathscr{S}_0$ and affine magnification $k>0$. Choose an affine coordinate α along $\mathscr{M}$ such that $\alpha=0$ at $\mathscr{S}_0$, the field $L=\partial/\partial\alpha$, and the field M tangentially to $\mathscr{S}_0$ satisfying $\nabla_LM=0$ everywhere on $\mathscr{M}$. If $\beta=\lg\alpha$, then $\beta(\varphi(p))=\beta(p)+\lg k$ for any $p\in\mathscr{M}$ with $\alpha(p)\neq 0$. Choose another frame L', M' as follows:

$$L'=\partial/\partial\beta,\qquad M'=M\quad\text{at}\quad\mathscr{S}_0,\qquad [L',M']=0\quad\text{on}\quad\mathscr{M}.$$

It follows that $L'=\alpha L$, $M'=M+\alpha\Omega L$, because of $M\alpha=-\alpha\Omega$. To show the last relation, we observe that $L(M\alpha)=M(L\alpha)+[L,M]\alpha$, and, according to (12), $L(M\alpha)=-\Omega$. On the other hand, $(M\alpha)_{\mathscr{S}_0}=0$. Clearly, $\varphi_*(L')=L'$, $\varphi_*(M')=M'$; the rotation coefficients with respect to the primed frame are given by

$$\nabla_{L'}L'=1\cdot L',\qquad \nabla_{L'}M'=\Omega L',\qquad \nabla_{M'}L'=\Omega L',\tag{28}$$

$$\nabla_{M'}M'=(e^{-\beta}\lambda^++M\Omega+\Gamma\Omega+\Omega^2)L'-\Gamma M',\tag{29}$$

$$\nabla_{M'}M'^+=(e^{-\beta}\mu_0+\Psi_2+M\Omega^++\Gamma\Omega^++\Omega\Omega^+)L'+\Gamma M'^+,\tag{30}$$

where μ_0 is defined by $\mu=\mu_0+\Psi_2\alpha$ according to (14). From the Eqs. (14), it follows at once that the primed rotation coefficients will be invariant under $\beta\to\beta+\lg k$ only if $\mu_0=\lambda=0$. Therefore, on $\mathscr{S}_0$, $\nabla_{M'}M'=-\Gamma M'$, $\nabla_{M'}M'^+=\Gamma M'^+$, i.e., $\mathscr{S}_0$ is totally autoparallel. (28)–(30) together with the Lemma 4 imply that the transformation $\beta\to-\beta$ is a collineation with $k=-1$. A combination of all these results with the Lemma 5 gives

Lemma 6. *The existence of a totally autoparallel Cauchy surface* $\mathscr{S}_0$ *on* $\mathscr{M}$ *implies the existence of a two-component* ($k<0$, $k>0$) *one-dimensional Lie group of collineations fixing* $\mathscr{S}_0$. *The locus of fixed points of any collineation on* $\mathscr{M}$ *is a totally autoparallel Cauchy surface.*

Let, next, φ be a translation. Choose an arbitrary Cauchy surface $\mathcal{S}$ and an affine coordinate α such that $\alpha=0$ at $\mathcal{S}$ and $\alpha(\varphi(p))=\alpha(p)+1$ for any $p\in\mathcal{M}$. Let $L=\partial/\partial\alpha$, M be tangential to $\mathcal{S}$ and $\nabla_L M=0$ everywhere on $\mathcal{M}$. Another frame L', M' is chosen such that $L'=L$, $M'=M+\alpha\Omega L$. Again, $\varphi_*(L')=L'$, $\varphi_*(M')=M'$; the new rotation coefficients are given by

$$\nabla_{L'}L'=0, \qquad \nabla_{L'}M'=\Omega L', \qquad \nabla_{M'}L'=\Omega L',$$

$$\nabla_{M'}M'=(\lambda^+ +(M\Omega+\Gamma\Omega+\Omega^2)\alpha)L'-\Gamma M',$$

$$\nabla_{M'}M'^+ =(\mu_0+(\Psi_2+M\Omega^+ -\Gamma\Omega^+ +\Omega\Omega^+)\alpha)L'+\Gamma M'^+ .$$

Invariance under $\alpha\to\alpha+1$ is equivalent to

$$M\Omega+\Gamma\Omega+\Omega^2=0, \tag{31}$$

$$\Psi_2+M\Omega^+ -\Gamma\Omega^+ +\Omega\Omega^+ =0. \tag{32}$$

Combination with the Lemma 5 and 6 yields

Lemma 7. *The existence of a longitudinal HS φ implies the existence of a one-dimensional group of longitudinal HS containing φ.*

Lemma 8. *If two one-dimensional groups φ_t and ψ_t of longitudinal HS contain a common element $\varphi_{T_1}=\psi_{T_2}$, $T_1 \neq 0$, then*

$$\varphi_{t\cdot T_1\cdot T_2^{-1}}=\psi_t \quad \textit{for any } t.$$

Lemma 9. *Two different totally autoparallel Cauchy surfaces $\mathcal{S}_1$, $\mathcal{S}_2$ in $\mathcal{M}$ imply the existence of a translation φ such that $\varphi(\mathcal{S}_1)=\mathcal{S}_2$.*

Proof. Choose $\alpha=0$ at $\mathcal{S}_1$ and let $\alpha=\alpha(x^A)$ at $\mathcal{S}_2$. The collineations implied by Lemma 5 can be written

$$\alpha\to k\alpha, \qquad \alpha-\alpha(x^A)\to l(\alpha-\alpha(x^A)).$$

Combining these collineations with $k=2$, $1=1/2$, we have

$$\alpha\to 2\big(\alpha(x^A)+\tfrac{1}{2}(\alpha-\alpha(x^A))\big)=\alpha+\alpha(x^A), \quad \text{q.e.d.}$$

Lemma 10. *There cannot be two different one-dimensional translational groups on a horizon $\mathcal{M}$.*

Proof. If the groups are different, we find, for any ray $\tilde{p}$, a combination of translations which is not the identity and which keeps $\tilde{p}$ fixed. In fact, we shall have a whole one-dimensional group $\Phi_{\tilde{p}}$ of such translations. Obviously, there is a dense subset $\tilde{\mathcal{S}}'$ of $\tilde{\mathcal{S}}$ such that, if $\tilde{p}\in\tilde{\mathcal{S}}$, then $\Phi_{\tilde{p}}$ induces a one-dimensional group of null rotations in $T_p(\mathcal{M})$ for every $p\in\tilde{p}$. Therefore, components of all quantities related to a frame L, M must be numerically the same as those related to the frames (L', M')

$= \Phi_{\tilde{p}*}(L, M): L' = L,\ M' = M + \xi(t) L$, t is the parameter of the group $\Phi_{\tilde{p}}$. Now, the corresponding components of the left hand side of (16) are related as follows

$$(\text{l.h.s. of (16)})' = (\text{l.h.s. of (16)}) - \xi(t)\Psi_2\ ;$$

invariance is possible only if

$$\Psi_2 = 0 \tag{33}$$

on $\tilde{p}$. But $\tilde{p}$ are dense in $\tilde{\mathscr{S}}$, so (33) holds everywhere. Equation (22) and (23) then implies that there is a rescaling after which $\Omega = 0$. In the corresponding frame L, M

$$\nabla_L L = 0, \quad \nabla_L M = 0, \quad \nabla_M L = 0$$

$$\nabla_M M = \lambda^+ L - \Gamma M, \quad \nabla_M M^+ = \mu L + \Gamma M^+\ ,$$

μ, λ, Γ being constants along rays and $[L, M] = 0$. According to Lemma 5, L generates a one-dimensional translational group. There must be another linearly independent translation generator, say, $L' = \eta L$, where η is a non-constant real function on $\mathscr{M}$, $L\eta = 0$. In the frame $L', M' = M$, $\Omega' = M(\lg|\eta|)$, and must satisfy (31) and (32) with (33). For $y = \lg|\eta|$, these equations read in the component formalism

$$\frac{\partial^2 y}{\partial x^A \partial x^B} = \frac{\partial y}{\partial x^A} \frac{\partial y}{\partial x^B} + \Gamma^C_{AB} \frac{\partial y}{\partial x^C}. \tag{34}$$

The integrability conditions of (34) yield either $y = \text{const}$ or $\tilde{K} = 0$. In the second case, $\tilde{\mathscr{S}}$ is a torus and periodic coordinates x^A can be chosen so that $\Gamma^C_{AB} = 0$. The only non-zero solution of (34) is, then

$$y = \lg(\eta^0 - \eta^0_A x^A)$$

with η^0, η^0_A constants. This function shows no periodicity unless all $\eta^0_A = 0$, giving $y = \text{const}$. But this is the old group generated by L, q.e.d.

(The compactness of $\tilde{\mathscr{S}}$ was used here; our linear solutions for η work all very well in the null plane of Minkowski space-time.)

Theorem 1. *The group $\mathscr{H}$ of all longitudinal HS of a given horizon $\mathscr{M}$ can be of the following four types, at most:*

1) *identity*
2) *one-dimensional two-component collineation group,*
3) *one-dimensional translation group,*
4) *two-dimensional two-component group generated by* 2) *and* 3).

We shall see later that all three types really exist. Next, we turn our attention to the so-called transversal groups. In general, any HS φ must map null curves into null curves, i.e. rays into rays. Thus, it induces a diffeomorphism $\tilde{\varphi}: \tilde{\mathscr{S}} \to \tilde{\mathscr{S}}$, $\tilde{\varphi} = \pi \circ \varphi \circ \pi^{-1}$, on $\tilde{\mathscr{S}}$, which clearly must be an isometry. Let $\mathscr{G}$ be the group of all HS of $\mathscr{M}$. Denote by $\tilde{\mathscr{G}} = \pi \circ \mathscr{G} \circ \pi^{-1}$ and by $\pi_g : \mathscr{G} \to \tilde{\mathscr{G}}$ the homomorphism defined by $\pi_g(\varphi) = \pi \circ \varphi \circ \pi^{-1}$. The kernel of π_g is the group $\mathscr{H}$ of all longitudinal HS of $\mathscr{M}$. $\tilde{\mathscr{G}}$ is called transversal group of $\mathscr{M}$. The possible structures of isometry groups of a compact orientable surface are well-known, we have, therefore

Theorem 2. *The component of the identity of the transversal group $\tilde{\mathscr{G}}$ of a given horizon $\mathscr{M}$ can be of the following four types*

1) *identity,*
2) $SO(2)$ *(axial symmetry),*
3) $SO(3)$ *(sphere),*
4) $SO(2) \times SO(2)$ *(torus).*

The discrete groups, also well-known, are not interesting for us. Our investigation on HS is nicely completed by proving the following "Decomposition Theorem":

Theorem 3. *Let $\mathscr{G}$ be the group of all HS of a given horizon $\mathscr{M}$, $\mathscr{H}$ the subgroup of all longitudinal symmetries and $\tilde{\mathscr{G}}$ the transversal group. Then,*

$$\mathscr{G} \cong \mathscr{H} \times \tilde{\mathscr{G}} .$$

First, we show the following

Lemma 11. *Let φ_t be a one-dimensional group of HS on $\mathscr{M}$ with space-like trajectories which are not closed. Then, there is a whole two-dimensional group of HS on $\mathscr{M}$ containing φ_t as a subgroup, a one-dimensional longitudinal subgroup ψ_t and a one-dimensional subgroup Φ_t with space-like closed trajectories such that $\tilde{\Phi}_t = \tilde{\varphi}_t$.*

Proof. According to the Theorem 2, $\tilde{\varphi}_t$ is isomorphic to $SO(2)$, so there is a minimum parameter value $t = T > 0$ such that $\tilde{\varphi}_T$ is identity on $\tilde{\mathscr{S}}$. Because the trajectories of φ_t are not closed, φ_T is not the identity on $\mathscr{M}$ and must, therefore, be purely longitudinal. The Lemma 7 then implies, that there is a whole group, ψ_s, say, of longitudinal HS; let us choose s such that $\psi_T = \varphi_T$. For a fixed t, $\varphi_{-t} \circ \psi_s \circ \varphi_t$ is again a longitudinal group, containing $\psi_T = \varphi_{-t} \circ \psi_T \circ \varphi_t$, hence $\varphi_{-t} \circ \psi_s \circ \varphi_t = \psi_s$, because of the Lemma 8. But the last identity means that any φ_t commutes with any ψ_s. Then $\Phi_t = \psi_{-t} \circ \varphi_t$ must be a one-dimensional group with space-like closed trajectories in $\mathscr{M}$, satisfying $\tilde{\Phi}_t = \tilde{\varphi}_t$, q.e.d.

In fact, we have shown the Theorem 3 in case that $\tilde{\mathscr{G}}$ is one-dimensional. If $\tilde{\mathscr{G}}$ has more dimensions, we can construct, for any one-dimensional subgroup of $\tilde{\mathscr{G}}$, the corresponding group with space-like closed trajectories in $\mathscr{M}$, as in the proof of the Lemma 11. Then, it must be shown that i) all elements of these "closed" subgroups commute with all elements of $\mathscr{H}$, ii) these "closed" subgroups generate a group whose trajectories in $\mathscr{M}$ are two-dimensional.

i) Denote by N the dimension of $\mathscr{G}$, by $N-n$ the dimension of $\mathscr{H}$ and choose a basis for the Lie algebra of $\mathscr{G}$ such that the last $N-n$ elements $H_{n+1}, \ldots, H_N$ form a basis for the subalgebra of $\mathscr{H}$. Each G_i from the remaining n elements $G_1, \ldots, G_n$ generates a one-dimensional subgroup $\mathscr{G}_i$ of $\mathscr{G}$ and the vector fields $\tilde{G}_i = \pi_{g*}(G_i)$ form a basis of $\tilde{\mathscr{G}}$. The generator G_i can be chosen in such a way that $\mathscr{G}_i$ all have closed trajectories, according to the Lemma 11. Then, we show that all elements of $\mathscr{G}_i$, $i=1, \ldots, n$, commute with all elements of $\mathscr{H}$. Choose $h \in \mathscr{H}$; $\mathscr{G}'_i = h^{-1} \cdot \mathscr{G}_i \cdot h$ is another one-dimensional transversal subgroup of $\mathscr{G}$ such that $\pi_g(\mathscr{G}'_i) = \pi_g(\mathscr{G}_i)$, so there is a generator G'_i of $\mathscr{G}'_i$ fulfilling $\pi_*(G'_i) = \pi_*(G_i)$. Hence, the vector field $G_i - G'_i$ generates a subgroup $\mathscr{H}_i$ of $\mathscr{H}$, if $G_i - G'_i \neq 0$. The trajectories of $\mathscr{G}_i$ and $\mathscr{G}'_i$ are both closed; starting from a common point p, they must have, according to the Theorem of Rolle, equal tangents at some point, $\tilde{q}$, say, before returning to p. But this means that $G_i(q_1) = G'_i(q_2)$ for any two points q_1, q_2 on $\tilde{q}$ and $\tilde{q}$ is a fixed ray of $\mathscr{H}_i$. This is possible only if $\mathscr{H}_i$ is trivial or a translation group. In the second case, $\mathscr{H}_i$ is the only translational subgroup of $\mathscr{H}$ according to the Lemma 10, and its fixed ray is a property of the horizon which must be symmetric with respect to $\mathscr{G}_i$. Hence, $\mathscr{H}_i$ is trivial, $G_i = G'_i$, and $\mathscr{G}_i = \mathscr{G}'_i$, q.e.d.

ii) First, we need the following results on HS:

Lemma 12. *The generator* $X = xL + z^+M + zM^+$ *of a HS must satisfy the equations*

$$(M+\Gamma)z=0,\ (M-\Gamma)z^+ + (M^+ - \Gamma^+)z = 0,\ Lz=0\,, \tag{35}$$

$$L(Lx)=0,\ M(Lx) = -(\Psi_2 - \Psi_2^+)z\,. \tag{36}$$

Proof. The Eqs. (35) are nothing but the Killing equation (17) written in the complex formalism. The Eq. (36) follows from the first equation of [8], p. 236, setting $Y=Z=L$ and $Y=M$, $Z=L$, and from Eqs. (35) and (14).

To show ii), we choose a special case $\tilde{\mathscr{G}} = SO(3)$. The proof for a torus, which is, according to the Theorem 2, the only rest to be removed, then, is similar and even simpler (and less physically interesting, [6]), so we shall not give it here.

Choose spherical coordinates ϑ and φ on $\tilde{\mathscr{S}}$. The generators of $\tilde{\mathscr{G}}$ can be chosen

$$\tilde{G}_1 = \sin\varphi \frac{\partial}{\partial\vartheta} + \operatorname{ctg}\vartheta \cos\varphi \frac{\partial}{\partial\varphi}\,,$$

$$\tilde{G}_2 = \cos\varphi \frac{\partial}{\partial\vartheta} - \operatorname{ctg}\vartheta \sin\varphi \frac{\partial}{\partial\varphi}\,,$$

$$\tilde{G}_3 = \frac{\partial}{\partial\varphi}\,.$$

If α is an affine coordinate on $\mathscr{M}$, generators G_1, G_2, G_3 can be of the form

$$G_1 = \sin\varphi \frac{\partial}{\partial\vartheta} + \operatorname{ctg}\vartheta \cos\varphi \frac{\partial}{\partial\varphi} + x_1 \frac{\partial}{\partial\alpha}, \tag{37}$$

$$G_2 = \cos\varphi \frac{\partial}{\partial\vartheta} - \operatorname{ctg}\vartheta \sin\varphi \frac{\partial}{\partial\varphi} + x_2 \frac{\partial}{\partial\alpha}, \tag{38}$$

$$G_3 = \frac{\partial}{\partial\varphi} + x_3 \frac{\partial}{\partial\alpha}. \tag{39}$$

The symmetry represented by (37), (38), and (39) implies $\Psi_2 = \text{const}$ over the horizon, and, because of (23), $\Psi_2 - \Psi_2^+ = 0$. We can, therefore, choose α in such a way that $\Omega = 0$, if $L = \partial/\partial\alpha$. α is determined up to a transformation $\alpha' = \eta\alpha + \alpha(\vartheta, \varphi)$, where η is a constant. The vector field M taken tangent to the surface $\alpha = 0$ remains, then, tangent to any surface $\alpha = \text{const}$. In such a coordinate system

$$M = M^\vartheta \frac{\partial}{\partial\vartheta} + M^\varphi \frac{\partial}{\partial\varphi},$$

where M^ϑ and M^φ are some complex functions independent of α, and we have from (37), (38) and (39) $G_i = x_i L + z_i^+ M + z_i M^+$. The Eqs. (36) give for x_i

$$x_i = \xi_i \cdot \alpha + a_i(\vartheta, \varphi) \qquad i = 1, 2, 3, \tag{40}$$

ξ_i being constant. Each group $\mathscr{G}_i$ has two fixed rays ($\mathscr{G}_1$, e.g., $\vartheta = \pi/2$, $\varphi = 0, \pi$), and because its trajectories are closed, at these rays, the corresponding x_i must be zero. Hence,

$$\xi_1 = \xi_2 = \xi_3 = 0, \tag{41}$$

$$a_1\left(\frac{\pi}{2}, 0\right) = a_1\left(\frac{\pi}{2}, \pi\right) = 0, \tag{42}$$

$$a_2\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = a_2\left(\frac{\pi}{2}, \frac{3\pi}{2}\right) = 0, \tag{43}$$

$$a_3(0, 0) = a_3(\pi, 0) = 0.$$

In addition to this, a_i's must be continuous:

$$\frac{\partial a_i(0, \varphi)}{\partial\varphi} = \frac{\partial a_i(\pi, \varphi)}{\partial\varphi} = 0, \qquad i = 1, 2, 3. \tag{44}$$

The general form of a generator of $\mathscr{H}$ is $H = (\eta \cdot \alpha + l(\vartheta, \varphi))\, \partial/\partial\alpha$, where η is a constant, because of Eq. (36). If there is an H with $\eta \neq 0$, we can specify α further as follows

$$\alpha' = \alpha + \frac{1}{\eta}\left(l(\vartheta, \varphi) - 1\right),$$

keeping everything valid and reducing H to

$$H=(\eta\cdot\alpha'+1)\,\partial/\partial\alpha'\,. \tag{45}$$

The commutators $[G_i, H]$ must be zero because of i), or, with the help of (37)–(40), (42), and (45), $\eta\cdot a_i=0$. Let $a_i=0$ for all $i=1,2,3$. Then, the Theorem follows immediately. Suppose, therefore, that one a_i differs from zero for any choice of the affine coordinate α. Then, all generators of $\mathscr{H}$ must have the form $H=l(\vartheta,\varphi)\,\partial/\partial\alpha$ and generate translations. According to the Lemma 10, there is at most one linearly independent H. i) yields

$$[H, G_i]=-(G_i l)\,\partial/\partial\alpha=0\,, \qquad i=1,2,3\,,$$

or, $l=\text{const}$. Thus, we can choose the affine parameter α so that the following relations hold

$$H=\frac{\partial}{\partial\alpha}\,,$$

$$a_1\left(\vartheta,\frac{\pi}{2}\right)=0\,, \tag{46}$$

$$a_3(\vartheta,\varphi)=0\,. \tag{47}$$

(46) and (47) reduce to the following construction of the surfaces $\alpha=\text{const}$: Choose a point p with $\vartheta=0$. Let $C(p,q)$ be the segment of the trajectory of $\mathscr{G}_1$ going through p, on which $\varphi=\pi/2$ and $0\leqq\vartheta\leqq\pi$, $\vartheta(q)=\pi$. Then, the surface $\alpha=\text{const}$ is $\mathscr{G}_3\cdot C(p,q)$.

The following must be generators of some HS in $\mathscr{M}$

$$[G_3, G_1]-G_2=\left(\frac{\partial a_1}{\partial\varphi}-a_2\right)\frac{\partial}{\partial\alpha}\,,$$

$$[G_2, G_3]-G_1=\left(-\frac{\partial a_2}{\partial\varphi}-a_1\right)\frac{\partial}{\partial\alpha}\,,$$

$$[G_1, G_2]-G_3=\left(\sin\varphi\frac{\partial a_2}{\partial\vartheta}-\cos\varphi\frac{\partial a_1}{\partial\vartheta}+\operatorname{ctg}\vartheta\cos\varphi\frac{\partial a_2}{\partial\vartheta}\right.$$
$$\left.+\operatorname{ctg}\vartheta\sin\varphi\frac{\partial a_1}{\partial\varphi}\right)\frac{\partial}{\partial\alpha}\,.$$

They all generate longitudinal groups, therefore

$$\frac{\partial a_1}{\partial\varphi}-a_2=A\,, \qquad \frac{\partial a_2}{\partial\varphi}+a_1=B\,, \tag{48}$$

$$\sin\varphi\frac{\partial a_2}{\partial\vartheta}-\cos\varphi\frac{\partial a_1}{\partial\vartheta}+\operatorname{ctg}\vartheta\left(\cos\varphi\frac{\partial a_2}{\partial\varphi}+\sin\varphi\frac{\partial a_1}{\partial\varphi}\right)=C\,, \tag{49}$$

where A, B, C are some reals. (48) have the general solution

$$a_1=B+f(\vartheta)\sin(\varphi-g(\vartheta))\,, \qquad a_2=-A+f(\vartheta)\cos(\varphi-g(\vartheta))\,, \tag{50}$$

where f and g are arbitrary functions of ϑ. With (50), (49) is equivalent to

$$(f \operatorname{ctg} \vartheta + f') \sin g - f g' \cos g = C . \tag{51}$$

(44), (42), (43), and (50) imply

$$f(0) = f(\pi) = 0 , \tag{52}$$

$$A = B = 0, \quad f\left(\frac{\pi}{2}\right) \sin g \left(\frac{\pi}{2}\right) = 0 . \tag{53}$$

(46), (50), and (53) imply

$$f(\vartheta) \cos g(\vartheta) = 0 . \tag{54}$$

From (54), we have either

$$f(\vartheta) = 0 , \tag{55}$$

or

$$\cos g = 0 . \tag{56}$$

(53) and (55) yield $a_1 = a_2 = 0$, (51) and (56) imply $f' + f \operatorname{ctg} \vartheta = \pm C$, which has the general solution of the form

$$f = \pm C \operatorname{ctg} \vartheta - D \frac{1}{\sin \vartheta} .$$

Then, (52) can only be satisfied, if $C = D = 0$, and we have again $a_1 = a_2 = 0$, which is a contradiction.

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